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Research Article

Hilbert Algebras of Fractions

Christina-Theresia Dan

The Faculty of Mathematics and Computer Science, University of Craiova, Al.I.Cuza street nr.13, 200585 Craiova, Romania

Correspondence should be addressed to Christina-Theresia Dan, christi_the_d@yahoo.com

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Let A be a bounded Hilbert algebra and S a $\veebar$-closed subset of A. The Hilbert algebra of fractions A_S is studied regarding maximal and irreducible deductive systems. As important results, we can mention a necessary and sufficient condition for a Hilbert algebra of fractions to be local and the characterization of this kind of algebras as inductive limits of some particular directed systems.

1. Introduction

The positive implication algebras characterize positive implicative logic, as it is mentioned in [1]. The theory of this kind of algebras has been developed by Diego in [2]. He also called them Hilbert algebras.

The notion of deductive system, defined by Monteiro, is equivalent to the notion of implicative filter used by Rasiowa in [1]. The maximal and irreducible deductive systems play an important role in the study of these algebras as the representation theorem for Hilbert algebras (see Theorem 4.4) states.

Maximal deductive systems are studied by Buşneag in [3]. He is also the one who studied the Hilbert algebras of fractions with respect to a $\veebar$-closed subset in [4, 5].

This paper is structured as follows. In "Preliminaries" are presented some fundamental results concerning Hilbert algebras and deductive systems. Next, there are established some properties referring to maximal and irreducible deductive systems. These particular deductive systems are connected to $\veebar$-closed subsets. It is also defined the local Hilbert algebra and we provide a necessary and sufficient condition for a Hilbert algebra to be local. The following section presents the spaces $\mathrm{Max}(A)$ and $\mathrm{Ir}(A)$. In the last section we study the Hilbert algebras of fractions, specially the one corresponding to maximal deductive systems. We prove that many results from ring theory maintain for Hilbert algebras as well. For example, we characterize a Hilbert algebra of fractions to be local. Finally, using as a guideline the construction of the dual of the category of bounded distributive lattice, presented in

[6], we define a presheaf on the base of the topological space $\mathrm{Max}(A)$ and we prove that a local Hilbert algebra of fractions is isomorphic to an inductive limit of a directed system. For irreducible deductive systems we offer a similar result.

2. Preliminaries

Definition 2.1. Let A be a non empty set, $\rightarrow$ a binary operation on A and $1 \in A$.

A triplet $(A, \rightarrow, 1)$ is called Hilbert algebra if the following axioms hold, for each $x, y, z \in A$:

(h_1) $x \rightarrow (y \rightarrow x) = 1$,

(h_2) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$,

(h_3) $x \rightarrow y = y \rightarrow x = 1$ implies $x = y$.

A Hilbert algebra A becomes a *poset* by defining an order relation $\leq$ such that $x \leq y$ if and only if $x \rightarrow y = 1$. 1 is the largest element of A with respect to this order.

If the algebra A has a smallest element, denoted by 0, it is called a *bounded Hilbert algebra*.

Definition 2.2. If A_1, A_2 are bounded Hilbert algebras, $f : A_1 \rightarrow A_2$ is said to be a morphism of Hilbert algebras if $f(x \rightarrow y) = f(x) \rightarrow f(y)$ and $f(0) = 0$.

Theorem 2.3 (see [2]). *In a Hilbert algebra A the following relations hold for each $x, y, z \in A$:*

(1) $x \leq y \rightarrow x$,

(2) $x \rightarrow 1 = 1$,

(3) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(4) $(x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) = (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow y)$,

(5) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(6) $x \leq (x \rightarrow y) \rightarrow y; ((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$,

(7) $1 \rightarrow x = x$,

(8) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,

(9) *if* $x \leq y$, *then* $z \rightarrow x \leq z \rightarrow y$ *and* $y \rightarrow z \leq x \rightarrow z$.

We define $x \veebar y = x^* \rightarrow y$ *and* $x \barwedge y = (x \rightarrow y^*)^*$ *where* $x^* = x \rightarrow 0$.

Theorem 2.4 (see [3]). *In a bounded Hilbert algebra A, the following relations hold for each of the elements $x, y, z \in A$:*

(1) $0^* = 1$ *and* $1^* = 0$,

(2) $x \rightarrow y^* = y \rightarrow x^*$,

(3) $x \rightarrow x^* = x^*$ *and* $x^* \rightarrow x = x^{**}$,

(4) $x \rightarrow y \leq y^* \rightarrow x^*$,

(5) $x \leq y$ *implies that* $y^* \leq x^*$,

(6) $(x \rightarrow y)^{**} = x \rightarrow y^{**} = x^{**} \rightarrow y^{**}$,

(7) $x \leq x \veebar y$; $y \leq x \veebar y$,

(8) $x \veebar x = x^{**}$; $x \veebar 1 = 1$; $1 \veebar x = 1$; $x \veebar x^* = 1$,

(9) $x \veebar (y \to z) = (x \veebar y) \to (x \veebar z)$,

(10) $x \veebar (y \veebar z) = y \veebar (x \veebar z) = (x \veebar y) \veebar z$,

(11) $x \barwedge y = y \barwedge x$,

(12) $x \barwedge y \leq x^{**}$, $x \barwedge y \leq y^{**}$.

(13) $x^* \barwedge y^* = (x \veebar y)^*$.

Definition 2.5. A subset D of a Hilbert algebra A is called a deductive system if $1 \in D$ and from $x, x \to y \in D$ it results that $y \in D$.

In a Hilbert algebra A, for $x, y \in A$, Jun defines in [7] the deductive system $A(x,y) = \{z \in A \mid x \to (y \to z) = 1\}$. Using this notion, he gives the following characterization of deductive systems.

Theorem 2.6 (see [7]). *Let D be a nonempty subset of a Hilbert algebra. Then, D is a deductive system of A if and only if for any $x, y \in D$, $A(x,y) \subseteq D$.*

We denote the set of all deductive systems in A with $Ds(A)$.

If X is a subset of A, the deductive system generated by X is the least deductive system containing X. We will denote it by $[X)$.

Theorem 2.7 (see [1]). *Let A be a Hilbert algebra and S a subset of A. The deductive system generated by S is the set of the elements $x \in A$ for which there exist $a_1, \ldots, a_n \in S$ such that $a_1 \to (a_2 \to (\cdots (a_n \to x) \cdots)) = 1$.*

In particular, if we consider the set $S = \{a\}$, the deductive system

$$[a) = \{x \in A \mid \ a \leq x\} \tag{2.1}$$

is called the deductive system generated by a.

