Research Article

A Note on the Range of the Operator $X \mapsto TX - XT$ Defined on $\mathcal{C}_2(\mathcal{H})$

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We show how a proof of J. Stampfli can be extended to prove that the operator $X \mapsto TX - XT$ defined on the Hilbert-Schmidt class, when T is an M -hyponormal, p -hyponormal, or log-hyponormal operator, has a closed range if and only if $\sigma(T)$ is finite.

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1. Introduction

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all linear bounded operators on $\mathcal{H}$. The Hilbert-Schmidt class, denoted by $\mathcal{C}_2(\mathcal{H})$, is a Hilbert space with the $\|\cdot\|_2$ -norm that arises from the inner product $\langle X, Y \rangle = \text{tr}(XY^*)$, where tr is the scalar-valued trace. For $T \in \mathcal{L}(\mathcal{H})$, define $\Delta_T : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ by $\Delta_T(X) = TX - XT$, and let $\sigma(T)$ denote the spectrum of T . Let the range of a linear operator S be denoted by $\mathcal{R}(S)$. For a normal operator $N \in \mathcal{L}(\mathcal{H})$, Anderson and Foiaş [1] proved that $\mathcal{R}(\Delta_N)$ is norm closed if and only if $\sigma(N)$ is a finite set. In [2], Stampfli extended this result to the class of hyponormal operators.

Theorem A ([2]). *Let $T \in \mathcal{L}(\mathcal{H})$ be a hyponormal operator. Then $\mathcal{R}(\Delta_T)$ is norm closed if and only if $\sigma(T)$ is finite.*

In fact, Stampfli provided a proof of the “only if” implication which extends to a larger class of operators than the class of hyponormal operators (see Proposition 2.2). For an operator $T \in \mathcal{L}(\mathcal{H})$, let $\sigma_{\text{nap}}(T)$ denote its *normal approximate point spectrum*, that is, the set of those $\lambda \in \mathbb{C}$ for which there exists an orthonormal sequence $\{\phi_n\}_n$ in $\mathcal{H}$ such that

$$\|(T - \lambda)\phi_n\| + \|(T - \lambda)^*\phi_n\| \longrightarrow 0. \quad (1.1)$$

Define the class $\mathcal{G}(\mathcal{H})$ as follows:

$$\mathcal{G}(\mathcal{H}) := \{T \in \mathcal{L}(\mathcal{H}) \mid \sigma_{\text{nap}}(T) \text{ is an infinite set}\}. \quad (1.2)$$

Some classes of hyponormal related operators, such as M -hyponormal operators, that is,

$$m \cdot \|(T - \lambda)^* \phi\| \leq \|(T - \lambda)\phi\|, \quad (\forall) \phi \in \mathcal{H}, (\forall) \lambda \in \mathbb{C}, \text{ for some } m > 0, \quad (1.3)$$

p -hyponormal operators, that is, $(T^*T)^p \geq (TT^*)^p$ for some $p > 0$, or log-hyponormal operators, that is, invertible operators such that $\log(T^*T) \geq \log(TT^*)$, have spectrum that is finite or they belong to $\mathcal{G}(\mathcal{H})$. Particularly, the hyponormal operators (i.e., 1-hyponormal) have this property.

In [3] Stampfli proved the following lemma which will be used in Section 2.

Lemma B. Let $T \in \mathcal{G}(\mathcal{H})$ and let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of distinct points of $\sigma_{\text{nap}}(T)$. Then for any sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive numbers converging to zero, there exists an orthonormal sequence $\{\phi_n\}_{n=1}^\infty$ of vectors in $\mathcal{H}$ such that

$$\|(T - \lambda_n)\phi_n\| + \|(T - \lambda_n)^*\phi_n\| < \varepsilon_n \quad \text{for } n = 1, 2, \dots, \quad (1.4)$$

$$\langle \phi_n, T\phi_k \rangle = 0 \quad \text{for } k = 1, \dots, n-1. \quad (1.5)$$

2. The Closedness of the Range of $\Delta_T^{(2)}$

The operator Δ_T defined on the Hilbert-Schmidt class will be denoted in the remainder of this note by $\Delta_T^{(2)}$, that is, $\Delta_T^{(2)} : \mathcal{C}_2(\mathcal{H}) \rightarrow \mathcal{C}_2(\mathcal{H})$, $\Delta_T^{(2)}(X) = TX - XT$. Let $H^M(\mathcal{H})$ denote the set of M -hyponormal operators.

Proposition 2.1. Let $T \in H^M(\mathcal{H})$. If $\sigma(T)$ is finite, then $\mathcal{R}(\Delta_T^{(2)})$ is closed.

