Research Article

On the Tensor Products of Maximal Abelian $JW$-Algebras

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It is well known in the work of Kadison and Ringrose (1983) that if $A$ and $B$ are maximal abelian von Neumann subalgebras of von Neumann algebras $M$ and $N$, respectively, then $A \overline{\otimes} B$ is a maximal abelian von Neumann subalgebra of $M \overline{\otimes} N$. It is then natural to ask whether a similar result holds in the context of $JW$-algebras and the $JW$-tensor product. Guided to some extent by the close relationship between a $JW$-algebra $M$ and its universal enveloping von Neumann algebra $W^*(M)$, we seek in this article to investigate the answer to this question.

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1. Introduction

A $JC$-algebra $A$ is a norm (uniformly) closed Jordan subalgebra of the Jordan algebra $B(H)_{s,a}$ of all bounded self adjoint operators on a Hilbert space $H$. The Jordan product is given by $a \circ b = (ab + ba)/2$. A subspace $I$ of a $JC$-algebra $A$ is called a Jordan ideal if $a \circ b \in I$ for every $a \in A$ and every $b \in I$. A $JC$-algebra is said to be simple if it has no nontrivial norm closed Jordan ideals. A $JW$-algebra $M \subseteq B(H)_{s,a}$ is a weakly closed $JC$-algebra. If $M$ is a $JC$-algebra (resp., $JW$-algebra), let $C^*(M)$ (resp., $W^*(M)$) be the universal enveloping $C^*$-algebra (resp., von Neumann algebra) of $M$, and let $\theta_M$ (resp., $\Phi_M$) be the canonical involutive $\ast$-antiautomorphism of $C^*(M)$ (resp., $W^*(M)$). Usually we will regard $M$ as a generating Jordan subalgebra of $C^*(M)$ and $W^*(M)$ so that $\theta_M$ and $\Phi_M$ fix each point of $M$. The real $C^*$-algebra $R^*(M) = \{ x \in C^*(M) : \theta_M(x) = x^* \}$ satisfies

$$R^*(M) \cap iR^*(M) = 0, \quad C^*(M) = R^*(M) \oplus iR^*(M), \quad (1.1)$$

and the real von Neumann algebra $RW^*(M) = \{ x \in W^*(M) : \Phi_M(x) = x^* \}$ satisfies

$$RW^*(M) \cap iRW^*(M) = 0, \quad W^*(M) = RW^*(M) \oplus iRW^*(M). \quad (1.2)$$
The reader is referred to [1–5] for a detailed account of the theory of JC-algebras and JW-algebras. The relevant background on the theory of C*-algebras and von Neumann algebras can be found in [6–8].

A projection $e$ of a JW-algebra $M$ is said to be abelian if $eMe$ is associative, and it is called minimal if it is nonzero and contains no other nonzero projections of $M$, or equivalently, $e$ is minimal if and only if $eMe = \mathbb{R}e$. A JW-factor is a JW-algebra with trivial centre; a Type I JW-factor is a JW-factor which contains a minimal projection. A JW-algebra is said to be of Type I, if there is a family of abelian projections $(e_a)_{a \in I}$ such that the central support $c_M(e_a)$ of $e_a$ in $M$ equals the unit $1_M$ of $M$, $\sum_{a \in I} e_a = 1_M$ and card $I = n$ (see [1, Section 5.3]). A spin factor $V = H \oplus \mathbb{R}1_V$ is a real Jordan algebra with identity $1_V$, where $H$ is a real Hilbert space of dimension at least two. The Jordan product on $V$ is defined by

$$
(a + \lambda 1_V) \circ (b + \mu 1_V) = (\mu a + \lambda b) + (\langle a, b \rangle + \lambda \mu)1_V, \quad a, b \in V, \lambda, \mu \in \mathbb{R},
$$

and the norm on $V$ is given by

$$
\|a + \lambda 1_V\| = \langle a, a \rangle^{1/2} + |\lambda|.
$$

A spin factor $V$ is universally reversible when $\dim V = 3$ or 4, nonreversible when $\dim V \neq 3,4$ or 6, and it can be either reversible or nonreversible when $\dim V = 6$. A spin factor is a simple reflexive JW-algebra and constitutes the Type $I_2$ JW-factor (see [2, Section 6.1]).