In [8] it is proved that $Ds(A)$ is a Heyting algebra with respect to the following operations:

$$D_1 \wedge D_2 = D_1 \cap D_2,$$

$$D_1 \vee D_2 = [D_1 \cup D_2),$$

$$D_1 \to D_2 = \{a \in A \mid [a) \cap D_1 \subseteq D_2\}.$$

Corollary 2.8 (see [9]). *If A is a Hilbert algebra, $D \in Ds(A)$, and $a \in A$, then*

$$[a) \vee D = \{x \in A \mid a \longrightarrow x \in D\}. \tag{2.2}$$

In [10] is defined the concept of ideals in a Hilbert algebra as follows.

Definition 2.9 (see [10]). A nonempty subset I of a Hilbert algebra A is called an ideal of A if:

(1) $1 \in I$,

(2) $x \to y \in I$, for all $x \in A, y \in I$,

(3) $(y_2 \to (y_1 \to x)) \to x \in I$, for all $x \in A$ and $y_1, y_2 \in I$.

In [11, 12] it is proved that the notions of deductive systems and ideals in a Hilbert algebra are equivalent. Also, the final result from [12] states that every congruence on a Hilbert algebra is uniquely determined by its kernel which is a deductive system. Hence, ideals, deductive systems and congruence kernels in a Hilbert algebra coincide.

In [13] it is proved much more: $Ds(A)$ is an algebraic lattice which is distributive and isomorphic with $\mathrm{Con}(A)$, the lattice of all congruences on A.

Definition 2.10 (see [1]). Let A be a Hilbert algebra and D a proper deductive system. D is called irreducible if for any two proper deductive systems D_1, D_2 such that $D = D_1 \cap D_2$ either $D = D_1$ or $D = D_2$.

Definition 2.11 (see [1]). A deductive system of a Hilbert algebra is said to be maximal if it is proper and it is not a proper subset of any proper deductive system in A.

For a Hilbert algebra A we will denote the set of all maximal deductive systems of A by $\mathrm{Max}(A)$ and the set of all irreducible deductive systems of A with $\mathrm{Ir}(A)$. According to their definitions, it is obvious that $\mathrm{Max}(A) \subseteq \mathrm{Ir}(A)$.

Theorem 2.12 (see [1]). *Let A be a Hilbert algebra.*

(i) *If D is a proper deductive system and $a \in A \setminus D$, there exists an irreducible deductive system I such that $D \subseteq I$ and $a \in A \setminus I$.*

(ii) *If $a \not\leq b$, there exists an irreducible deductive system I such that $a \in I$ and $b \in A \setminus I$.*

(iii) *If A is bounded, each proper deductive system is included in a maximal one.*

Lemma 2.13 (see [3]). *Let A be a bounded Hilbert algebra. The following statements are equivalent:*

(i) *M is a maximal deductive system in A.*

(ii) *For $x, y \in A$ and $x \veebar y \in M$, either $x \in M$ or $y \in M$.*

Corollary 2.14 (see [3]). *M is a maximal deductive system in the bounded Hilbert algebra A if and only if for any $x \in A \setminus M$, $x^* \in M$.*

Lemma 2.15 (see [3]). *The deductive system $\mathfrak{D}(A) = \{x \in A \mid x^* = 0\}$, formed with the dense elements of a bounded Hilbert algebra A, is the intersection of all maximal deductive systems in A.*

Theorem 2.16 (see [2]). *I is an irreducible deductive system in the bounded Hilbert algebra A if and only if for every $x, y \notin I$ there exists $z \notin I$ such that $x, y \leq z$.*

3. Maximal and Irreducible Deductive Systems

A Hilbert algebra is said to be *local* if it has a single maximal deductive system. Using Lemma 2.15, we offer a necessary and sufficient condition for a bounded Hilbert algebra to be local.

Proposition 3.1. *A bounded Hilbert algebra A is local if and only if for $x, y \in A$, $x \veebar y = 1$ implies $x^{**} = 1$ or $y^{**} = 1$.*

Proof. If A is local, from Lemma 2.15, $\mathfrak{D}(A)$ is a maximal deductive system. Hence, if $x \veebar y = 1 \in \mathfrak{D}(A)$, then $x \in \mathfrak{D}(A)$ or $y \in \mathfrak{D}(A)$ as it is stated in Lemma 2.13. Thus, $x^{**} = 1$ or $y^{**} = 1$. Conversely, let $x \veebar y \in \mathfrak{D}(A)$. Then, $1 = (x \veebar y)^{**} = x \veebar y^{**}$. It results that $x^{**} = 1$ or $y^{**} = 1$ and so, $x \in \mathfrak{D}(A)$ or $y \in \mathfrak{D}(A)$. Since $\mathfrak{D}(A)$ is maximal, from Lemma 2.15, it is the unique maximal deductive system in A. □

Lemma 3.2. *Let A be a local bounded Hilbert algebra with $\mathfrak{D}(A)$ its single maximal deductive system. Then, $\mathfrak{D}(A) = A \setminus \{0\}$.*

Proof. Let $x \in A \setminus \{0\}$. Since $[x)$ is a proper deductive system, from Theorem 2.12, $[x) \subseteq \mathfrak{D}(A)$. Hence, $x \in \mathfrak{D}(A)$. □

Proposition 3.3 (see [9]). *Every proper deductive system D of a Hilbert algebra is the intersection of the irreducible deductive systems which contain D.*

Proposition 3.4. *If D is a proper deductive system in the bounded Hilbert algebra A, then $x \barwedge y \in D$, for $x, y \in D$.*

Proof. If we suppose that there exist $x, y \in D$ such that $x \barwedge y = (x \to y^*)^* \notin D$, from Corollary 2.8, $0 \notin D \vee [(x \to y^*))$. From Theorem 2.12, there exists a maximal deductive system M such as $D \vee [(x \to y^*)) \subseteq M$. But, $y^* \in M$ since $x \in M$ and $x \to y^* \in M$. Then, $y^*, y \in M$ imply that $M = A$, a contradiction. □

Definition 3.5 (see [5]). Let S be a subset of a bounded Hilbert algebra A. S is said to be $\veebar$-closed if, for each $x, y \in S$, $x \veebar y \in S$.

Proposition 3.6. *Let D be a proper deductive system in a bounded Hilbert algebra A. Then, D is maximal if and only if $A \setminus D$ is a $\veebar$-closed subset.*

Proof. If we consider D to be a maximal deductive system and $x, y \in A \setminus D$, from Lemma 2.13, it results that $x \veebar y \in A \setminus D$. Conversely, let $A \setminus D$ be a $\veebar$-closed set and let $x \veebar y \in D$. From Definition 3.5 it results that $x \notin A \setminus D$ or $y \notin A \setminus D$. To prove that D is a maximal deductive system, we apply once again Lemma 2.13. □

Proposition 3.7. *Let A be a bounded Hilbert algebra, S a $\veebar$-closed subset of A and D a proper deductive system such that $S \cap D = \emptyset$. Then, there exists a maximal deductive system M such as $D \subseteq M$ and $M \cap S = \emptyset$.*

Proof. We consider the nonempty family

$$\mathbf{T} = \{F \mid F \text{ deductive system}, D \subseteq F, F \cap S = \emptyset\}. \tag{3.1}$$

It is easily verified that each chain in $\mathbf{T}$ has an upper bound in $\mathbf{T}$. Then, by Zorn's lemma, $\mathbf{T}$ has a maximal element M. All we have to do, furthermore, is to prove that M is a maximal deductive system. If we presume that $x \veebar y \in M$ and $x, y \in A \setminus M$, then $M \vee [x)$ and $M \vee [y)$ intersect S, as M is a maximal element in $\mathbf{T}$. Let $s, t \in S$, $s \in M \vee [x)$, $t \in M \vee [y)$. Then, from Corollary 2.8, $x \to s, y \to t \in M$.