Proof. It is well known that an operator $T \in H^M(\mathcal{H})$ with finite spectrum is normal. Indeed, for such an operator, the restriction to an invariant subspace $\mathcal{M}$ belongs to $H^M(\mathcal{M})$. On the other hand, if $T \in H^M(\mathcal{H})$ with $\sigma(T) = \{\lambda\}$, then $T = \lambda I$, (cf. [4]). Thus, we can write $T = \sum_{i=1}^{n_0} \lambda_i E_i$, where E_i 's are the spectral projections.

Let X_n and C be in $\mathcal{C}_2(\mathcal{H})$ such that $\|\Delta_T^{(2)}(X_n) - C\|_2 \rightarrow 0$. Therefore $\Delta_T(X_n) - C \rightarrow 0$ in the $\mathcal{L}(\mathcal{H})$ norm, and according to Theorem A, there exists $X^0 \in \mathcal{L}(\mathcal{H})$ such that $C = TX^0 - X^0T$. For an arbitrary $X \in \mathcal{L}(\mathcal{H})$, let $[X_{ij}]$ be the block-matrix representation of X relative to the decomposition $\mathcal{H} = \sum_{i=1}^{n_0} \oplus E_i \mathcal{H}$. Thus

$$C_{ij} = (\lambda_i - \lambda_j)X_{ij}^0, \quad (2.1)$$

for all $i, j = 1, \dots, n_0$. This implies that each $X_{ij}^0 = (1/(\lambda_i - \lambda_j))C_{ij}$ is a Hilbert-Schmidt operator. Moreover X_{ii}^0 can be chosen 0, and thus $X^0 \in \mathcal{C}_2(\mathcal{H})$. $\square$

Proposition 2.2. Let $T \in \mathcal{C}(\mathcal{H})$. Then $\mathcal{R}(\Delta_T^{(2)})$ is not closed.

Proof. We will use same notation and circle of ideas as in [2]. Let $\{\lambda_n\}_{n \geq 1}$ be sequence of distinct points of $\sigma_{\text{nap}}(T)$ so that $\lambda_n \rightarrow \lambda_0$. Let

$$\eta_n = \max \left\{ |\lambda_{j+1} - \lambda_j|^{-1/2} \mid j = 1, \dots, n \right\}, \quad (2.2)$$

and choose a nonincreasing sequence $\{\varepsilon_n\}_{n \geq 1}$ so that $0 < \varepsilon_n \leq |\lambda_{n+1} - \lambda_n|^2$, $n \geq 1$, and $\sum_{n \geq 1} \varepsilon_n^2 \eta_n^2 < \infty$. According to Lemma B, there exists an orthonormal sequence $\{\phi_n\}_{n \geq 1}$ that satisfies (1.4) and (1.5). Let $\mathcal{H}_1 = \vee \{\phi_n \mid n \geq 1\}$, $\mathcal{H}_2 = \mathcal{H}_1^\perp$, and let δ_n such that

$$T\phi_n = \mu_n \phi_n + \delta_n, \quad \delta_n \perp \phi_n, \quad n \geq 1. \quad (2.3)$$

It results that

$$|\mu_n - \lambda_n| < \varepsilon_n, \quad \|\delta_n\| < 2\varepsilon_n, \quad n \geq 1. \quad (2.4)$$

Define $V : \mathcal{H} \rightarrow \mathcal{H}$ by $V\phi_n = |\lambda_{j+1} - \lambda_j|^{-1/2} \phi_{n+1}$, $n \geq 1$, and $Vg = 0$, $g \in \mathcal{H}_2$. Let $\mathcal{M}_n = \vee \{\phi_j \mid j = 1, \dots, n\}$ and let P_n be the orthogonal projection onto $\mathcal{M}_n$, and define $V_n = VP_n$. A tedious calculation shows that

$$\Delta_T(V_n)\phi_j = \begin{cases} v_j(\mu_{j+1} - \mu_j)\phi_{j+1} + v_j\delta_{j+1} - V_n\delta_j, & j \leq n, \\ -V_n\delta_j, & j > n, \end{cases} \quad (2.5)$$

where $v_j = |\lambda_{j+1} - \lambda_j|^{-1/2}$. Denoting $\Delta_T(V_n) - \Delta_T(V_m)$ by $\Delta_T^{n,m}$, then for $n < m$,

$$\Delta_T^{n,m}\phi_j = \begin{cases} 0, & j \leq n, \\ -v_j(\mu_{j+1} - \mu_j)\phi_{j+1} + v_j\delta_{j+1} + (V_m - V_n)\delta_j, & n < j \leq m, \\ (V_m - V_n)\delta_j, & j > m. \end{cases} \quad (2.6)$$

Furthermore, from (2.3) it results that

$$\delta_j \perp \phi_j, \phi_{j+1}, \phi_{j+2}, \dots \quad (2.7)$$

and from (2.4)

$$\|V_n\delta_j\| \leq 2\eta_j\varepsilon_j, \quad \forall j, n \geq 1. \quad (2.8)$$

We will show next that $\|\Delta_T^{n,m}\|_2 \rightarrow 0$ when $m, n \rightarrow \infty$, thus there exists $C \in \mathcal{C}_2(\mathcal{H})$ such that $\|\Delta_T(V_n) - C\|_2 \rightarrow 0$, that is, $C \in \overline{\mathcal{R}(\Delta_T^{(2)})}$.

