Research Article

An Application of Differential Subordination

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We apply the general theory of differential subordination to obtain certain interesting criteria for p-valent starlikeness and strong starlikeness. Some applications of these results are also discussed.

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1. Introduction

Let \( \mathcal{A}_p \) \((p \in \mathbb{N} = \{1, 2, 3, \ldots \})\) be the class of functions \( f(z) \) of the form

\[
f(z) = z^p + \sum_{m=1}^{\infty} a_{p+m}z^{p+m}
\]  

which are analytic in the open unit disk \( \Delta := \{ z : |z| < 1 \} \).

Let \( \mathcal{P} \) be the class of functions \( p(z) \) of the form

\[
p(z) = 1 + \sum_{n=1}^{\infty} p_nz^n
\]  

which are analytic in \( \Delta \). If \( p(z) \in \mathcal{P} \) satisfies \( \Re p(z) > 0 \) \((z \in \Delta)\), then we say that \( p(z) \) is a Carathéodory function.

With a view to recalling the principle of subordination between analytic functions, let the functions \( f \) and \( g \) be analytic in \( \Delta \). Then we say that the function \( f \) is subordinate to \( g \) if there exists a Schwarz function \( w(z) \), analytic in \( \Delta \) with

\[
w(0) = 0, \quad |w(z)| < 1 \quad (z \in \Delta),
\]
such that

\[ f(z) = g(\omega(z)) \quad (z \in \Delta). \]  

We denote this subordination by

\[ f \prec g, \quad f(z) \prec g(z) \quad (z \in \Delta). \]  

In particular, if the function \( g \) is univalent in \( \Delta \), the above subordination is equivalent to

\[ f(0) = g(0) \quad \text{or} \quad f(\Delta) \subset g(\Delta). \]

For \(-1 \leq b < a \leq 1\) and \(0 < \gamma \leq 1\), a function \( f \in A_p \) is said to be in the class \( S_p^\gamma(\gamma, a, b) \) if it satisfies

\[ \frac{zf'(z)}{f(z)} < p\left(\frac{1 + az}{1 + bz}\right)^\gamma. \]  

Also, we write \( S_p^\gamma(1, -1) = SS_p^\gamma(\gamma) \), the class of strongly starlike \( p \)-valent functions of order \( \gamma \) in \( \Delta \). \( S_p^\gamma(1, a, b) = S_p^\gamma(a, b) \), the class of Janowski starlike \( p \)-valent function, \( S_p^\gamma(1, -1) = S_p^\gamma \), the class of \( p \)-valent starlike function, and \( S_p^\gamma(1 - 2\gamma, 1) = S_p^\gamma(\gamma) (0 \leq \gamma < 1) \), the class of \( p \)-valent starlike function of order \( \gamma \).

For Carathéodory functions, Miller [1] obtained certain sufficient conditions applying the differential inequalities. Recently, Nunokawa et al. [2] have given some improvement of result by Miller [1]. Recently Ravichandran and Jayamala [3] studied some subordination results for Carathéodory functions. In this paper by extending the result of Ravichandran and Jayamala [3], we find sufficient conditions for the subordination \( p(z) \prec q(z) \) to hold for given \( q(z) \) and criteria for \( p \)-valent starlikeness. Our results include results obtained by Nunokawa et al. [2]. We also give some criteria for \( p \)-valently starlikeness and strong starlikeness.

To prove our result we need the following lemma due to Miller and Mocanu [4].

**Lemma 1.1** (see [4, Theorem 3.4h, page 132]). Let \( q(z) \) be analytic and univalent in the unit disk \( \Delta \) and \( \theta(\omega) \) and let \( \phi(\omega) \) be analytic in a domain \( D \) containing \( q(\Delta) \) with \( \phi(\omega) \neq 0 \) when \( \omega \in q(\Delta) \). Set

\[ Q(z) = zq'(z)\phi(q(z)), \quad h(z) = \theta(q(z)) + Q(z). \]  

Suppose that

(i) \( Q(z) \) is starlike univalent in \( \Delta \),

(ii) \( \Re\{zh'(z)/Q(z)\} = \Re\{\theta'(q(z))/\phi(q(z)) + zQ'(z)/Q(z)\} > 0 \) for \( z \in \Delta \).