The relation $y \to t \leq x \veebar (y \to t)$ implies that $x \veebar (y \to t) = x \veebar y \to x \veebar t \in M$. Since $x \veebar y \in M$, $x \veebar t \in M$. Hence, $s \veebar (x \veebar t) \in M$. But, $s \veebar (x \veebar t) = s^* \to (x^* \to t) = (s^* \to x^*) \to (s^* \to t) = (x \to s^{**}) \to s \veebar t = (x \to s)^{**} \to s \veebar t$.

Since $x \to s \leq (x \to s)^{**}$ and $x \to s \in M$, it results that $(x \to s)^{**} \in M$. From $(x \to s)^{**} \to (s \veebar t) \in M$ we obtain that $s \veebar t \in M$. This result contradicts the fact that $M \cap S = \emptyset$ since $s \veebar t \in S$. □

Definition 3.8. Let S be a subset of the bounded Hilbert algebra A. We say that S is a decreasing subset if for $s \in S$ and $x \in A$, $x \leq s$ implies that $x \in S$.

Remark 3.9. It is easy to verify that $x \veebar y \in S$ if and only if $x, y \in S$ for a decreasing $\veebar$-closed subset S.

For an arbitrary element x of a bounded Hilbert algebra A we consider $[x^{**})$ to be the deductive system generated by x^{**}. If D is a proper deductive system in A, let us define the set $\widetilde{D}$ as follows:

$$\widetilde{D} = \{x \in A \mid D \vee [x^{**}) = A\}. \tag{3.2}$$

Then, $x \in \widetilde{D}$ if and only if for each $a \in A$, $x^{**} \to a \in D$, as in Corollary 2.8. The last relation is equivalent to $x^* \veebar a \in D$, for all $a \in A$. We can see that $1 \notin \widetilde{D}$, otherwise $D = A$.

Lemma 3.10. *Let D be a deductive system in a bounded Hilbert algebra A. Then, $x \in \widetilde{D}$ if and only if $x^* \in D$.*

Proof. If $x \in \widetilde{D}$ then, using the relation (8) of Theorem 2.4, $x^* \veebar x^* = x^* \in D$. Conversely, let $x^* \in D$. Since for each $a \in A$ we have $x^* \leq (x^* \veebar a)$, it results that $x^* \veebar a \in D$, for each $a \in A$. Thus, $x \in \widetilde{D}$. □

Remark 3.11. If D and $\widetilde{D}$ intersect, $0 \in D$ and then $A = D$. Hence, if we consider D to be a proper deductive system, $D \cap \widetilde{D} = \emptyset$. Thus, $\widetilde{D} \subseteq A \setminus D$. The equality holds if and only if D is a maximal deductive system. Indeed, $\widetilde{D} = A \setminus D$ if and only if $A \setminus D \subseteq \widetilde{D}$. Hence, for all $x \notin D, x \in \widetilde{D}$ which is equivalent with $x^* \in D$. Applying Corollary 2.14, this property is equivalent with the condition that D is a maximal deductive system.

Proposition 3.12. *$\widetilde{D}$ defined as above is a decreasing $\veebar$-closed subset if D is a proper deductive system in A.*

Proof. Firstly, because $1 \in D$, we see that $0 \in \widetilde{D}$. Let $x \leq y$ and $y \in \widetilde{D}$. Since $y^* \leq x^*$ and $y^* \in D$, we get $x^* \in D$. Thus, $x \in \widetilde{D}$. Now we consider $x, y \in \widetilde{D}$. Then, $x^*, y^* \in D$ and, from Proposition 3.4, $x^* \barwedge y^* \in D$.

But, from Theorem 2.4 (13), $x^* \barwedge y^* = (x \veebar y)^* \in D$ which proves that $x \veebar y \in \widetilde{D}$. □

For a proper deductive system D we define

$$D_0 = \{x \in A \mid x^{**} \in D\}. \tag{3.3}$$

We obtain another deductive system which contains D. Since $0 \notin D$, D_0 is also proper. If D is maximal, $D = D_0$. Let M be a maximal deductive system containing D. If $x \in D_0$, we get $x^{**} \in M$, since $x^{**} \in D \subseteq M$. Thus, $x \in M$. Hence, $D_0 \subseteq M$.

Remark 3.13. Let us consider D a proper deductive system. Using the following chain of equivalences

$$x \in \widetilde{D} \Longleftrightarrow x^* \in D \Longleftrightarrow x^* \in D_0 \Longleftrightarrow x \in \widetilde{D}_0, \tag{3.4}$$

we obtain that $\widetilde{D} = \widetilde{D}_0$.

Proposition 3.14. *For a proper deductive system D in A there is an irreducible deductive system J such that $D \subseteq J \subseteq A \setminus \widetilde{D}$.*

Proof. If we consider $x \in D_0$, then $x^{**} \in D$. Hence, $x^* \notin D$, else $0 \in D$ and $D = A$. Then $x \notin \widetilde{D}$ and so $D \subseteq D_0 \subseteq A \setminus \widetilde{D}$. Thus, $\mathbf{F}$, the family of all deductive systems which contain D and are included in $A \setminus \widetilde{D}$, is non empty. It is easy to verify that every chain in $\mathbf{F}$ has an upper bound in $\mathbf{F}$. From Zorn's Lemma, there exists J, a maximal element in $\mathbf{F}$. To complete the proof, we show that J is irreducible. Let's suppose that $J = J_1 \cap J_2$ and $J \neq J_1$, $J \neq J_2$. Then, $J_1, J_2 \notin \mathbf{F}$. Since $D \subseteq J \subseteq J_1, J_2$ we have $J_1, J_2 \not\subseteq A \setminus \widetilde{D}$. Let $a \in J_1, b \in J_2$ with $a, b \notin A \setminus \widetilde{D}$. From Proposition 3.12, $\widetilde{D}$ is $\veebar$-closed. Hence, since $a, b \in \widetilde{D}$ we obtain $a \veebar b \in \widetilde{D}$. But $a \veebar b \in J_1 \cap J_2 = J$ and this contradicts the fact that $J \in \mathbf{F}$. □

4. The Spaces $\mathbf{Max}(A)$ and $\mathbf{Ir}(A)$

Let A be a bounded Hilbert algebra. For each $x \in A$, we define the set:

$$r(x) = \{M \in \mathrm{Max}(A) \mid x \in M\}. \tag{4.1}$$

We will consider $\mathrm{Max}(A)$ as a topological space with the class $\{r(x) \mid x \in A\}$ as a base.

