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Research Article

A Quantization Procedure of Fields Based on Geometric Langlands Correspondence

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We expose a new procedure of quantization of fields, based on the Geometric Langlands Correspondence. Starting from fields in the target space, we first reduce them to the case of fields on one-complex-variable target space, at the same time increasing the possible symmetry group ${}^LG$. Use the sigma model and momentum maps, we reduce the problem to a problem of quantization of trivial vector bundles with connection over the space dual to the Lie algebra of the symmetry group ${}^LG$. After that we quantize the vector bundles with connection over the coadjoint orbits of the symmetry group ${}^LG$. Use the electric-magnetic duality to pass to the Langlands dual Lie group G. Therefore, we have some affine Kac-Moody loop algebra of meromorphic functions with values in Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Use the construction of Fock space reprsentations to have representations of such affine loop algebra. And finally, we have the automorphic representations of the corresponding Langlands-dual Lie groups G.

1. Introduction

A quantization procedure can be described as a covariant functor from the category of classical Hamiltonian systems to the category of quantum systems:

$$\begin{gathered}\{\text{Classical systems}\} \longrightarrow \{\text{Quantum systems}\}\\ \text{e.g.,}\\ \text{classical observables} \longmapsto \text{quantum observables}\\ \text{i.e.,}\\ \text{real (complex) valued functions} \longmapsto \text{autoadjoint (normal) operators}\end{gathered} \tag{1.1}$$

The main rule of a quantization procedure is that when the Planck constant $\hbar$ approaches 0 the system does become the starting classical system, that is, the *classical limit*.

There are some well-known rules of quantization, namely, *Weyl quantization*, related with the canonical representation of the commutation relations; *pseudodifferential operators quantization*, regarding the functions of position q^i and momentum variables p_i as symbols of some pseudodifferential operators; *geometric quantization*, thinking of the symplectic gradients of functions as vector fields acting on sections following some connection, and so forth.

Let us concentrate on the geometric quantization and explain in some more detail.

1.1. Geometric Quantization

In the general model, a Hamiltonian system is modeled as some symplectic manifold (M, ω) with a flat action ∇ of some Lie group of symmetry G; see [1, 2] for more details.

If f is a function on the symplectic manifold, its symplectic gradient is the vector field ξ_f such that

$$\imath(\xi_f)\omega + df \equiv 0. \tag{1.2}$$

Conversely, every element X of the Lie algebra of symmetry $\mathfrak{g} = \operatorname{Lie} G$ provides a vector field $\xi_X(m) = d/dt|_{t=0} \exp(tX)m$. The condition is that the potential f always exists

$$\xi_X \longmapsto f_X, \tag{1.3}$$

and that it can be lifted to a homomorphism of corresponding Lie algebras

$$\begin{tikzcd}
 & & & 0 & \\
 & & & \mathrm{H}^1_{DR}(M, \mathbf{R}) \arrow[u] & \\
 & & \mathfrak{g} \arrow[r] \arrow[d] & \mathrm{Ham}(M, \omega) \arrow[u] & \\
0 \arrow[r] & \mathbf{R} \arrow[r] & C^\infty(M) \arrow[r] & \mathrm{Ham}_0(M, \omega) \arrow[u] \arrow[r] & 0 \\
 & & & 0 \arrow[u] &
\end{tikzcd} \tag{1.4}$$

On an arbitrary symplectic manifold, there exist the so-called Darboux coordinate systems $(q^1, \ldots, q^n; p_1, \ldots, p_n)$ at some neighborhood of every point. A global system of such separations of variables is the so-called *polarization*. In more general context, it is given by some G-invariant integrable tangent distribution of the complexified tangent bundle. If in each coordinate chart one uses the pseudodifferential quantization through the oscillating

integral Fourier and then glue the results between the chart, one meets some obstacle, which is the so-called *Arnold-Maslov index* classes.