A linear map $\varphi : A \to B$ between JC-algebras $A$ and $B$ is called a (Jordan) homomorphism if it preserves the Jordan product. A Jordan homomorphism which is one to one is called a Jordan isomorphism. A factor representation of a JC-algebra $A$ is a (Jordan) homomorphism of $A$ onto a weakly dense subalgebra of a JW-factor $M$. Type I factor representations are defined accordingly.

A JC-algebra $A$ is said to be reversible if $a_1a_2 \cdots a_n + a_na_{n-1} \cdots a_1 \in A$ whenever $a_1, a_2, \ldots, a_n \in A$ and is said to be universally reversible if $\pi(A)$ is reversible for every representation $\pi$ of $A$ [2, page 5]. The only universally reversible spin factors are $V_2 = M_2(\mathbb{R})_{s.a}$ and $V_3 = M_2(\mathbb{C})_{s.a}$ [2, Theorem 2.1]. A JC-algebra $A$ is universally reversible if and only if it has no spin factor representations other than onto $V_2$ and $V_3$ [2, Theorems 2.2]. Every JW-algebra without a direct summand of Type $I_2$ is universally reversible [1, 5.1.5, 5.3.5, 6.2.3].

Two elements $a$ and $b$ of a JC-algebra $A$ are said to operator commute if $T_aT_b = T_bT_a$, where $T_a : A \to A$ is the multiplication operator defined by $T_a(x) = a \circ x$, for all $x \in A$. A JW-algebra $M$ is called associative if all its elements operators commute. A JW-subalgebra $A$ of a JW-algebra $M$ is called maximal associative if it is not contained in any larger associative JW-subalgebra of $M$. If $A$ is a JW-subalgebra of a JW-algebra $M \subseteq B(H)_{s.a}$ and $A'$ is the set of all elements of $B(H)_{s.a}$ which operator commutes with all elements of $A$, then $A$ is a maximal associative JW-subalgebra of $M$ if and only if $A = A' \cap M$. Indeed, since $A$ is associative, $A \subseteq A' \cap M$ and $A$ together with any element of $A' \cap M$ generates an associative JW-subalgebra of $M$ which implies that $A' \cap M \subseteq A$ since $A$ is maximal abelian. In particular, if $A \subseteq B(H)_{s.a}$ is an associative JW-algebra, then $A$ is maximal associative if and only if $A = A'$.

This article aims to study the relationship between the maximality of an associative JW-subalgebra $B$ of a JW-algebra $M$ and that of $W^*(B)$ in $W^*(M)$. We give a counterexample which rules out the establishing of a result in the theory of JW-tensor products analog to that
given in [6, Theorem 11.2.18] for von Neumann tensor products (cf. Example 2.2). Then we prove that a Jordan analog of Theorem 11.2.18 in [6] can be established in some particular cases.

**Theorem 1.1** (see [9, Proposition 1]). Let \( M \subseteq B(H)_{s,a} \) be a JW-algebra, and let \( a, b \in M \). Then the following are equivalent:

(i) \( ab = ba \);

(ii) \( T_aT_b = T_bT_a \);

(iii) \( a^2 \circ b = aba \).

That is, \( a \) and \( b \) operators commute if and only if they commute under ordinary operator multiplication.

**Definition 1.2.** Let \( M \) and \( N \) be a pair of JW-algebras canonically embedded in their respective universal enveloping von Neumann algebras \( W^*(M) \) and \( W^*(N) \). Then the JW-tensor product \( JW(M \otimes N) \) of \( M \) and \( N \) is the JW-algebra generated by \( M \otimes N \) in \( W^*(M) \otimes W^*(N) \). The reader is referred to [10] for the properties of the JW-tensor product of JW-algebras.

**Theorem 1.3** (see [10, Theorem 2.9]). Let \( M \) and \( N \) be JW-algebras. If \( JW(M \otimes N) \) is universally reversible, then

\[
W^*(JW(M \otimes N)) = W^*(M) \overline{\otimes} W^*(N). \tag{1.5}
\]

**2. Maximal Abelian JW-algebras**

Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be maximal abelian von Neumann subalgebras of von Neumann algebras \( \mathfrak{M} \) and \( \mathfrak{N} \), respectively, then \( \mathfrak{A} \otimes \mathfrak{B} \) is a maximal abelian von Neumann algebra of \( \mathfrak{M} \otimes \mathfrak{N} \) (see [6, 11.2.18]). In Example 2.2, we show that the Jordan analog of this result, in the context of JW-algebras and the JW-tensor product, is not true in general. However, it is shown in Theorem 2.11 that the result does hold in special circumstances.