Theorem 4.1 (see [3]). *For each $x, y \in A$ one has:*

(1) $r(0) = \emptyset$ *and* $r(1) = \mathrm{Max}(A)$,

(2) $x \leq y \Rightarrow r(x) \subseteq r(y)$,

(3) $r(x^*) = \mathrm{Int}(\mathrm{Max}(A) \setminus r(x))$,

(4) $r(x^{**}) = r(x)$,

(5) $r(x \to y) = r(x) \to r(y)$ *where* $r(x) \to r(y) = \mathrm{Int}((\mathrm{Max}(A) \setminus r(x)) \cup r(y))$,

(6) $r(x) \cap r(y) = r(x \barwedge y)$,

(7) $r(x) \cup r(y) = r(x \veebar y)$.

Proposition 4.2. *For $x, y \in A$, $r(x) = r(y)$ if and only if $x^{**} = y^{**}$.*

Proof. Let $r(x) = r(y)$. It means that for an arbitrary maximal deductive system M, $x \in M$ if and only if $y \in M$. Assuming that $x^{**} \not\leq y^{**}$, there exists an irreducible deductive system

I, as in Theorem 2.12, such that $x^{**} \in I$ and $y^{**} \notin I$. Then, $0 \notin I \vee [y^*)$. Let M_1 be a maximal deductive system with $I \vee [y^*) \subseteq M_1$. Since $y^* \in M_1$, then $y \notin M_1$. Yet, since M_1 is maximal and $x^{**} \in I \subseteq M_1$, then $x \in M_1$. From our supposition, $r(x) = r(y)$, we get that $y \in M_1$ which contradicts the former result.

Thus, we obtain $x^{**} \leq y^{**}$. In a similar way, we prove that $y^{**} \leq x^{**}$, hence $x^{**} = y^{**}$. From the previous theorem the converse implication is also true. □

It is known that a set $\mathbf{B}$ of open subsets of a topological space (X, τ) is a base for the topology τ if and only if for each $x \in X$, the set $B_x = \{B \mid B \in \mathbf{B}, x \in B\}$ is a fundamental system of neighborhoods for x.

In our case, we have considered on $\mathrm{Max}(A)$ the topology generated by the basis $\{r(x) \mid x \in A\}$. Using this property, the following lemma results.

Lemma 4.3. *Let $M \in \mathrm{Max}(A)$. Then, $V_M = \{r(x) \mid x \in M\}$ is a fundamental system of neighborhoods for M.*

Theorem 4.4 (see [9]). *Every Hilbert algebra A is isomorphic with a subalgebra of the Hilbert algebra of all open subsets of the topological T_0-space* $\mathrm{Ir}(A)$.

In the proof of this theorem, A is isomorphic to the class $\{s(x) \mid x \in A\}$ where $s(x) = \{I \in \mathrm{Ir}(A) \mid x \in I\}$, for all $x \in A$. This class is a subbase of the topological space $\mathrm{Ir}(A)$.

Moreover, $s(x \to y) = \mathrm{Int}((\mathrm{Ir}(A) \setminus s(x)) \cup s(y))$ for all $x, y \in A$.

The following lemma corresponds to Lemma 4.3.

Lemma 4.5. *Let I be an irreducible deductive system in the bounded Hilbert algebra A. Then,*

$$V_I = \left\{ \bigcap_{i=1}^{n} s(x_i) \mid x_i \in I,\ 1 \leq i \leq n,\ n \geq 1 \right\} \tag{4.2}$$

is a fundamental system of neighborhoods for I.

5. Hilbert Algebras of Fractions

The Hilbert algebra of fractions with respect to a $\veebar$-closed subset is constructed in [4, 5] as follows.

Let A be a bounded Hilbert algebra and S be a $\veebar$-closed subset in A. A congruence relation θ_S on A is defined by $(x, y) \in \theta_S \Leftrightarrow \exists\, s \in S,\ s \veebar x = s \veebar y$. The corresponding quotient Hilbert algebra $A/_{\theta_S}$ is denoted by A_S and it is called *the Hilbert algebra of fractions of A with respect to the $\veebar$-closed subset S*. The congruence class of x in A_S will be denoted by $[x]_S$.

If D is a deductive system in A, it is easy to verify that $D_S = \{[x]_S \mid x \in D\}$ is a deductive system in A_S.

Let M be a maximal deductive system in A. From Proposition 3.6, $A \setminus M$ is a $\veebar$-closed subset and, in this case, we will denote the Hilbert algebra of fractions with respect to $A \setminus M$ by A_M.

We know that a deductive system D induces a congruence relation θ_D on A defined by: $(x, y) \in \theta_D \Leftrightarrow x \to y,\ y \to x \in D$, as it is proved in [1]. The resulting quotient Hilbert algebra will be denoted by $A/_D$ and the congruence class of x will be denoted by $x/_D$.

Lemma 5.1. *Let A be a Hilbert algebra, D a deductive system and S a $\veebar$-closed subset in A. On $A/_D$ we define the following relation:*

$$(x/_D, y/_D) \in \theta'_S \Longleftrightarrow \exists\, s \in S, \quad (s \veebar x)/_D = (s \veebar y)/_D. \tag{5.1}$$

Then, θ'_S is a congruence relation on $A/_D$.

Proof. Let $p_D : A \to A/_D$ be the canonical morphism of Hilbert algebras. We easily see that $S_D = p_D(S) = \{p_D(s) \mid s \in S\}$ is a $\veebar$-closed subset of $A/_D$. Then, $(x/_D, y/_D) \in \theta_{S_D} \Leftrightarrow \exists\, t \in S_D$ such that $t \veebar x/_D = t \veebar y/_D \Leftrightarrow \exists\, s \in S$, $t = s/_D$, such that $(s \veebar x)/_D = t \veebar x/_D = t \veebar y/_D = (s \veebar y)/_D \Leftrightarrow (x/_D, y/_D) \in \theta'_S$.

Hence, θ'_S is exactly the congruence defined in the Hilbert algebra $A/_D$ by means of the $\veebar$-closed subset $S_D = p_D(S)$. □

The quotient Hilbert algebra regarding the congruence relation θ'_S will be denoted by $(A/_D)_S$ and the corresponding congruence classes will be denoted by $[x/_D]_S$.

For a maximal deductive system M, let $S = A \setminus M$ be the related $\veebar$-closed subset. Let $p_M : A \to A_M$ be the canonical morphism defined by $p_M(x) = [x]_M$ for each $x \in A$.