If \( p(z) \) is analytic in \( \Delta \) with \( p(0) = q(0), p(\Delta) \subseteq D \), and

\[ \theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)), \]  

then \( p(z) \prec q(z) \) and \( q(z) \) is the best dominant.
2. Application of Differential Subordination

By making use of Lemma 1.1, we first prove the following theorem.

**Theorem 2.1.** Let \( 0 \neq \alpha \in \mathbb{C} \) and \( \lambda \) be a positive real number. Let \( q(z) \) be convex univalent in \( \Delta \) and \( \Re((1 - \alpha) \backslash \alpha + m(q(z))^{m-1}) > 0, m \in \mathbb{N} \setminus \{1\} \). If \( p \in \mathcal{D} \) satisfies

\[
(1 - \alpha)p(z) + \alpha(p(z))^m + \alpha\lambda p'(z) < h(z),
\]

where

\[
h(z) = (1 - \alpha)q(z) + \alpha(q(z))^m + \alpha\lambda q'(z),
\]

then

\[
p(z) < q(z),
\]

and \( q(z) \) is the best dominant of (2.1).

**Proof.** Let

\[
\theta(w) = (1 - \alpha)w + \alpha w^m, \quad \phi(w) = \alpha \lambda.
\]

Then clearly \( \theta(w) \) and \( \phi(w) \) are analytic in \( \mathbb{C} \) and \( \phi(w) \neq 0 \). Also let

\[
Q(z) = zq'(z)\phi(q(z)) = \alpha\lambda q'(z),
\]

\[
h(z) = \theta(q(z)) + Q(z)
\]

\[
= (1 - \alpha)q(z) + \alpha(q(z))^m + \alpha\lambda q'(z).
\]

Since \( q(z) \) is convex univalent, \( zq'(z) \) is starlike univalent. Therefore \( Q(z) \) is starlike univalent in \( \Delta \), and

\[
\Re\left( \frac{zQ'(z)}{Q(z)} \right) = \frac{1}{\lambda} \Re\left\{ \frac{1 - \alpha}{\alpha} + m(q(z))^{m-1} + \lambda \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0
\]

for \( z \in \Delta \).

From (2.1)–(2.6) we see that

\[
\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)) = h(z).
\]

Therefore, by applying Lemma 1.1, we conclude that \( p(z) < q(z) \) and \( q(z) \) is the best dominant of (2.1). The proof of the theorem is complete.

By taking \( \alpha \) as real and \( q(z) = ((1 + az)/(1 + bz))^\gamma \) in Theorem 2.1, we get the following corollary.
Corollary 2.2. Let $-1 < b < a \leq 1$, $m \in \mathbb{N} \setminus \{1\}$, $0 < \gamma \leq 1/(m-1)$, $\lambda > 0$ and $0 < a \leq 1$. If $p \in \mathcal{P}$ satisfies

$$
(1 - \alpha)p(z) + \alpha (p(z))^m + \alpha \lambda z p'(z) < h(z),
$$

where

$$
h(z) = (1 - \alpha) \left( \frac{1 + az}{1 + bz} \right)^\gamma + \alpha \left( \frac{1 + az}{1 + bz} \right)^m + \frac{\alpha \lambda \gamma (a - b) z}{(1 + az)^1(1 + bz)^1 + \gamma},
$$

then

$$
p(z) < \left( \frac{1 + az}{1 + bz} \right)^\gamma,
$$

and $((1 + az)/(1 + bz))^\gamma$ is the best dominant of (2.8).