It is given also by a maximal commutative subalgebra of functions, with respect to the Poisson brackets

$$\{f, g\} = \omega(\xi_f, \xi_g), \tag{1.5}$$

and one defines a momentum map from and arbitrary heomegeneous strictly Hamiltonian symplectic manifolds to the coadjoint orbits of the universal covering of Lie group G or its central extension $\widetilde{G}$

$$0 \longrightarrow \mathbf{R} \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 0 \tag{1.6}$$

in the dual space of the Lie algebra $\widetilde{\mathfrak{g}}$.

The sheaf of sections of a vector bundle vanishing along the direction $p_1, \ldots, p_n$ provides the so-called *quantum vector bundle* and the space of square integrable sections depending only on the space direction $q^1, \ldots, q^n$ is the *Hilbert space of quantum states*.

Geometric Quantization of Hamiltonian systems is given by the rule of assigning to each real- (or complex-) valued function an autoadjoint (or normal, resp.) operator

$$f \longmapsto \widehat{f} = \frac{\hbar}{i}\nabla_{\xi_f} + f, \tag{1.7}$$

such that

$$\begin{gathered} \left[\widehat{f}, \widehat{g}\right] = \frac{i}{\hbar}\widehat{\{f, g\}}, \\ \widehat{1} = \mathrm{Id}_{\mathscr{H}}. \end{gathered} \tag{1.8}$$

Application to group representations: $X \in \mathfrak{g} := \operatorname{Lie} G$ can be considered as a function on an arbitrary coadjoint G-orbit Ω in the vector space $\mathfrak{g}^*$ dual to $\mathfrak{g}$.

1.2. Fields Theories

The general conception of (physical) fields is physical systems (movements, forces, interactions, etc.) located in some parametrized region of space-time, for example, the null-dimensional fields are the same as particles, and the one-dimensional fields are the fields in the quantum mechanics. We refer the reader to [3] for discussion of the cases of one-dimensional and $(1+1)$-dimensional fields.

In field theory one defines the *partition function* as

$$\langle f(X) \rangle = \frac{\int f(X) e^{-S[X]} DX}{\int e^{-S[X]} DX}, \tag{1.9}$$

where DX is the Wiener measure of the space of paths from a point to another one, where

$$S[X] = \int_{t_1}^{t_2} L(X, \dot{X})\, d\tau \tag{1.10}$$

is the action which is the integral of the Lagrangian. The *general field equation* is obtained from the variation principle.

The sigma model for the general field theory is started with reduction to reduce the 4-dimensional Minkowski space M_4 to the product $M_4 = C \times \Sigma$ of a possibly noncompact Riemann surface C and a compact Riemann surface Σ. The Klein reduction requires to compactify C and have some effective theory on Σ.

Therefore one needs to consider the sigma-model on Σ with target space $\mathcal{M}_H(G)$ or $\mathcal{M}_H({}^LG)$ the moduli space of semistable Higgs bundles $(\mathcal{E}, \Phi)$ on C, that is, the holomorphic vector bundles $\mathbf{E} \twoheadrightarrow C$ over the hyperKähler manifold C, with self-dual connection ∇. They can be obtained from the principal bundle $G_{\mathbf{C}} \twoheadrightarrow C$, a finite-dimensional $G_{\mathbf{C}}$-module $\mathbf{E}$, and $\Phi \in H^0(C, \mathbf{E} \times_G \mathfrak{g})$. The operator of taking partial trace Tr_R related with the representation R give the space B of values of partial traces and has a pair of fibrations

$$\begin{tikzcd} \mathcal{M}_H({}^LG) \arrow[rr, leftrightarrow, "\text{Mirror Symmetry}"] \arrow[dr] & & \mathcal{M}_H(G) \arrow[dl] \\ & \text{B vector space} & \end{tikzcd} \tag{1.11}$$

The fibers of these two fibrations are pairwise-dual tori. There are 3 complex structures I, J, K on the corresponding spaces. The moduli space $\mathcal{M}_H({}^LG)$ of semistable ${}^LG$-bundles with holomorphic connection $\mathcal{E} = (\mathbf{E}, \nabla)$ on C endowed with the complex structure J is denoted by $Y({}^LG)$. The $\mathcal{B}$-branes on C are the same as these objects $\mathcal{E}$.

The manifold $\mathcal{M}_H(G)$ is endowed with the complex structure K and become a symplectic manifold with respect to the symplectic structure ω_K.