Lemma 5.2. *If D is a deductive system in the bounded Hilbert algebra A, then*

$$D = \bigcap_{M \in \mathrm{Max}(A)} p_M^{-1}(D_M). \tag{5.2}$$

Proof. Since $D \subseteq p_M^{-1}(D_M)$ for each maximal deductive system, we obtain one inclusion. To prove the converse inclusion, let us consider $x \in p_M^{-1}(D_M)$, for each maximal deductive system M. Then, $p_M(x) \in \bigcap_{M \in \mathrm{Max}(A)} D_M$. There exists $a \in D$ such that $[x]_M = [a]_M$ which means that there exist $a \in D$, $t \notin M$, for all M, such that $t \veebar x = t \veebar a$. Since $a \in D$, $t \veebar a \in D$ and then $t \veebar x \in D$. From $t \notin M$ it results that $[t) \vee D = \{y \in A \mid t \to y \in D\} \not\subseteq M$, for each M. But each proper deductive system is included in a maximal one. Hence, $[t) \vee D = A$. Thus $0 \in [t) \vee D$, that is $t^* = t \to 0 \in D$. Finally, $t \veebar x = t^* \to x \in D$ implies that $x \in D$. □

Lemma 5.3. *Let D_1, D_2 be two deductive systems in the Hilbert algebra A. Then*

(i) *$D_1 \subseteq D_2$ if and only if $(D_1)_M \subseteq (D_2)_M$ for each maximal deductive system M in A,*

(ii) *$D_1 = D_2$ if and only if $(D_1)_M = (D_2)_M$ for each maximal deductive system M in A,*

(iii) *$(D_1)_M = (D_2)_M$ if and only if $(A/_{D_1})_M \simeq (A/_{D_2})_M$ for each maximal deductive system M in A,*

(iv) *$A/_{D_1} \simeq A/_{D_2}$ if and only if $(A/_{D_1})_M \simeq (A/_{D_2})_M$ for each maximal deductive system M in A.*

Proof. (i) We presume that $(D_1)_M \subseteq (D_2)_M$ for each maximal system M. Then, from Lemma 5.2,

$$D_1 = \bigcap_{M \in \mathrm{Max}(A)} p_M^{-1}((D_1)_M) \subseteq \bigcap_{M \in \mathrm{Max}(A)} p_M^{-1}((D_2)_M) = D_2. \tag{5.3}$$

(ii) results from the previous one.

(iii) We presume that $(A/_{D_1})_M \simeq (A/_{D_2})_M$ for each maximal deductive system M in A. Let $x, y \in A$. Then, $[x/_{D_1}]_M = [y/_{D_1}]_M$ if and only if $[x/_{D_2}]_M = [y/_{D_2}]_M$. For $x \in D_1$ we get that $x/_{D_1} = 1/_{D_1}$, that is, $[x/_{D_1}]_M = [1/_{D_1}]_M$. This is equivalent to $[x/_{D_2}]_M = [1/_{D_2}]_M$. Hence, there exists $t \notin M$ such that $(t \veebar x)/_{D_2} = (t \veebar 1)/_{D_2} = 1/_{D_2}$. So, there exists $t \notin M$ such that $t \veebar x \in D_2$. Since $t \veebar x = t \veebar (t \veebar x)$ we obtain that $[x]_M = [(t \veebar x)]_M \in (D_2)_M$.

We suppose now that $(D_1)_M = (D_2)_M$. Let $[x/_{D_1}]_M = [y/_{D_1}]_M$. Hence, there exists $t \notin M$ with $(t \veebar x)/_{D_1} = (t \veebar y)/_{D_1}$. Then, $t \veebar (x \to y), t \veebar (y \to x) \in D_1$. Since $(D_1)_M = (D_2)_M$, we obtain that $[t \veebar (x \to y)]_M = [u]_M$ for $u \in D_2$. Thus, there exists $s \notin M$ such that $s \veebar t \veebar (x \to y) = s \veebar u$. But M is maximal and $s, t \notin M$ implies that $s \veebar t \notin M$. Then, $(s \veebar t) \veebar (x \to y) = s \veebar u \in D_2$. In a similar way, $[t \veebar (y \to x)]_M = [v]_M$ for $v \in D_2$. It results that there exists $s_1 \notin M$ such that $(s_1 \veebar t) \veebar (y \to x) = s_1 \veebar v \in D_2$. Hence,

$$s_1 \veebar s \veebar t \notin M, \qquad (s_1 \veebar s \veebar t) \veebar (y \longrightarrow x), (s_1 \veebar s \veebar t) \veebar (x \longrightarrow y) \in D_2. \tag{5.4}$$

These relations imply that $((s_1 \veebar s \veebar t) \veebar x)/_{D_2} = ((s_1 \veebar s \veebar t) \veebar y)/_{D_2}$, showing that $[x/_{D_2}]_M = [y/_{D_2}]_M$.

(iv) From (ii) and (iii) it results that $A/_{D_1} \simeq A/_{D_2}$ if and only if $D_1 = D_2$ if and only if $(D_1)_M = (D_2)_M$ for each maximal deductive system M in A which is equivalent with $A/_{D_1} \simeq A/_{D_2}$ if and only if $(A/_{D_1})_M \simeq (A/_{D_2})_M$ for each maximal deductive system M in A. □

Proposition 5.4. *If M is a maximal deductive system in the Hilbert algebra A, the Hilbert algebra of fractions A_M is local.*

Proof. We show that $\mathfrak{D}(A_M)$ is maximal using the result from Proposition 3.1. To do this, let's consider $x, y \in A$ such that $[x \veebar y]_M = [1]_M$. Then, there is $t \notin M$ such that $t \veebar (x \veebar y) = t \veebar 1 = 1$. Hence, $t^* \leq x \veebar y$. But, from Corollary 2.14, $t^* \in M$ and then $x \veebar y \in M$. The fact that M is maximal implies either $x \in M$ or $y \in M$. We get that $x^* \notin M$ or $y^* \notin M$. Finally, $x^* \veebar x^{**} = x^* \veebar 1$ or $y^* \veebar y^{**} = y^* \veebar 1$ imply that $[x^{**}]_M = [1]_M$ or $[y^{**}]_M = [1]_M$. □

Lemma 5.5. *In the bounded Hilbert algebra of fractions A_S, $[x]_S = [0]_S$ if and only if there exists $s \in S$ such as $x \leq s$.*

Proof. If $[x]_S = [0]_S$ there exists $t \in S$ with $t \veebar x = t \veebar 0 = t^{**} = t \veebar t \in S$. Hence, there exists $s = t \veebar x \in S$ and $x \leq s$. Conversely, if there is $s \in S$ such that $x \leq s$, then $s \veebar x \leq s \veebar s = s^{**}$. Hence, $(s \veebar x)^{**} \leq s^{**}$ which means that

(1) $s^* \leq (s \veebar x)^*$.

