Research Article

Vanishing Power Values of Commutators with Derivations on Prime Rings

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Let R be a prime ring of char $R \neq 2$, d a nonzero derivation of R and ρ a nonzero right ideal of R such that $[[d(x), x]_n, [y, d(y)]_m]^t = 0$ for all $x, y \in \rho$, where $n \geq 0, m \geq 0, t \geq 1$ are fixed integers. If $[\rho, \rho]\rho \neq 0$, then $d(\rho)\rho = 0$.

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1. Introduction

Throughout this paper, unless specifically stated, R always denotes a prime ring with center $Z(R)$ and extended centroid C , Q the Martindale quotients ring. Let n be a positive integer. For given $a, b \in R$, let $[a, b]_0 = a$ and let $[a, b]_1$ be the usual commutator $ab - ba$, and inductively for $n > 1$, $[a, b]_n = [[a, b]_{n-1}, b]$. By d we mean a nonzero derivation in R .

A well-known result proven by Posner [1] states that if $[[d(x), x], y] = 0$ for all $x, y \in R$, then R is commutative. In [2], Lanski generalized this result of Posner to the Lie ideal. Lanski proved that if U is a noncommutative Lie ideal of R such that $[[d(x), x], y] = 0$ for all $x \in U, y \in R$, then either R is commutative or $\text{char } R = 2$ and R satisfies S_4 , the standard identity in four variables. Bell and Martindale III [3] studied this identity for a semiprime ring R . They proved that if R is a semiprime ring and $[[d(x), x], y] = 0$ for all x in a non-zero left ideal of R and $y \in R$, then R contains a non-zero central ideal. Clearly, this result says that if R is a prime ring, then R must be commutative.

Several authors have studied this kind of Engel type identities with derivation in different ways. In [4], Herstein proved that if $\text{char } R \neq 2$ and $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. In [5], Filippis showed that if R is of characteristic different from 2 and ρ a non-zero right ideal of R such that $[\rho, \rho]\rho \neq 0$ and $[[d(x), x], [d(y), y]] = 0$ for all $x, y \in \rho$, then $d(\rho)\rho = 0$.

In continuation of these previous results, it is natural to consider the situation when $[[d(x), x]_n, [y, d(y)]_m]^t = 0$ for all $x, y \in R$, $n, m \geq 0$, $t \geq 1$ are fixed integers. We have studied this identity in the present paper.

It is well known that any derivation of a prime ring R can be uniquely extended to a derivation of Q , and so any derivation of R can be defined on the whole of Q . Moreover Q is a prime ring as well as R and the extended centroid C of R coincides with the center of Q . We refer to [6, 7] for more details.

Denote by $Q *_C C\{X, Y\}$ the free product of the C -algebra Q and $C\{X, Y\}$, the free C -algebra in noncommuting indeterminates X, Y .

2. The Case: R Prime Ring

We need the following lemma.

Lemma 2.1. *Let ρ be a non-zero right ideal of R and d a derivation of R . Then the following conditions are equivalent: (i) d is an inner derivation induced by some $b \in Q$ such that $b\rho = 0$; (ii) $d(\rho)\rho = 0$ (for its proof refer to [8, Lemma]).*

We mention an important result which will be used quite frequently as follows.

Theorem 2.2 (see Kharchenko [9]). *Let R be a prime ring, d a derivation on R and I a non-zero ideal of R . If I satisfies the differential identity $f(r_1, r_2, \dots, r_n, d(r_1), d(r_2), \dots, d(r_n)) = 0$ for any $r_1, r_2, \dots, r_n \in I$, then either (i) I satisfies the generalized polynomial identity*

$$f(r_1, r_2, \dots, r_n, x_1, x_2, \dots, x_n) = 0, \quad (2.1)$$

or (ii) d is Q -inner, that is, for some $q \in Q$, $d(x) = [q, x]$ and I satisfies the generalized polynomial identity

$$f(r_1, r_2, \dots, r_n, [q, r_1], [q, r_2], \dots, [q, r_n]) = 0. \quad (2.2)$$

Theorem 2.3. *Let R be a prime ring of char $R \neq 2$ and d a derivation of R such that $[[d(x), x]_n, [y, d(y)]_m]^t = 0$ for all $x, y \in R$, where $n \geq 0$, $m \geq 0$, $t \geq 1$ are fixed integers. Then R is commutative or $d = 0$.*