Corollary 2.3. Let $-1 < b < a \leq 1$, $\lambda > 0$. If $f \in \mathcal{A}_p$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and

$$
\frac{zf'(z)}{pf(z)} \left[ 1 - \alpha + \frac{\alpha}{p \gamma} (1 - \lambda p) \frac{zf'(z)}{f(z)} + \alpha \lambda \left( 1 + \frac{zf''(z)}{f(z)} \right) \right] < h(z),
$$

where

$$
h(z) = (1 - \alpha) \left( \frac{1 + az}{1 + bz} \right)^\gamma + \alpha \left( \frac{1 + az}{1 + bz} \right)^2 + \frac{\alpha \lambda \gamma (a - b) z}{(1 + az)^1(1 + bz)^1 + \gamma'},
$$

then

$$
\frac{zf'(z)}{pf(z)} < \left( \frac{1 + az}{1 + bz} \right)^\gamma.
$$

Proof. Let $p(z) = zf'(z)/pf(z)$, then $p \in \mathcal{P}$ and (2.11) can be written as

$$
(1 - \alpha)p(z) + \alpha p^2(z) + \alpha \lambda z p'(z)
$$

$$
< (1 - \alpha) \left( \frac{1 + az}{1 + bz} \right)^\gamma + \alpha \left( \frac{1 + az}{1 + bz} \right)^2 + \frac{\alpha \lambda \gamma (a - b) z}{(1 + az)^1(1 + bz)^1 + \gamma'}.
$$

Taking $m = 2$ in Corollary 2.2 and using (2.14), we have

$$
\frac{zf'(z)}{pf(z)} < \left( \frac{1 + az}{1 + bz} \right)^\gamma.
$$
By taking \( p = \lambda = \gamma = a = 1 \) and \( b = -1 \) in Corollary 2.3, we get the following result of Padmanabhan [5].

**Corollary 2.4.** Let \( f \in \mathcal{A} \) and

\[
\frac{zf'(z)}{f(z)} + a\frac{z^2f''(z)}{f'(z)} < \frac{2\alpha(z^2 + 2z) + 1 - z^2}{(1 - z)^2} \quad (0 < \alpha \leq 1),
\]

then

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > 0.
\]

**Theorem 2.5.** Let \( \alpha, \beta, \xi, \eta \in \mathbb{C} \) and \( \eta \neq 0 \). Let \( q(z) \) be convex univalent in \( \Delta \) and satisfy

\[
\Re \left[ \frac{1}{\eta} (\beta + 2\xi q(z)) \right] > 0.
\]

If \( p \in \mathcal{P} \) satisfies

\[
\alpha + \beta p(z) + \xi p^2(z) + \eta z p'(z) < \alpha + \beta q(z) + \xi q^2(z) + \eta z q'(z) = h(z),
\]

then

\[
p(z) < q(z),
\]

and \( q(z) \) is the best dominant of (2.19)

**Proof.** By setting \( \theta(w) := \alpha + \beta w + \xi w^2 \) and \( \phi(w) := \eta \) it can be easily observed that \( \theta(w) \) and \( \phi(w) \) are analytic in \( \mathbb{C} \) and that \( \phi(w) \neq 0 \) (\( w \in \mathbb{C} \setminus \{0\} \)).

Also, by letting

\[
Q(z) = zq'(z)\phi(q(z)) = \eta z q'(z),
\]

\[
h(z) = \theta(q(z)) + Q(z)
\]

\[
= \alpha + \beta q(z) + \xi q^2(z) + \eta z q'(z),
\]

we find that \( Q(z) \) is starlike univalent in \( \Delta \) and that

\[
\Re \left( \frac{zh'(z)}{Q(z)} \right) = \Re \left[ \frac{1}{\eta} (\beta + 2\xi q(z)) + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right] > 0.
\]

The differential subordination

\[
\alpha + \beta p(z) + \xi p^2(z) + \eta z p'(z) < \alpha + \beta q(z) + \xi q^2(z) + \eta z q'(z)
\]
becomes

\[ \theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)). \]  \hspace{1cm} (2.24)

Now, the result follows as an application of Lemma 1.1.