Since $s \leq s \veebar x$, we get

(2) $(s \veebar x)^* \leq s^*$.

From these two relations, we obtain that $(s \veebar x)^* = s^*$. Thus,

$$s \veebar x = (s \veebar x) \veebar x = s \veebar (x \veebar x) = s \veebar x^{**} \tag{5.5}$$

and then,

$$s \veebar x = s \veebar x^{**} = (s \veebar x)^{**} = s^{**} = s \veebar 0. \tag{5.6}$$

Hence, $[x]_S = [0]_S$. □

Proposition 5.6. *Let S be a $\veebar$-closed subset in the bounded Hilbert algebra A. One denotes $(S] = \{x \in A \mid \exists s \in S,\ x \leq s\}$ the least decreasing subset generated by S. Then, $(S]$ is a $\veebar$-closed subset and it is the complement in A of the union of all maximal systems which do not intersect S.*

Proof. It is obvious that $(S]$ is a $\veebar$-closed subset. We denote the complement in A of the union of all maximal systems which do not intersect S by S'. We shall prove that $(S] = S'$. Let $x \in (S]$. Then, $x \leq s$ for some $s \in S$. If we suppose that $x \notin S'$, there exists a maximal system M which does not intersect S and $x \in M$. But then, $s \in M$, false. Conversely, if $x \in S'$, then $x \notin M$ for each maximal system M with $M \cap S = \emptyset$. If $[x) \cap S = \emptyset$, from Proposition 3.7, there exists a maximal deductive system M_1 such that $x \in M_1$ and M_1 does not intersect S, false. Thus, $[x)$ intersects S which means that $x \in (S]$. □

In [6], it is proved that the lattice of fractions verifies a property of universality. In the same way, we can easily deduce that the Hilbert algebras of fractions verify the following property of universality.

Proposition 5.7. *Let A be a bounded Hilbert algebra and S a $\veebar$-closed subset in A. Then, for each bounded Hilbert algebra B and for any morphism of bounded Hilbert algebras $f : A \to B$ with $f(s) = 0$, for all $s \in S$, there exists an unique morphism $g : A_S \to B$ such that $g \circ p_S = f$ which means that the following diagram is commutative:*

$$\begin{tikzcd} A \arrow[rr, "p_S"] \arrow[dr, "f"'] & & A_s \arrow[dl, "g"] \\ & B & \end{tikzcd} \tag{5.7}$$

Proposition 5.8. *Let S and T be two $\veebar$-closed subsets in the bounded Hilbert algebra A, $S \subseteq T$ and we consider the morphism of Hilbert algebras $f : A_S \to A_T$ defined by $f([x]_S) = [x]_T$, for all $x \in A$. The following statements are equivalent:*

(i) *f is bijective,*

(ii) $T \subseteq (S]$,

(iii) *for each maximal deductive system M with $M \cap T \neq \emptyset$, $M \cap S \neq \emptyset$.*

Proof. (i) → (ii) Let $t \in T$. Since $f([t]_S) = [t]_T = [0]_T$ and f is bijective, $[t]_S = [0]_S$. Hence, from Lemma 5.5, $t \in (S]$.

(ii) → (iii) Let M be a maximal system with $M \cap T \neq \emptyset$. Let $t \in T \cap M$. Since $T \subseteq (S]$, there exists $s \in S$ with $t \leq s$. Hence, $s \in S \cap M$ and we get $M \cap S \neq \emptyset$.

(iii) $\to$ (ii) Let $t \in T$. If $t \notin (S]$, then $[t) \cap S = \emptyset$. From Proposition 3.7, there exists a maximal deductive system M with $[t) \subseteq M$ and $M \cap S = \emptyset$. Then, since $t \in T \cap M$, we reach a contradiction.

(ii) $\to$ (i) Let $[x]_T = [y]_T$. Thus, there exists $t \in T$ such that $t \veebar x = t \veebar y$. Since $t \leq s$ for some $s \in S$, we obtain that $s^* \leq t^*$. It results that:

$$\begin{aligned}&(s \veebar t) \veebar x\\ &\quad = s \veebar (t \veebar x) = s^* \longrightarrow (t^* \longrightarrow x) = (s^* \longrightarrow t^*) \longrightarrow (s^* \longrightarrow x) = 1 \longrightarrow (s \veebar x) = s \veebar x.\end{aligned} \tag{5.8}$$

Then, $s \veebar x = (s \veebar t) \veebar x = (s \veebar t) \veebar y = s \veebar y$. Hence, $[x]_S = [y]_S$ and f is bijective. □

Proposition 5.9. *Let A be a bounded Hilbert algebra and S a $\veebar$-closed subset in A. If A_S is local, there exists a maximal system M such as A_S and A_M are isomorphic.*

Proof. Let $M = \{x \in A \mid [x]_S \in \mathfrak{D}(A_S)\}$. Because M is the inverse image of the deductive system $\mathfrak{D}(A_S)$ by means of the canonical map, it is a deductive system in A. In our case, $\mathfrak{D}(A_S)$ is maximal in A_S. Hence, M is also maximal.

Let us consider now $(S]$, the least decreasing $\veebar$-closed subset generated by S. We prove that $(S] = A \setminus M$. Let $x \in A \setminus M$. Then, $x \notin M$ if and only if $[x]_S \notin \mathfrak{D}(A_S)$ which is equivalent to $[x]_S = [0]_S$, as Lemma 3.2 states. Then, from Lemma 5.5, $x \in (S]$. Finally, from Proposition 5.8, $A_S \simeq A_M$. □

Proposition 5.4 and Proposition 5.9 lead to the following result.

Theorem 5.10. *A bounded Hilbert algebra of fractions A_S is local if and only if there exists a maximal deductive system M in A such that A_S and A_M are isomorphic.*

Now, we return to the previous subsets $r(x)$ in $\mathrm{Max}(A)$ and $s(x)$ in $\mathrm{Ir}(A)$ where A is a bounded Hilbert algebra. For each $x \in A$, one defines the sets:

$$S_x = A \setminus \bigcup_{M \in r(x)} M, \qquad T_x = A \setminus \bigcup_{I \in s(x)} I. \tag{5.9}$$

Lemma 5.11. *For each $x \in A$, the following statements hold:*

(1) *S_x are decreasing $\veebar$-closed subsets. In fact, $S_x = \{a \in A \mid a \leq x^*\}$.*

(2) *$T_x = S_x$.*

Proof. (1) Let $a, b \in S_x$. Then, $a, b \notin M$, for each $M \in r(x)$. Since M is maximal, $a \veebar b \notin M$, for each M and so, $a \veebar b \in S_x$. Let now consider $a \leq b$ with $b \in S_x$. If $a \notin S_x$, there is a maximal deductive system $M \in r(x)$ such that $a \in M$. Then, $b \in M$ and so, $b \notin S_x$, false.