First, we will show that $\|\Delta_T^{n,m}|_{\mathcal{L}_1}\|_2^2 \rightarrow 0$, when $m, n \rightarrow \infty$. Indeed,

$$\begin{aligned} \|\Delta_T^{n,m}|_{\mathcal{L}_1}\|_2^2 &= \sum_{j=1}^{\infty} \|\Delta_T^{n,m}\phi_j\|^2 \stackrel{(2.6)}{=} \\ &= \sum_{j=n+1}^m \|-v_j(\mu_{j+1} - \mu_j)\phi_{j+1} + v_j\delta_{j+1} + (V_m - V_n)\delta_j\|^2 \\ &\quad + \sum_{j=m+1}^{\infty} \|(V_m - V_n)\delta_j\|^2. \end{aligned} \quad (2.9)$$

The first sum of the right-hand side of the above can be majorized by

$$2 \cdot \sum_{j=n+1}^m \|-v_j(\mu_{j+1} - \mu_j)\phi_{j+1} + v_j\delta_{j+1}\|^2 + 2 \cdot \sum_{j=n+1}^m \|(V_m - V_n)\delta_j\|^2. \quad (2.10)$$

Since $\phi_{j+1} \perp \delta_{j+1}$, we have

$$\|\Delta_T^{n,m}|_{\mathcal{L}_1}\|_2^2 \leq 2 \left[\sum_{j=n+1}^m (v_j^2 |\mu_{j+1} - \mu_j|^2 + v_j^2 \|\delta_{j+1}\|^2) + \sum_{j=n+1}^{\infty} \|(V_m - V_n)\delta_j\|^2 \right]. \quad (2.11)$$

According to (2.8),

$$\|(V_m - V_n)\delta_j\|^2 \leq 16\eta_j^2 \varepsilon_j^2, \quad (2.12)$$

and according to (2.4),

$$\begin{aligned} v_j^2 \|\delta_{j+1}\|^2 &\leq 4\eta_j^2 \varepsilon_{j+1}^2 \leq 4\eta_j^2 \varepsilon_j^2, \\ |\mu_{j+1} - \mu_j|^2 &\leq (2\varepsilon_j + |\lambda_{j+1} - \lambda_j|)^2 \leq 8\varepsilon_j^2 + 2|\lambda_{j+1} - \lambda_j|^2, \end{aligned} \quad (2.13)$$

which implies

$$v_j^2 |\mu_{j+1} - \mu_j|^2 \leq 8\eta_j^2 \varepsilon_j^2 + 2|\lambda_{j+1} - \lambda_j|. \quad (2.14)$$

Therefore

$$\|\Delta_T^{n,m}|_{\mathcal{L}_1}\|_2^2 \leq c_1 \cdot \sum_{j=n+1}^{\infty} \eta_j^2 \varepsilon_j^2 + c_2 \cdot \sum_{j=n+1}^m |\lambda_{j+1} - \lambda_j|, \quad (2.15)$$

where c_1 and c_2 are some constants. After a careful review of the proof, one can see that the sequence $\{\lambda_n\}$ can be assumed to converge fast enough (otherwise choose a subsequence of it), more precisely $\sum_{j=n+1}^m |\lambda_{j+1} - \lambda_j| \rightarrow 0$ when $n, m \rightarrow \infty$.

We show next that $\|\Delta_T^{n,m}|_{\mathcal{L}_2}\|_2^2 \rightarrow 0$, when $m, n \rightarrow \infty$. Indeed, we can write

$$T^*\phi_n = \bar{\mu}_n\phi_n + \gamma_n \quad \text{with } \langle \gamma_n, \phi_n \rangle = 0, \quad \|\gamma_n\| \leq 2\varepsilon_n, \quad n \geq 1. \quad (2.16)$$