**Theorem 2.6.** Let \( \alpha, \beta, \xi, \eta, \) and \( \delta \) be complex numbers, \( \delta \neq 0. \) Let \( 0 \neq q(z) \) be univalent in \( \Delta \) and satisfy the following conditions for \( z \in \Delta: \)

1. \( Q(z) = \delta zq'(z)/q(z) \) be starlike,
2. \( \Re\{(\beta/\delta)q(z) + (2\xi/\delta)q^2(z) - (\eta/\delta q(z)) + zQ'(z)/Q(z)\} > 0. \)

If \( p \in \mathcal{P} \) satisfies

\[ \alpha + \beta p(z) + \xi(p(z))^2 + \frac{\eta}{p(z)} + \delta zp'(z)\frac{p'(z)}{p(z)} < \alpha + \beta q(z) + \xi(q(z))^2 + \frac{\eta}{q(z)} + \delta zq'(z)\frac{q'(z)}{q(z)}, \]  \hspace{1cm} (2.25)

then

\[ p(z) < q(z), \]  \hspace{1cm} (2.26)

and \( q(z) \) is the best dominant.

**Proof.** The proof of this theorem is much akin to the proof of Theorem 2.5 and hence can be omitted. \( \square \)

**Remark 2.7.** By taking \( \alpha = \beta = 0, \xi = (\lambda/\mu) \mu > 0, \lambda > -\mu/2, \eta = 1, \) and \( q(z) = (1 + z)/(1 - z) \) in Theorem 2.5 we get the result of Nunokawa et al. [2] which was proved by a different method.

**Remark 2.8.** For the choices of \( \alpha = \beta = 0 \) in Theorem 2.5, we get the result of [3, Theorem 1, page 192] and for \( \alpha = \xi = \eta = 0 \) in Theorem 2.6 we get the result of [3, Theorem 2, page 194].

**Corollary 2.9.** Let \(-1 \leq b < a \leq 1, 0 < \gamma \leq 1 \) and \( \lambda > 0. \) If \( f \in \mathcal{A}_p \) satisfies \( f(z)f'(z) \neq 0 \) in \( 0 < |z| < 1, \) then

\[ (1 - \lambda)\frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)}\right) < p\left(1 + a\frac{z}{1 + bz}\right)^\gamma + \frac{\lambda \gamma (a-b)z}{(1 + a\frac{z}{1 + bz})}, \]  \hspace{1cm} (2.27)

implies

\[ f \in S_p^\gamma(\gamma, a, b). \]  \hspace{1cm} (2.28)

Also,

\[ 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} < \frac{\gamma (a-b)z}{(1 + a\frac{z}{1 + bz})}, \]  \hspace{1cm} (2.29)
implies

\[ f \in S_p^*(\gamma, a, b). \quad (2.30) \]

**Proof.** By taking \( \alpha = \xi = \eta = 0, \beta = p/\lambda, \delta = 1, p(z) = zf'(z)/pf(z), \) and \( q(z) = ((1 + az)/(1 + bz))^t \) in Theorem 2.6, we get the first part.

Proof of the second part follows, by setting \( \alpha = \beta = \xi = \eta = 0, \delta = 1, p(z) = zf'(z)/pf(z), \) and \( q(z) = ((1 + az)/(1 + bz))^t. \) □

For \( \alpha = \xi = 0, \beta = 1, p(z) = zf'(z)/f(z), \) and \( q(z) = (1 + az)/(1 - z), -1 < a \leq 1 \) in Theorem 2.5, we have the following result.