Since $r(x^{**}) = r(x)$, we have that $S_{x^{**}} = S_x$. As $x^* \in S_x$ and S_x is a decreasing $\veebar$-closed set, each $a \leq x^*$ is an element of S_x. Next, let's consider that $a \leq x^*$ and there exists $M \in r(x)$ such that $a \in M$. Then, $x^* \in M$ and, since $x \in M$, we get $M = A$, false.

(2) $T_x \subseteq S_x$, since each maximal deductive system is an irreducible one. Let $a \leq x^*$. If we suppose that $a \notin T_x$, there is an irreducible deductive system $I \in s(x)$ with $a \in I$. Then, $x^* \in I$ and, finally, $I = A$, false. □

For each $x_1, \ldots, x_n \in A$, we define also the sets $T_{x_1,\ldots,x_n}$ as the complement in A of the union of all irreducible deductive systems $I \in \bigcap_{i=1}^{n} s(x_i)$.

Lemma 5.12. *With the previous notations,*

$$T_{x_1,\ldots,x_n} = T_{\overline{\bigwedge}_{i=1,n} x_i}. \tag{5.10}$$

Proof. Let $a \in T_{x_1,\ldots,x_n}$. Then, $a \notin I$ for all $I \in s(x_i)$ with $i = 1, \ldots, n$. Presuming that $a \notin T_{\overline{\bigwedge}_{i=1}^{n} x_i}$, from Lemma 5.11, $a \not\leq (\overline{\bigwedge}_{i=1}^{n} x_i)^*$. Hence, there is an irreducible deductive system J with $a \in J$ and $(\overline{\bigwedge}_{i=1}^{n} x_i)^* \notin J$, as Theorem 2.12 states. Since $J \vee [\overline{\bigwedge}_{i=1}^{n} x_i)$ is a proper deductive system, it is included into a maximal one. Let M be such a maximal deductive system.

From Theorem 2.4, $\overline{\bigwedge}_{i=1}^{n} x_i \leq x_i^{**}$ for all $1 \leq i \leq n$. From $\overline{\bigwedge}_{i=1}^{n} x_i \in M$ we obtain that $x_i^{**} \in M$, hence $x_i \in M$, for all i. Then $M \in \bigcap_{i=1}^{n} s(x_i)$, and since $a \in J \subseteq M$ we contradict the assumption that $a \in T_{x_1,\ldots,x_n}$.

For the converse inclusion, let us presume that $a \in T_{\overline{\bigwedge}_{i=1}^{n} x_i}$ and $a \notin T_{x_1,\ldots,x_n}$. Then, $a \leq (\overline{\bigwedge}_{i=1}^{n} x_i)^*$ and there exists $J \in \bigcap_{i=1}^{n} s(x_i)$ with $a \in J$. From Proposition 3.4, since each $x_i \in J$, $\overline{\bigwedge}_{i=1}^{n} x_i \in J$. Hence, $(\overline{\bigwedge}_{i=1}^{n} x_i)^* \notin J$ and then $a \notin J$. This contradiction ends the proof. □

On the base of the topological space $\mathrm{Max}(A)$ we can define a presheaf which means a contravariant functor from $\mathrm{Max}(A)$ in the category of Hilbert algebras. We consider the open subsets $r(x) \subseteq r(y)$. Then, $S_y \subseteq S_x$.

For $x \in A$, let A_x be the Hilbert algebra of fractions with respect to the $\veebar$-closed subset S_x.

Taking into account the result obtained in Proposition 5.7, we can consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[rr, "p_y"] \arrow[dr, "p_x"'] & & A_y \arrow[dl, "\varphi_{y,x}"] \\ & A_x & \end{tikzcd} \tag{5.11}$$

where p_x, p_y are the canonical surjections and $\varphi_{y,x} : A_y \to A_x$ is defined by $\varphi_{y,x}([a]_{S_y}) = [a]_{S_x}$, for all $a \in A$.

Hence, each $r(x)$ is put into correspondence with A_x and the morphism $\varphi_{y,x} : A_y \to A_x$ commutes the following diagram:

$$\begin{tikzcd} r(x) \arrow[r, hook] \arrow[d] & r(y) \arrow[d] \\ A_x & A_y \arrow[l, "\varphi_{y,x}"'] \end{tikzcd} \tag{5.12}$$

Lemma 5.13. *For $M \in \mathrm{Max}(A)$, the set $\{S_x \mid x \in M\}$ is directed.*

Proof. For $x, y \in M$, using Proposition 3.4, $x \barwedge y \in M$. Hence, $S_x, S_y \subseteq S_{x \barwedge y}$ since $r(x \barwedge y) = r(x) \cap r(y)$. □

We define now the $\veebar$-closed subset $S_M = \bigcup_{x\in M} S_x$, for each maximal deductive system M. Since $S_x \subseteq S_M$, for each $x \in M$, there exists a morphism $\theta_x : A_x \to A_{S_M}$ with $\theta_x \circ p_x = p_{S_M}$ as Proposition 5.7 states. It is obvious that each morphism θ_x is surjective.

Let us consider the family of morphisms $(\theta_x : A_x \to A_{S_M})_{x\in M}$. For each $x, y \in M$ such that $S_y \subseteq S_x$, we get

$$(\theta_x \circ \varphi_{y,x})\left([a]_{S_y}\right) = \theta_x([a]_{S_x}) = (\theta_x \circ p_x)(a) = [a]_{S_M} = \theta_y\left([a]_{S_y}\right). \tag{5.13}$$

Hence, $\theta_x \circ \varphi_{y,x} = \theta_y$, for all $x, y \in M$ with $S_y \subseteq S_x$.

Let $(i_x : A_x \to \varinjlim_{x\in M} A_x)_{x\in M}$ be the direct limit of the directed system $(A_x, \varphi_{y,x})$. Since it verifies also a property of universality, there exists a unique morphism $\theta : \varinjlim_{x\in M} A_x \to A_{S_M}$ which commutes the diagram:

$$\begin{tikzcd} A_x \arrow[rr, "i_x"] \arrow[dr, "\theta_x"'] & & \varinjlim_{x\in M} A_x \arrow[dl, "\theta"] \\ & A_{S_M} & \end{tikzcd} \tag{5.14}$$

Theorem 5.14. *With the previous notations,* $A_M \simeq \varinjlim_{x\in M} A_x$.

Proof. Let $s \in S_M$. Then, there exists $x_0 \in M$ with $s \in S_{x_0}$ which means that $s \notin \bigcup_{J\in r(x_0)} J$. Hence, $s \notin J$, for all $J \in r(x_0)$. We define $\Psi : A \to \varinjlim_{x\in M} A_x$ by $\Psi = i_{x_0} \circ p_{x_0}$.