Proof. Let R be noncommutative. If d is not Q -inner, then by Kharchenko's Theorem [9]

$$g(x, y, u, v) = [[u, x]_n, [y, v]_m]^t = 0, \quad (2.3)$$

for all $x, y, u, v \in R$. This is a polynomial identity and hence there exists a field F such that $R \subseteq M_k(F)$ with $k > 1$, and R and $M_k(F)$ satisfy the same polynomial identity [10, Lemma 1]. But by choosing $u = e_{12}$, $x = e_{11}$, $v = e_{11}$ and $y = e_{21}$, we get

$$0 = [[u, x]_n, [y, v]_m]^t = (-1)^{tn} (e_{11} + (-1)^t e_{22}), \quad (2.4)$$

which is a contradiction.

Now, let d be Q -inner derivation, say $d = ad(a)$ for some $a \in Q$, that is, $d(x) = [a, x]$ for all $x \in R$, then we have

$$[[a, x]_{n+1}, [y, [a, y]]_m]^t = 0, \quad (2.5)$$

for all $x, y \in R$. Since $d \neq 0$, $a \notin C$ and hence R satisfies a nontrivial generalized polynomial identity (GPI). By [11], it follows that RC is a primitive ring with $H = \text{Soc}(RC) \neq 0$, and eHe is finite dimensional over C for any minimal idempotent $e \in RC$. Moreover we may assume that H is noncommutative; otherwise, R must be commutative which is a contradiction.

Notice that H satisfies $[[a, x]_{n+1}, [y, [a, y]]_m]^t = 0$ (see [10, Proof of Theorem 1]). For any idempotent $e \in H$ and $x \in H$, we have

$$0 = [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^t. \quad (2.6)$$

Right multiplying by e , we get

$$\begin{aligned} 0 &= [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^t e \\ &= [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^{t-1} \\ &\quad \cdot \{ [a, e]_{n+1} ([ex(1-e), [a, ex(1-e)]]_m) e - ([ex(1-e), [a, ex(1-e)]]_m) [a, e]_{n+1} e \} \\ &= [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^{t-1} \\ &\quad \cdot \left\{ [a, e]_{n+1} \left(\sum_{j=0}^m (-1)^j \binom{m}{j} [a, ex(1-e)]^j ex(1-e) [a, ex(1-e)]^{m-j} \right) e \right. \\ &\quad \left. - \left(\sum_{j=0}^m (-1)^j \binom{m}{j} [a, ex(1-e)]^j ex(1-e) [a, ex(1-e)]^{m-j} \right) [a, e]_{n+1} e \right\} \\ &= [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^{t-1} \\ &\quad \cdot \left\{ 0 - \left(\sum_{j=0}^m (-1)^j \binom{m}{j} (-ex(1-e) a)^j ex(1-e) (aex(1-e))^{m-j} \right) ae \right\} \\ &= -[[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^{t-1} \left(\sum_{j=0}^m \binom{m}{j} (ex(1-e) a)^{m+1} \right) e \\ &= -2^m [[a, e]_{n+1}, [ex(1-e), [a, ex(1-e)]]_m]^{t-1} (ex(1-e) a)^{m+1} e \\ &= (-)^t 2^{mt} (ex(1-e) a)^{(m+1)t} e. \end{aligned} \quad (2.7)$$

This implies that $0 = (-)^t 2^{mt} ((1-e)aex)^{(m+1)t+1}$. Since $\text{char } R \neq 2$, $((1-e)aex)^{(m+1)t+1} = 0$. By Levitzki's lemma [12, Lemma 1.1], $(1-e)aex = 0$ for all $x \in H$. Since H is prime ring, $(1-e)ae = 0$, that is, $ea = ae$ for any idempotent $e \in H$. Now replacing e with $1-e$, we get that $ea(1-e) = 0$, that is, $ea = ea$. Therefore for any idempotent $e \in H$, we have $[a, e] = 0$.