**Remark 2.1.** (i) Note that any JW-subalgebra of a spin factor which is not a spin factor is of dimension at most 2. Indeed, let \( A \) be a JW-subalgebra of a spin factor \( V \subseteq B(H)_{s,a} \). If \( 1_A \neq 1_V \), then \( 1_A \) is the only projection in \( A \) since every projection in \( V \) is minimal, and hence \( \dim A = 1 \). If \( 1_A = 1_V \), then any family of orthogonal central projections of \( A \) contains at most two projections. Indeed if \( e_1 + e_2 + e_3 = 1_A, e_i \in Z(A), i = 1, 2, 3 \), then \( e_2 + e_3 \leq 1_A - e_1 \). Since \( 1_A - e_1 \) is a minimal projection, we see that one of \( e_i, i = 1, 2, 3 \) must be zero. It is clear that if \( A \) is a factor, then it is of Type II\(_2\), and hence it is a spin factor. (ii) Recall that \( W^*(V) = M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \), where \( V \) is the 4-dimensional spin factor \( M_2(\mathbb{C})_{s,a} \) [1, 6.2.1]:

\[
M_2(\mathbb{C}) = \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right\} = \mathbb{C} \overline{\otimes} M_2(\mathbb{R}), \tag{2.1}
\]

which is an 8-dimensional real C*-algebra.
Example 2.2. Let \( A \) be a maximal abelian JW-subalgebra of \( V = M_2(\mathbb{C})_{s,a} \). Then \( JW(A \boxplus A) \) is not a maximal abelian subalgebra of \( JW(V \boxplus V) \).

Proof. By the above remark, \( \dim A = 2 \), and hence \( A = \mathbb{R}e + \mathbb{R}f \) for some minimal projections \( e, f \). Therefore,

\[
JW(A \boxplus A) = A \boxplus A = \mathbb{R}(e \oplus e) \oplus \mathbb{R}(e \oplus f) \oplus \mathbb{R}(f \oplus e) \oplus \mathbb{R}(f \oplus f),
\]

and hence \( \dim JW(A \boxplus A) = 4 \), since \( \dim A \boxplus A = \dim A \cdot \dim A \) (see [11, Corollary 7.5]). On the other hand, \( JW(V \boxplus V) \) is universally reversible, by [10, Proposition 2.7] which implies that

\[
JW(V \boxplus V) = RW^*(JW(V \boxplus V))_{s,a}
\]

\[
= \left( RW^*(V) \boxplus RW^*(V) \right)_{s,a}
\]

\[
= \left( M_2(\mathbb{C}) \boxplus M_2(\mathbb{C}) \right)_{s,a}
\]

\[
= M_2(C)_{s,a} \oplus M_2(C)_{s,a},
\]

since \( RW^*(M_2(\mathbb{C})_{s,a}) = M_2(\mathbb{C}) \) [3, page 385]. It can be seen that a maximal abelian JW-subalgebra of \( JW(V \boxplus V) \) is of dimension 8, which implies that \( JW(A \boxplus A) \) is not maximal abelian in \( JW(V \boxplus V) \). \( \square \)

Remark 2.3. Note that if \( B \) is an associative JW-subalgebra of a JW-algebra \( M \) such that \( W^*(B) \) is a maximal abelian subalgebra of \( W^*(M) \), then \( B \) is a maximal associative JW-subalgebra of \( M \), since \( B = W^*(B) \cap M \).

Lemma 2.4. Let \( B \) be an associative JW-subalgebra of a JW-algebra \( M \). Then,

\[
W^*(B) = [B] = B \oplus iB,
\]

is an abelian von Neumann algebra, where \([B]\) is the weak*-closure of the \( C^*\)-subalgebra \([B]\) of \( W^*(M) \) generated by \( B \).