We prove that the definition of Ψ is not depending on the choosing of $x_0 \in M$. To realize that, let $y_0 \in M$ such that $s \in S_{y_0}$. We have to verify that $i_{x_0} \circ p_{x_0} = i_{y_0} \circ p_{y_0}$. Since $x_0, y_0 \in M$, as in Lemma 5.13, $z_0 = x_0 \barwedge y_0 \in M$ and then $S_{x_0}, S_{y_0} \subseteq S_{z_0}$.

The following diagram is a commutative one:

$$\begin{tikzcd} & A_{x_0} \arrow[d, "\varphi_{x_0,z_0}"] \arrow[dr, "i_{x_0}"] & \\ A \arrow[ur, "p_{x_0}"] \arrow[r, "p_{z0}"] \arrow[dr, "p_{y_0}"'] & A_{z_0} \arrow[r, "i_{z_0}"] & \varinjlim_{x\in M} A_x \\ & A_{y_0} \arrow[u, "\varphi_{y_0,z_0}"'] \arrow[ur, "i_{y_0}"'] & \end{tikzcd} \tag{5.15}$$

Indeed, $i_{x_0} \circ p_{x_0} = (i_{z_0} \circ \varphi_{x_0,z_0}) \circ p_{x_0} = i_{z_0} \circ p_{z_0} = i_{z_0} \circ (\varphi_{y_0,z_0} \circ p_{y_0}) = i_{y_0} \circ p_{y_0}$ because the family of morphisms $(p_x : A \to A_x)_{x\in M}$ verifies the relation $p_x = \varphi_{y,x} \circ p_y$, for each pairs (x, y) such that $S_y \subseteq S_x$. In conclusion, Ψ is well defined.

Let $x_0 \in M$ with $s \in S_{x_0}$. Then $\Psi(s) = i_{x_0}(p_{x_0}(s)) = i_{x_0}([0]_{S_{x_0}}) = \tilde{0}$ where $\tilde{0}$ is the equivalence class of 0 in the inductive limit. From the property of universality of algebras of fractions, there exists an unique morphism $\theta' : A_{S_M} \to \varinjlim_{x\in M} A_x$ such that $\theta'([a]_{S_M}) = \Psi(a)$.

From $\theta' \circ p_{S_M} = \Psi$ and $\theta \circ \Psi = p_{S_M}$, we get that $\theta \circ \theta' = 1_{A_{S_M}}$. The relation $(\theta' \circ \theta) \circ \Psi = \theta' \circ p_{S_M} = \Psi$ implies that $(\theta' \circ \theta) \circ i_x = i_x$, for each $x \in M$. Since $i_x \circ \varphi_{y,x} = i_y$ for all $x, y \in M$ such that $S_y \subseteq S_x$, we have obtained two morphisms $\theta' \circ \theta$ and $1_{\varinjlim_{x\in M} A_x}$ which commute the diagram:

$$\begin{tikzcd} A_x \arrow[rr, "i_x"] \arrow[dr, "i_x"'] & & \varinjlim_{x\in M} A_x \arrow[dl] \\ & \varinjlim_{x\in M} A_x & \end{tikzcd} \tag{5.16}$$

Using the property of universality of the inductive limit of the directed system $(A_x, \varphi_{y,x})$, we get that $\theta' \circ \theta = 1_{\varinjlim_{x\in M} A_x}$. Hence, θ' is the inverse of the morphism θ and $A_{S_M} \simeq \varinjlim_{x\in M} A_x$.

To end the proof, we show that $S_M = A \setminus M$. Let $y \in S_M$. Then, $y \notin K$, where $K \in r(x')$ with $x' \in M$. Hence, $y \notin M$.

Conversely, since M is maximal, for $y \notin M$ we have $y^* \in M$. From Lemma 5.11 we obtain $y \in S_{y^*} \subseteq S_M$. □

We now repeat the previous construction for irreducible deductive systems. From Lemmas 5.11 and 5.12 we know that for each $n \geq 1$ and each $x_1, \ldots, x_n \in A$, $T_{x_1,\ldots,x_n}$ is a decreasing $\veebar$-closed subset in A.

Lemma 5.15. *For $I \in \mathrm{Ir}(A)$, the class $\{T_{x_1,\ldots,x_n} \mid \bigcap_{i=1}^n s(x_i) \in V_I\}$ is directed.*

Proof. If we consider $T_{\overline{\bigwedge}_{i=1}^n x_i}$ and $T_{\overline{\bigwedge}_{j=1}^m y_j}$ be two elements of this class, Lemmas 5.11 and 5.13 imply that $z = \overline{\bigwedge}_{i,j=1}^{n,m}(x_i \barwedge y_j) \in I$ and $T_{\overline{\bigwedge}_{i=1}^n x_i}, T_{\overline{\bigwedge}_{j=1}^m y_j} \subseteq T_z$. □

Now, let I be an irreducible deductive system in A. We denote the union of all the $\veebar$-subsets $T_{x_1,\ldots,x_n}$, when $\bigcap_{i=1}^n s(x_i) \in V_I$ and $n \geq 1$ with T_I. Then, T_I is a decreasing $\veebar$-subset in A.

Theorem 5.16. *With the previous notations, for an irreducible deductive system I in the bounded Hilbert algebra A,*

$$A_{\widetilde{I}} \simeq A_{T_I} \simeq \varinjlim_{\bigcap_{i=1}^n s(x_i)\in V_I} T_{\overline{\bigwedge}_{i=1}^n x_i}. \tag{5.17}$$

Proof. From Proposition 3.12, $\widetilde{I}$ is a decreasing $\veebar$-closed subset.

Since T_I is a decreasing $\veebar$-closed subset in A, we obtain, as in the proof of Theorem 5.14, that

$$A_{T_I} \simeq \varinjlim_{\bigcap_{i=1}^n s(x_i)\in V_I} T_{\overline{\bigwedge}_{i=1}^n x_i}. \tag{5.18}$$

Finally, we prove that $T_I = \widetilde{I}$.

For $x \in \widetilde{I}$, $x^* \in I$. Then, $x \in T_{x^*} = \{a \in A, a \leq x^{**}\}$. Since $T_{x^*} \subseteq T_I$, we obtain that $x \in T_I$. Conversely, let $x \in T_I$. This implies that there exists an element $\overline{\bigwedge}_{i=1}^{k} y_i \in I$ such that $x \in T_{\overline{\bigwedge}_{i=1}^{k} y_i}$. Hence, $x \leq (\overline{\bigwedge}_{i=1}^{k} y_i)^*$. But, $\overline{\bigwedge}_{i=1}^{k} y_i = (\overline{\bigwedge}_{i=1}^{k} y_i)^{**} \leq x^*$. As $\overline{\bigwedge}_{i=1}^{k} y_i \in I$, we get $x^* \in I$ which is equivalent to $x \in \widetilde{I}$. Thus, $T_I = \widetilde{I}$. □

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