So a commutes with all idempotents in H . Since H is a simple ring, either H is generated by its idempotents or H does not contain any nontrivial idempotents. The first case gives $a \in C$ contradicting $d \neq 0$. In the last case, H is a finite dimensional division algebra over C . This implies that $H = RC = Q$ and $a \in H$. By [10, Lemma 2], there exists a field F such that $H \subseteq M_k(F)$ and $M_k(F)$ satisfies $[[a, x]_{n+1}, [y, [a, y]]_m]^t$. Then by the same argument as earlier, a commutes with all idempotents in $M_k(F)$, again giving the contradiction $a \in C$, that is, $d = 0$. This completes the proof of the theorem. $\square$

Theorem 2.4. *Let R be a prime ring of char $R \neq 2$, d a non-zero derivation of R and ρ a non-zero right ideal of R such that $[[d(x), x]_n, [y, d(y)]_m]^t = 0$ for all $x, y \in \rho$, where $n \geq 0, m \geq 0, t \geq 1$ are fixed integers. If $[\rho, \rho]\rho \neq 0$, then $d(\rho)\rho = 0$.*

We begin the proof by proving the following lemma.

Lemma 2.5. *If $d(\rho)\rho \neq 0$ and $[[d(x), x]_n, [y, d(y)]_m]^t = 0$ for all $x, y \in \rho$, $m, n \geq 0, t \geq 1$ are fixed integers, then R satisfies nontrivial generalized polynomial identity (GPI).*

Proof. Suppose on the contrary that R does not satisfy any nontrivial GPI. We may assume that R is noncommutative; otherwise, R satisfies trivially a nontrivial GPI. We consider two cases.

Case 1. Suppose that d is Q -inner derivation induced by an element $a \in Q$. Then for any $x \in \rho$,

$$[[a, xX]_{n+1}, [xY, [a, xY]]_m]^t \quad (2.8)$$

is a GPI for R , so it is the zero element in $Q *_C C\{X, Y\}$. Expanding this, we get

$$\begin{aligned} & \left([a, xX]_{n+1} \sum_{j=0}^m (-1)^j \binom{m}{j} [a, xY]^j xY [a, xY]^{m-j} \right. \\ & \left. - \sum_{j=0}^m (-1)^j \binom{m}{j} [a, xY]^j xY [a, xY]^{m-j} [a, xX]_{n+1} \right) A(X, Y) = 0, \end{aligned} \quad (2.9)$$

where $A(X, Y) = [[a, xX]_{n+1}, [xY, [a, xY]]_m]^{t-1}$. If ax and x are linearly C -independent for some $x \in \rho$, then

$$\begin{aligned} & \left((axX)^{n+1} \sum_{j=0}^m (-1)^j \binom{m}{j} [a, xY]^j xY [a, xY]^{m-j} \right. \\ & \left. - \sum_{j=0}^m (-1)^j \binom{m}{j} (axY)^j xY [a, xY]^{m-j} [a, xX]_{n+1} \right) A(X, Y) = 0. \end{aligned} \quad (2.10)$$

Again, since ax and x are linearly C -independent, above relation implies that

$$(-xY[a, xY]^m[a, xX]_{n+1})A(X, Y) = 0, \quad (2.11)$$

and so

$$(-xY(axY)^m(axX)^{n+1})A(X, Y) = 0. \quad (2.12)$$

Repeating the same process yields

$$(-xY(axY)^m(axX)^{n+1})^t = 0 \quad (2.13)$$

in $Q *_C C\{X, Y\}$. This implies that $ax = 0$, a contradiction. Thus for any $x \in \rho$, ax and x are C -dependent. Then $(a - \alpha)\rho = 0$ for some $\alpha \in C$. Replacing a with $a - \alpha$, we may assume that $a\rho = 0$. Then by Lemma 2.1, $d(\rho)\rho = 0$, contradiction.

Case 2. Suppose that d is not Q -inner derivation. If for all $x \in \rho$, $d(x) \in xC$, then $[d(x), x] = 0$ which implies that R is commutative (see [13]). Therefore there exists $x \in \rho$ such that $d(x) \notin xC$, that is, x and $d(x)$ are linearly C -independent.

By our assumption, we have that R satisfies

$$[[d(xX), xX]_n, [xY, d(xY)]_m]^t = 0. \quad (2.14)$$

By Kharchenko's Theorem [9],

$$[[d(x)X + xr_1, xX]_n, [xY, d(x)Y + xr_2]_m]^t = 0, \quad (2.15)$$

for all $X, Y, r_1, r_2 \in R$. In particular for $r_1 = r_2 = 0$,

$$[[d(x)X, xX]_n, [xY, d(x)Y]_m]^t = 0, \quad (2.16)$$

which is a nontrivial GPI for R , because x and $d(x)$ are linearly C -independent, a contradiction. □

We are now ready to prove our main theorem.