Proof. Being associative, \( B \) has no representation into a spin factor of the form \( V_{n+1} \) and is, therefore, universally reversible. It follows from [3, page 383] that

\[
B = RW^*(B)_{s,a}.
\]

Therefore, by [3, Corollary 3.2], \( RW^*(B) \) is isomorphic to the weak*-closure \( \overline{RB} \) of the real \( C^*\)-subalgebra \( R(B) \) of \( W^*(M) \) generated by \( B \), and the result follows. \( \square \)

Recall that if \( M \) is a JW-algebra isomorphic to the self adjoint part \( N_{s,a} \) of a von Neumann algebra \( N \) and has no one-dimensional representations, then \( W^*(M) \) is *-isomorphic to \( N \oplus N^\circ \), where \( N^\circ \) is the opposite algebra of \( N \) [2, 7.4.15]. A real \( C^*\)-algebra \( \mathfrak{A} \)
can be realized as a complex C*-algebra if there is a C*-algebra isomorphism \( \phi : \mathcal{B} \rightarrow \mathfrak{A} \) of a complex C*-algebra \( \mathcal{B} \) onto \( \mathfrak{A} \). In this case, the real linear isometry \( j \) on \( \mathfrak{A} \) defined, for each \( a \) in \( \mathcal{B} \), by

\[
j\phi(a) = \phi(ia)
\]

is such that \( j^2 \) and \(-id_\mathfrak{A}\) coincide.

**Lemma 2.5.** Let \( B \) be a maximal associative JW-subalgebra of a JW-algebra \( M \). Suppose that \( M \) is isomorphic to the self adjoint part \( \mathcal{A}_{s.a} \) of a von Neumann algebra \( \mathcal{A} \) and has no one-dimensional representations. Then \( W^*(B) \) is not a maximal abelian on Neumann subalgebra of \( W^*(M) \).

*Proof.* Identifying \( M \) with \( \mathcal{A}_{s.a} \), \([B]\) is a von Neumann subalgebra of both \( \mathcal{A} \) and \( \mathcal{A}_\sigma \), and hence, the von Neumann subalgebra \([B] \oplus [B] \) of \( \mathcal{A} \oplus \mathcal{A}_\sigma \equiv W^*(M) \) is abelian and contains \( W^*(B) = [B] \cong [B] \oplus \{0\} \), which implies that \( W^*(B) \) is not maximal abelian in \( W^*(M) \). \( \square \)

**Lemma 2.6.** Let \( B \) be a maximal associative JW-subalgebra of a JW-algebra \( M \). If \( RW^*(M) \) is *-isomorphic to a complex C*-algebra, then \( W^*(B) \) is not a maximal abelian von Neumann subalgebra of \( W^*(M) \).

*Proof.* Since \( C^*(M) \) is the complex C*-algebra \([M]\) generated by \( M \) in \( W^*(M) \) [12, Theorem 2.7], \( RW^*(M) \) is the weak*-closure of \( R^*(M) \) in \( W^*(M) \). Therefore, \( R^*(M) \) is a complex C*-algebra, which implies that \( C^*(M) = \mathcal{J} \oplus \Phi_M(\mathcal{J}) \) for some norm closed ideal \( \mathcal{J} \) of \( C^*(M) \) isomorphic to \( R^*(M) \) [13, Lemma 1], so that \( W^*(M) = \mathcal{J} \oplus \Phi_M(\mathcal{J}) \), where \( \mathcal{J} \) is the weak*-closure \( \overline{\mathcal{J}} \) of \( \mathcal{J} \) in \( W^*(M) \). Hence, \( \mathcal{J} \) is isomorphic to \( RW^*(M) \). Let \( \phi \) be the isomorphism of \( \mathcal{J} \) onto \( RW^*(M) \), and let \( j \) be the corresponding real linear operator on \( RW^*(M) \), defined above. Then, using Lemma 2.4, there exists an isomorphism \( \pi \) from the \( W^* \)-algebra \( B \oplus iB \) into \( RW^*(M) \) such that, for elements \( b_1 \) and \( b_2 \) in \( B \),

\[
\pi(b_1 + ib_2) = b_1 + jb_2.
\]

It follows that \( \phi^{-1} \circ \pi \) and \( \Phi_M \circ \phi^{-1} \circ \pi \) are *-isomorphisms of \([B] = B \oplus iB\) into \( \mathcal{J} \) and \( \Phi_M(\mathcal{J}) \), respectively. Since a *-isomorphism between C*-algebras is an isometry [7, Corollary 1.5.4], we may identify \([B]\) with \( \phi^{-1} \circ \pi([B]) \) and \( \Phi_M \circ \phi^{-1} \circ \pi([B]) \). It follows that \([B] \oplus [B]\) is an abelian von Neumann subalgebra of \( W^*(M) \), proving that \( W^*(B) = [B] \cong [B] \oplus \{0\} \) is not maximal abelian in \( W^*(M) \). \( \square \)