**Corollary 2.10.** If \( f \in A \) satisfies \( f(z) \neq 0, z \in \Delta \) and

\[ \frac{zf'(z)}{f(z)} \left[ \left(1 - \eta \frac{zf'(z)}{f(z)}\right) + \eta \left(1 + \frac{zf''(z)}{f'(z)}\right) \right] < h(z), \quad (2.31) \]

where

\[ h(z) = \frac{1 + az}{1 - z} + \eta \frac{(1 + a)z}{(1 - z)^2}, \quad (2.32) \]

then

\[ \frac{zf'(z)}{f(z)} < \frac{1 + az}{1 - z}. \quad (2.33) \]

One notes that if \( h(z) = u + iv, \) then \( h(\Delta) \) is the exterior of the parabola given by

\[ \nu^2 = -\frac{1 + a}{\eta} \left[ u - \frac{2 - 2a - \eta(1 + a)}{4}\right], \quad (2.34) \]

with its vertex as \((2 - 2a - \eta(1 + a)/4, 0)\) (see [5, 6]).

By taking \( \eta = a = 1 \) in Corollary 2.10, we obtain the following.

**Corollary 2.11.** If \( f \in A \) satisfies \( f(z) \neq 0, z \in \Delta, \) and

\[ \frac{zf'(z)}{f(z)} \left[ 2 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} \right] < \frac{1 + 2z - z^2}{(1 - z)^2}, \quad (2.35) \]

then

\[ \frac{zf'(z)}{f(z)} < \frac{1 + z}{1 - z}. \quad (2.36) \]

Region \( h(\Delta) \) has been shown shaded in Figure 1.
Letting $\alpha = \beta = 0, \xi = \eta = 1, p(z) = z f'(z)/f(z)$, and $q(z) = (1 + (1 - 2\gamma)z)/(1 - z)$ in Theorem 2.5, we get the following.

**Corollary 2.12.** If $f \in \mathcal{A}$ satisfies $f(z) \neq 0, 0 < |z| < 1$, and

\[
\frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < h(z),
\]

where

\[
h(z) = \frac{(1 - 2\gamma)^2 z^2 + 2(2 - 3\gamma)z + 1}{(1 - z)^2}
\]

for some $\gamma$ ($0 \leq \gamma < 1$), then

\[
\Re\left( \frac{zf'(z)}{f(z)} \right) > \gamma.
\]

For the univalent function $h(z)$ given by (2.38), One now finds the image $h(\Delta)$ of the unit disk $\Delta$. 

![Figure 1: $\eta = a = 1$.](image)
Let \( h = u + iv \), where \( u \) and \( v \) are real. One has

\[
\begin{align*}
  u &= -\frac{(2 - 3\gamma) + (1 + 2\gamma^2 - 2\gamma) \cos \theta}{(1 - \cos \theta)}, \\
  v &= \frac{2\gamma(1 - \gamma) \sin \theta}{1 - \cos \theta}.
\end{align*}
\] (2.40)

Elimination of \( \theta \) yields

\[

v^2 = -\frac{8\gamma^2(1 - \gamma)}{3 - 2\gamma} \left[ u - \frac{2\gamma^2 + \gamma - 1}{2} \right].
\] (2.41)

Therefore, one concludes that

\[

h(\Delta) = \left\{ w = u + iv; \quad v^2 > -\frac{8\gamma^2(1 - \gamma)}{3 - 2\gamma} \left[ u - \frac{2\gamma^2 + \gamma - 1}{2} \right] \right\},
\] (2.42)

which properly contains the half plane \( \Re w > (2\gamma^2 + \gamma - 1)/2 \).