Obviously, we can write $T^*\phi_n = \theta_n\phi_n + \gamma_n$ with $\langle \gamma_n, \phi_n \rangle = 0$, which implies

$$\begin{aligned} \theta_n &= \langle \theta_n\phi_n + \gamma_n, \phi_n \rangle = \langle T^*\phi_n, \phi_n \rangle = \langle \phi_n, T\phi_n \rangle = \langle \phi_n, \mu_n\phi_n + \delta_n \rangle = \bar{\mu}_n, \\ \|\gamma_n\| &= \|(T^* - \bar{\mu}_n)\phi_n\| \leq \|(T - \lambda_n)^*\phi_n\| + \left| \bar{\lambda}_n - \bar{\mu}_n \right| \stackrel{(1.4),(2.4)}{\leq} 2\varepsilon_n. \end{aligned} \quad (2.17)$$

For an orthonormal basis $\{\psi_i\}_{i \geq 1}$ of $\mathcal{L}_2$, we will show that

$$\sum_{i=1}^{\infty} \|\Delta_T^{n,m}\psi_i\|^2 \longrightarrow 0 \quad \text{when } n, m \longrightarrow \infty. \quad (2.18)$$

For each i , write $T\psi_i = \sum_{k=1}^{\infty} a_k^{(i)}\phi_k + w_i$ with $w_i \in \mathcal{L}_2$. Thus

$$V_m T\psi_i = \sum_{k=1}^m a_k^{(i)} V_m \phi_k + V_m w_i = \sum_{k=1}^m a_k^{(i)} v_k \phi_{k+1}. \quad (2.19)$$

Since $V_m \psi_i = 0$, we have $\Delta_T(V_m)\psi_i = -V_m T\psi_i$, and consequently, for $n < m$,

$$\Delta_T^{n,m}\psi_i = \sum_{k=n+1}^m a_k^{(i)} v_k \phi_{k+1}. \quad (2.20)$$

Since the sequence $\{\phi_k\}$ is orthonormal, we have

$$\|\Delta_T^{n,m}\psi_i\|^2 = \sum_{k=n+1}^m \left| a_k^{(i)} \right|^2 \cdot v_k^2. \quad (2.21)$$

Therefore

$$\sum_{i=1}^{\infty} \|\Delta_T^{n,m}\psi_i\|^2 = \sum_{i=1}^{\infty} \sum_{k=n+1}^m \left| a_k^{(i)} \right|^2 \cdot v_k^2 = \sum_{k=n+1}^m v_k^2 \left(\sum_{i=1}^{\infty} \left| a_k^{(i)} \right|^2 \right). \quad (2.22)$$

For a fixed k ,

$$\begin{aligned}
 \sum_{i=1}^{\infty} |a_k^{(i)}|^2 &= \sum_{i=1}^{\infty} |\langle T\psi_i, \phi_k \rangle|^2 = \sum_{i=1}^{\infty} |\langle \psi_i, T^* \phi_k \rangle|^2 \\
 &\stackrel{(2.16)}{=} \sum_{i=1}^{\infty} |\langle \psi_i, \bar{\mu}_k \phi_k + \gamma_k \rangle|^2 = \sum_{i=1}^{\infty} |\langle \psi_i, \gamma_k \rangle|^2 \leq \|\gamma_k\|^2 \\
 &\stackrel{(2.16)}{\leq} 4\varepsilon_k^2.
 \end{aligned} \tag{2.23}$$

Consequently, $\sum_{i=1}^{\infty} \|\Delta_T^{n,m} \psi_i\|^2 \leq 4 \sum_{k=n+1}^m v_k^2 \cdot \varepsilon_k^2 \rightarrow 0$ for $n, m \rightarrow \infty$.

The operator C is not in $\mathcal{R}(\Delta_T^{(2)})$ since, according to the proof of Theorem A in [2], $C \notin \mathcal{R}(\Delta_T)$. $\square$

Theorem 2.3. *Let $T \in H^M(\mathcal{H})$. Then $\mathcal{R}(\Delta_T^{(2)})$ is closed if and only if $\sigma(T)$ is finite.*

Proof. If $T \in H^M(\mathcal{H})$ and $\sigma(T)$ are finite, then according to Proposition 2.1, $\mathcal{R}(\Delta_T^{(2)})$ is closed. Conversely, if $T \in H^M(\mathcal{H})$ has an infinite spectrum, then there are infinitely many distinct points $\{\lambda_n\}_n$ that are either isolated points of the spectrum, in which case they are eigenvalues, or accumulation points of the spectrum, in which case they are in $\sigma_{\text{ap}}(T)$. Since $T \in H^M(\mathcal{H})$, we have $\sigma_p(T), \sigma_{\text{ap}}(T) \subseteq \sigma_{\text{nap}}(T)$. Thus $T \in \mathcal{G}(\mathcal{H})$ and according to Proposition 2.2, $\mathcal{R}(\Delta_T^{(2)})$ is not closed. $\square$

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