**Proposition 2.7.** Let \( M \) be a universally reversible JW-algebra not isomorphic to the self adjoint part of a von Neumann algebra and without direct summands of type \( I_1 \). If \( B \) is a maximal associative subalgebra of \( M \), then \( W^*(B) \) is a maximal abelian von Neumann subalgebra of \( W^*(M) \).

*Proof.* By Lemma 2.4, \( W^*(B) = [B] = B \oplus iB \hookrightarrow W^*(M) \). If \( W^*(B) \) is not maximal abelian in \( W^*(M) \), there exists an element \( z \in W^*(M) = RW^*(M) \oplus iRW^*(M) \), \( z \notin W^*(B) \) such that \( z \) together with \( W^*(B) \) generate an abelian von Neumann subalgebra \( Y \supsetneq W^*(B) \supsetneq B \) of \( W^*(M) \supsetneq M \). Let \( z = x + iy, x, y \in W^*(M)_{s.a} \). Since \( z \notin W^*(B) \), then either \( x \) or \( y \) (or both) does not belong to \( W^*(B) \). Suppose that \( x \notin W^*(B) \), since \( W^*(M) = RW^*(M) \oplus iRW^*(M) \), then \( x = a + ib \), for some \( a, b \in RW^*(M) \). Then either \( a \) or \( b \) (or both) does not belong to
\( W^*(B) \). Since \( x \in W^*(M)_{s,a} \), we have \( a = a^* \), and \( b = -b^* \), and so \( a \in M = RW^*(M)_{s,a} \) since \( M \) is a universally reversible [3, page 383]. Therefore, \( a \) must be the zero element, since it obviously commutes with all elements in \( B \). On the other hand, \( b^2 = -bb^* \in RW^*(M)_{s,a} = M \). Since \( bu = ub \) for all \( u \in W^*(B) \), \( b^2u = ubu = ub^2 \) for all \( u \in B \), and so \( b^2 \) and \( u \) operators commute relative to the Jordan product in \( B \) [9, Proposition 1. Hence \( b^2 \in B \subseteq W^*(B) \), since \( B \) is a maximal associative subalgebra of \( M \), which implies that \( b \in W^*(B) \). Therefore, \( x = ib \in W^*(B) \), a contradiction. This proves the result. \( \square \)

**Lemma 2.8.** Let \( M \) be a universally reversible JW-algebra not isomorphic to the self adjoint part of a von Neumann algebra. If \( B \) is a maximal associative subalgebra of \( M \), then \( W^*(B) \) is a maximal abelian von Neumann subalgebra of \( W^*(M) \).

**Proof.** Splitting \( M = M_{i1} \oplus M_{n,a} \) as the direct sum of a JW-algebra \( M_{i1} \) of type \( I \) (the abelian part) and a JW-algebra \( M_{n,a} \) without direct summands of type \( I \) (the nonabelian part). It is clear that \( B \supseteq M_{i1}, B_{n,a} = B \cap M_{n,a} \) is a maximal associative subalgebra of \( M_{n,a} \) and \( B = M_{i1} \oplus B_{n,a} \). By Proposition 2.7, \( W^*(B_{n,a}) \) is a maximal abelian von Neumann subalgebra of \( W^*(M_{n,a}) \), and hence \( W^*(B) = W^*(M_{i1}) \oplus W^*(B_{n,a}) \) is a maximal abelian von Neumann subalgebra of \( W^*(M) \), since \( W^*(M) = W^*(M_{i1}) \oplus W^*(M_{n,a}) \) [12, Lemma 2.6]. \( \square \)

**Proposition 2.9.** Let \( B_i \) be a maximal associative subalgebra of a JW-algebra \( M_i, i = 1,2 \), and suppose that \( M_i \) is universally reversible JW-algebra not isomorphic to the self adjoint part of a von Neumann algebra and without direct summands of type \( I \). Then \( JW(B_1 \otimes B_2) \) is a maximal associative JW-subalgebra of \( JW(M_1 \otimes M_2) \).