**Corollary 2.13.** Let \(-1 \leq b < a \leq 1 \) and \( \Re \beta \geq 0 \). If \( f \in \mathcal{A}_p \) satisfies \( f'(z) \neq 0 \) in \( 0 < |z| < 1 \) and

\[

(1 - \beta) \frac{f(z)}{zf'(z)} + \frac{f(z)f''(z)}{(f'(z))^2} < h(z),
\] (2.43)

where

\[

h(z) = \frac{b(pb - \beta a)z^2 + ((2p + 1 - \beta)b - (1 + \beta)a)z + p - \beta}{p(1 + bz)^2},
\] (2.44)

then

\[

f \in S_p^*(b, a).
\] (2.45)

**Proof.** If we let \( p(z) = pf(z)/zf'(z) \), then \( p \in \mathcal{D} \) and (2.43) can be expressed as

\[

\beta p(z) + zp'(z) < \beta \left( \frac{1 + az}{1 + bz} \right) + \frac{(a - b)z}{(1 + bz)^2}.
\] (2.46)

Hence, by taking \( a = \xi = 0, \eta = 1, q(z) = (1 + az)/(1 + bz) \) and \( \Re \beta \geq 0 \) in Theorem 2.5, we have \( p(z) < (1 + az)/(1 + bz) \). So, \( f(z) \in S_p^*(b, a) \).

Setting \( p = 1 \) and \( b = -1 \) in Corollary 2.13, we get the following corollary.
Corollary 2.14. Let $-1 < a \leq 1$ and $\Re \beta \geq 0$. If $f \in \mathcal{A}$ satisfies $f'(z) \neq 0$ in $0 < |z| < 1$ and

$$
(1 - \beta) \frac{f(z)}{zf'(z)} + \frac{f(z)f''(z)}{(f'(z))^2} < h(z),
$$

where

$$
h(z) = \frac{(1 + \beta a)z^2 + ((\beta - 3) - (1 + \beta)a)z + 1 - \beta}{(1 - z)^2},
$$

then

$$
\frac{f(z)}{zf'(z)} < \frac{1 + az}{1 - z}.
$$

Remark 2.15. For the function $h(z)$ given by (2.48), we have

$$
h(\Delta) = \left\{ w = u + iv; v^2 > a_0[u - b_0] \right\},
$$

which properly contains the half plane $\Re w > b_0$, where

$$
a_0 = (1 + a)\beta^2,
$$

$$
b_0 = \frac{5 + a + 2\beta(a - 1)}{4}.
$$

By putting $p = a = \beta = 1$ and $b = -1$ in Corollary 2.13, we get the following result of Tuneski [7].

Corollary 2.16. If $f(z) \in \mathcal{A}$ and

$$
\frac{f(z)f''(z)}{(f'(z))^2} \prec \frac{2z(z - 2)}{(1 - z)^2},
$$

then

$$
\Re \left( \frac{f(z)}{zf'(z)} \right) > 0.
$$

Remark 2.17. By putting $0 = b < a \leq 1, p = 1$, and $\beta = 0$ in Corollary 2.13, we get the result obtained by Singh [8], which refines the result of Silverman [9].
Corollary 2.18. Let $0 \neq \eta$ and $q(z)$ be convex univalent in $\Delta$ with $q(0) = 1$ and satisfy (2.18). Let $f \in \mathcal{A}$ and 
\[
\psi(z) := \alpha + \frac{\beta}{p} \left( \frac{f(z)}{z^p} \right)^{\mu} + \frac{\xi}{p^2} \left( \frac{f(z)}{z^p} \right)^{2\mu} + \eta \mu \left( \frac{f(z)}{z^p} \right)^{\mu} \left[ z f'(z) \frac{p}{p f(z)} - 1 \right].
\] (2.54)

If 
\[
\psi(z) < \alpha + \beta q(z) + \xi q^2(z) + \eta z q' (z),
\] (2.55)

then 
\[
\frac{1}{p} \left( \frac{f(z)}{z^p} \right)^{\mu} < q(z),
\] (2.56)

and $q(z)$ is the best dominant.