Proof of Theorem 2.4. Suppose that $d(\rho)\rho \neq 0$, then we derive a contradiction. By Lemma 2.5, R is a prime GPI ring, so is also Q by [14]. Since Q is centrally closed over C , it follows from [11] that Q is a primitive ring with $H = \text{Soc}(Q) \neq 0$.

By our assumption and by [7], we may assume that

$$[[d(x), x]_n, [y, d(y)]_m]^t = 0 \quad (2.17)$$

is satisfied by ρQ and hence by ρH . Let $e = e^2 \in \rho H$ and $y \in H$. Then replacing x with e and y with $ey(1-e)$ in (2.17), then right multiplying it by e , we obtain that

$$\begin{aligned} 0 &= [[d(e), e]_n, [ey(1-e), d(ey(1-e))]]_m^t e \\ &= [[d(e), e]_n, [ey(1-e), d(ey(1-e))]]_m^{t-1} \\ &\quad \cdot \left\{ [d(e), e]_n \sum_{j=0}^m (-1)^j \binom{m}{j} d(ey(1-e))^j ey(1-e) d(ey(1-e))^{m-j} e \right. \\ &\quad \left. - \sum_{j=0}^m (-1)^j \binom{m}{j} d(ey(1-e))^j ey(1-e) d(ey(1-e))^{m-j} [d(e), e]_n e \right\}. \end{aligned} \quad (2.18)$$

Now we have the fact that for any idempotent e , $d(y(1-e))e = -y(1-e)d(e)$, $ed(e)e = 0$ and so

$$\begin{aligned} 0 &= [[d(e), e]_n, [ey(1-e), d(ey(1-e))]]_m^{t-1} \\ &\quad \cdot \left\{ 0 - \sum_{j=0}^m (-1)^j \binom{m}{j} e(-y(1-e)d(e))^j y(1-e) d(ey(1-e))^{m-j} d(e)e \right\}. \end{aligned} \quad (2.19)$$

Now since for any idempotent e and for any $y \in R$, $(1-e)d(ey) = (1-e)d(e)y$, above relation gives

$$\begin{aligned} 0 &= [[d(e), e]_n, [ey(1-e), d(ey(1-e))]]_m^{t-1} \\ &\quad \cdot \left\{ -e \sum_{j=0}^m \binom{m}{j} (y(1-e)d(e))^j y(1-e) (d(e)y(1-e))^{m-j} d(e)e \right\} \\ &= [[d(e), e]_n, [ey(1-e), d(ey(1-e))]]_m^{t-1} \left\{ -e \sum_{j=0}^m \binom{m}{j} (y(1-e)d(e))^{m+1} e \right\} \\ &= [[d(e), e]_n, [ey(1-e), d(ey(1-e))]]_m^{t-1} \left\{ -2^m e (y(1-e)d(e))^{m+1} e \right\} \\ &= \left\{ -2^m e (y(1-e)d(e))^{m+1} \right\}^t e. \end{aligned} \quad (2.20)$$

This implies that $0 = (-1)^t 2^{mt} ((1-e)d(e)ey)^{(m+1)t+1}$ for all $y \in H$. Since $\text{char } R \neq 2$, we have by Levitzki's lemma [12, Lemma 1.1] that $(1-e)d(e)ey = 0$ for all $y \in H$. By primeness of H , $(1-e)d(e)e = 0$. By [15, Lemma 1], since H is a regular ring, for each $r \in \rho H$, there exists an idempotent $e \in \rho H$ such that $r = er$ and $e \in rH$. Hence $(1-e)d(e)e = 0$ gives $(1-e)d(e) = (1-e)d(e^2) = (1-e)d(e)e = 0$ and so $d(e) = ed(e) \in eH \subseteq \rho H$ and $d(r) = d(er) = d(e)er + ed(er) \in \rho H$. Hence for each $r \in \rho H$, $d(r) \in \rho H$. Thus $d(\rho H) \subseteq \rho H$. Set $J = \rho H$.