**Proof.** Note first that \( W^*(B_1) \otimes W^*(B_2) \) is a von Neumann \(*\)-subalgebra of \( W^*(M_1) \otimes W^*(M_2) \) [8, Theorem 11.2.10], and \( JW(B_1 \otimes B_2) \) is a JW-subalgebra of \( JW(M_1 \otimes M_2) \), since \( B_1 \otimes B_2 \subseteq M_1 \otimes M_2 \). By Proposition 5.2, \( W^*(B_1) \) is maximal abelian in \( W^*(M_1) \), and hence, \( W^*(B_1) \otimes W^*(B_2) \) is maximal abelian in \( W^*(M_1) \otimes W^*(M_2) \) [8, Corollary 11.2.18] and [10, Theorem 2.9]. The result is now obvious, since \( W^*(JW(B_1 \otimes B_2)) = W^*(B_1) \otimes W^*(B_2) \), and \( W^*(JW(M_1 \otimes M_2)) = W^*(M_1) \otimes W^*(M_2) \) [10, Theorem 2.9]. \( \square \)

**Proposition 2.10.** Let \( N \) be an associative JW-algebra, and let \( M \) be a universally reversible JW-algebra not isomorphic to the self adjoint part of a von Neumann algebra and without direct summands of type \( I \). If \( B \) is a maximal associative subalgebra of \( M \), then \( JW(N \otimes B) \) is a maximal associative JW-subalgebra of \( JW(N \otimes M) \).

**Proof.** Let \( M = M_{i1} \oplus M_{n,a} \) be the decomposition of \( M \) into abelian part \( M_{i1} \) and nonabelian part \( M_{n,a} \). Then \( B = M_{i1} \otimes B_{n,a} \), where \( B_{n,a} = B \cap M_{n,a} \) is obviously a maximal associative subalgebra of \( M_{n,a} \). By [10, Remark 2.14],

\[
JW(N \otimes M) = JW(N \otimes (M_{i1} \oplus M_{n,a})) = JW(A \otimes M_{i1}) \oplus JW(A \otimes M_{n,a}),
\]

\[
JW(N \otimes B) = JW(N \otimes (M_{i1} \oplus B_{n,a})) = JW(N \otimes M_{i1}) \oplus JW(N \otimes B_{n,a}).
\]

It is clear now that \( JW(N \otimes B) \) is a maximal associative JW-subalgebra of \( JW(N \otimes M) \), since \( JW(N \otimes M_{i1}) \) is obviously associative, and \( JW(N \otimes B_{n,a}) \) is maximal in \( JW(N \otimes M_{n,a}) \), by Proposition 2.9. \( \square \)
Theorem 2.11. Let $M$ and $N$ be universally reversible JW-algebras not isomorphic to the self adjoint parts of von Neumann algebras. If $A$ and $B$ are maximal associative subalgebra of $M$ and $N$, respectively, then $JW(A \boxtimes B)$ is a maximal associative JW-subalgebra of $JW(M \boxtimes N)$.

Proof. Let $M = M_{l_i} \oplus M_{n.a}, N = N_{l_i} \oplus N_{n.a}$ be the decomposition of $M, N$ into abelian parts $(M_{l_i}, N_{l_i})$, and nonabelian parts $(M_{n.a}, N_{n.a})$. Then $A = M_{l_i} \oplus A_{n.a}$ and $B = N_{l_i} \oplus B_{n.a}$, where $A_{n.a} = A \cap M_{n.a}$ and $B_{n.a} = B \cap N_{n.a}$. Therefore,

\[
JW(M \boxtimes N) = JW(M_{l_i} \boxtimes N_{l_i}) \oplus JW(M_{l_i} \boxtimes N_{n.a}) \\
\quad \oplus JW(M_{n.a} \boxtimes N_{l_i}) \oplus JW(M_{n.a} \boxtimes N_{n.a}),
\]

\[
JW(A \boxtimes B) = JW(M_{l_i} \boxtimes N_{l_i}) \oplus JW(M_{l_i} \boxtimes B_{n.a}) \\
\quad \oplus JW(A_{n.a} \boxtimes N_{l_i}) \oplus JW(A_{n.a} \boxtimes B_{n.a}),
\]

by [13, Remark 2.14]. The proof is complete, by Propositions 2.9 and 2.10.

References