Proof. By taking $p(z) = (1/p)(f(z)/z^p)^{\mu}$ in Theorem 2.5, we have the above corollary. \qed

Corollary 2.19. Let $0 \neq \lambda \in \mathbb{C}$ and $q(z)$ be convex univalent in $\Delta$ with $q(0) = 1$ and satisfy 
\[
\Re \left( \frac{\mu}{\lambda} \right) > 0.
\] (2.57)

(i) If $f \in \mathcal{A}$ satisfies 
\[
(1 - \lambda) \left( \frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} < q(z) + \frac{1}{\mu} z q' (z),
\] (2.58)

then 
\[
\left( \frac{f(z)}{z} \right)^{\mu} < q(z),
\] (2.59)

(ii) If $f \in \mathcal{A}$ satisfies 
\[
f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} - \left( \frac{f(z)}{z} \right)^{\mu} < \frac{1}{\mu} z q' (z),
\] (2.60)

then 
\[
\left( \frac{f(z)}{z} \right)^{\mu} < q(z),
\] (2.61)

and $q(z)$ is the best dominant.
Proof. Proof of the first part follows from Corollary 2.18, by taking $\beta = p = 1, \alpha = \xi = 0, \eta = \frac{\lambda}{\mu}$.

The proof of the second part follows from Corollary 2.18, by taking $\alpha = \beta = \xi = 0, p = 1$, and $\eta = 1/\mu$.

By taking $\lambda = \mu = n$ where $n$ is a positive integer and $q(z) = A + (1 - A)(-1 - (2/z) \log(1 - z))$ in the first part of Corollary 2.19, we get the following result of Ponnusamy [10].

**Corollary 2.20.** Let $f \in \mathcal{A}$, then for a positive integer $n$, one has that

$$\Re \left\{ (1 - n) \left( \frac{f(z)}{z} \right)^n + nf'(z) \left( \frac{f(z)}{z} \right)^{n-1} \right\} > \beta$$

implies

$$\left( \frac{f(z)}{z} \right)^n < A + (1 - A) \left( -1 - \frac{2}{z} \log(1 - z) \right),$$

and $A + (1 - A)[-1 - (2/z) \log(1 - z)]$ is the best dominant.

**Remark 2.21.** By taking $\mu = 1$ and $q(z) = 1 + (A/(1 + \delta))z$ in Corollary 2.19 and $\mu = \lambda = 1$ and $q(z) = A/B + (1 - A/B)(\log(1 + Bz)/Bz)$ we get the result of Ponnusamy and Juneja [11].

By taking $\beta = \xi = \eta = 0, \alpha = p = 1, \delta = 1/\mu, p(z) = (1/p)(f(z)/zp)^\mu$, and $q(z) = e^{\mu Az}$ in Theorem 2.5, we get the following result obtained by Owa and Obradović [12].

**Corollary 2.22.** Let $f \in \mathcal{A}$ and

$$zf'(z) < 1 + Az,$$

then

$$\left( \frac{f(z)}{z} \right)^\mu < e^{\mu Az},$$

and $e^{\mu Az}$ is the best dominant.

We remark here that $q(z) = e^{\mu Az}$ is univalent if and only if $|\mu A| < \pi$.

**Remark 2.23.** For a special case when $p(z) = (1/p)(f(z)/zp)^\mu, q(z) = 1/(1 - z)^2b$ where $b \in \mathbb{C} \setminus \{0\}$ and $\beta = \xi = \eta = 0, \alpha = \mu = p = 1$, and $\delta = 1/b$ in Theorem 2.6, we have the result obtained by Srivastava and Lashin [13].

**Corollary 2.24.** If $f \in \mathcal{A}$ satisfies

$$(1 + \lambda) \left( \frac{z}{f(z)} \right)^\mu - \lambda f'(z) \left( \frac{z}{f(z)} \right)^{\mu+1} < q(z) + \frac{1}{\mu} zq'(z),$$

(2.66)
then
\[
\left( \frac{z}{f(z)} \right)^\mu \prec q(z),
\]
and \( q \) is the best dominant.

Proof. By taking \( p(z) = (1/p)(z^n/f(z))^\mu \) and \( \alpha = \xi = 0, \beta = p = 1 \) and \( \eta = \lambda/\mu \) in Theorem 2.5, we get the previous corollary. \( \square \)

References