Then $\bar{J} = J/(J \cap I_H(J))$, a prime C-algebra with the derivation $\bar{d}$ such that $\bar{d}(\bar{x}) = \overline{d(x)}$, for all $x \in J$. By assumption, we have that

$$\left[[\bar{d}(\bar{x}), \bar{x}]_n, [\bar{y}, \bar{d}(\bar{y})]_m \right]^t = 0, \quad (2.21)$$

for all $\bar{x}, \bar{y} \in \bar{J}$. By Theorem 2.3, we have either $\bar{d} = 0$ or $\overline{\rho H}$ is commutative. Therefore we have that either $d(\rho H)\rho H = 0$ or $[\rho H, \rho H]\rho H = 0$. Now $d(\rho H)\rho H = 0$ implies that $0 = d(\rho\rho H)\rho H = d(\rho)\rho H\rho H$ and so $d(\rho)\rho = 0$. $[\rho H, \rho H]\rho H = 0$ implies that $0 = [\rho\rho H, \rho H]\rho H = [\rho, \rho H]\rho H\rho H$ and so $[\rho, \rho H]\rho = 0$, then $0 = [\rho, \rho\rho H]\rho = [\rho, \rho]\rho H\rho$ implying that $[\rho, \rho]\rho = 0$. Thus in all the cases we have contradiction. This completes the proof of the theorem. $\square$

3. The Case: R Semiprime Ring

In this section we extend Theorem 2.3 to the semiprime case. Let R be a semiprime ring and U be its right Utumi quotient ring. It is well known that any derivation of a semiprime ring R can be uniquely extended to a derivation of its right Utumi quotient ring U and so any derivation of R can be defined on the whole of U [7, Lemma 2].

By the standard theory of orthogonal completions for semiprime rings, we have the following lemma.

Lemma 3.1 (see [16, Lemma 1 and Theorem 1] or [7, pages 31-32]). *Let R be a 2-torsion free semiprime ring and P a maximal ideal of C . Then PU is a prime ideal of U invariant under all derivations of U . Moreover, $\bigcap \{PU \mid P \text{ is a maximal ideal of } C \text{ with } U/PU \text{ 2-torsion free}\} = 0$.*

Theorem 3.2. *Let R be a 2-torsion free semiprime ring and d a non-zero derivation of R such that $[[d(x), x]_n, [y, d(y)]_m]^t = 0$ for all $x, y \in R$, $n, m \geq 0$, $t \geq 1$ fixed are integers. Then d maps R into its center.*

Proof. Since any derivation d can be uniquely extended to a derivation in U , and R and U satisfy the same differential identities [7, Theorem 3], we have

$$[[d(x), x]_n, [y, d(y)]_m]^t = 0, \quad (3.1)$$

for all $x, y \in U$. Let P be any maximal ideal of C such that U/PU is 2-torsion free. Then by Lemma 3.1, PU is a prime ideal of U invariant under d . Set $\bar{U} = U/PU$. Then derivation d canonically induces a derivation $\bar{d}$ on $\bar{U}$ defined by $\bar{d}(\bar{x}) = \overline{d(x)}$ for all $x \in U$. Therefore,

$$\left[[\bar{d}(\bar{x}), \bar{x}]_n, [\bar{y}, \bar{d}(\bar{y})]_m \right]^t = 0, \quad (3.2)$$

for all $\bar{x}, \bar{y} \in \bar{U}$. By Theorem 2.3, either $\bar{d} = 0$ or $[\bar{U}, \bar{U}] = 0$, that is, $d(U) \subseteq PU$ or $[U, U] \subseteq PU$. In any case $d(U)[U, U] \subseteq PU$ for any maximal ideal P of C . By Lemma 3.1,

$\bigcap \{PU \mid P \text{ is a maximal ideal of } C \text{ with } U/PU \text{ 2-torsion free}\} = 0$. Thus $d(U)[U, U] = 0$. Without loss of generality, we have $d(R)[R, R] = 0$. This implies that

$$0 = d(R^2)[R, R] = d(R)R[R, R] + Rd(R)[R, R] = d(R)R[R, R]. \quad (3.3)$$

Therefore $[R, d(R)]R[R, d(R)] = 0$. By semiprimeness of R , we have $[R, d(R)] = 0$, that is, $d(R) \subseteq Z(R)$. This completes the proof of the theorem. $\square$

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