By the mirror symmetry transformation, the $\mathcal{B}$-branes become the so called $\mathcal{A}$-branes, those are the Lagrangian submanifolds on $\mathcal{M}_H(G)$.

1.3. Quantization of Fields

In the general scheme there are two models of D-branes: the $\mathcal{A}$ model with 't Hooft line operators and $\mathcal{B}$ model with Wilson loop operators, in one side and the $\mathfrak{D}$-modules on the stack Bun_G of vector G-bundles on X with Hecke operators, on another side. With reduction $X = C \times \Sigma$ the moduli stack Bun_G is reduced to the moduli stack of G-bundles on C with connection ∇, the curvature of which satisfies the Bogomolnyi system of equations

$$\begin{aligned} F_A - \Phi \wedge \Phi &= 0, \\ D_A \Phi - D_A \star \Phi &= 0. \end{aligned} \tag{1.12}$$

This system means that the curvature $F_{\tilde{A}} = 0$, where $\tilde{A} := A + i\Phi$. We have the general picture as

$$\begin{tikzcd}
\begin{array}{c}\text{'t Hooft opers} \circlearrowleft \{\mathcal{A}\text{-branes}\} \\ \text{on } Y_H({}^L G)\end{array} \arrow[rr, leftrightarrow, "\text{Mirror Sym.}"'] \arrow[dr, "\begin{array}{c}\text{Fukaya} \\ \text{Tsygan deformation}\end{array}"'] & & \begin{array}{c}\{\mathcal{B}\text{-branes}\} \circlearrowright \text{Wilson opers} \\ \text{on } \mathcal{M}_H(G)\end{array} \arrow[dl, "\text{Geometric Langlands}"] \\
& \begin{array}{c}\text{Hecke opers} \circlearrowleft \{\mathfrak{D}\text{-modules}\} \\ \text{on } \mathrm{Bun}_G\end{array} &
\end{tikzcd} \tag{1.13}$$

Our tash is to realize the flesh from $\mathcal{A}$-branes to $\mathfrak{D}$-modules which is equivalent to the Geometric Langlands Correspondence, through the mirror symmetry.

Going from the $\mathcal{A}$ model of D branes to Hecke eigensheaves of $\mathfrak{D}$ modules can be considered as *a quantization procedure* of fields, using the Fukaya category or mutidimensional Fedosov deformation quantization, or the B. Tsygan deformation quantization. The most deficulties are related with the complicated category or analytic transformation in Tsygan approach. From the $\mathcal{B}$ model to $\mathfrak{D}$ module Hecke eigensheaves can be considered as the *second quantization procedure* of fields, related much more with algebraic geometry. Our approach is related with ideas of geometric quantization.

The *new quantization procedure* we proposed consists of the following steps.

(i) Starting from a connection ${}^L\nabla$ associated with the ${}^L G$-bundle, use the *Electric-Magnetic GNO Duality* to obtain bundles with connection ∇ for the dual groups G.

(ii) Use the *Kaluza-Klein Reduction* to reduce the model to the case over complex curve C extending the symmetry group G.

(iii) From a connection ∇ construct the correspoding representation $\sigma : \pi_1(C) \to G$.

(iv) Construct the corresponding *Momentum Maps.*

(v) Use the *Orbit Method* to obtain the representations of Lie group G.

(vi) Use *ADHM construction* and the *Hitchin Fibration Construction* to have some holomorphic bundle on $\mathbf{CP}^3$.

(vii) Use the *positive energy representations of Virasoro algebras* to obtain representations of *loop algebras* (Fock space construction).

Together compose all the steps, we have the same *automorphic representations* those appeared in GLC. As *the main result* of this paper we have the following.

Theorem 1.1 (quantization procedure for fields). *The obtained automorphic representations are exactly the automorphic representations of G from the Geometric Langlands Correspondence.*

In our method, beside the other things, the new idea involved the orbit method to provide a quantization procedure. The rest of this paper is devoted to prove this in exposing the corresponding theories in a suitable form.

1.4. Structure of the Paper

We describe in more detail the conception of quantization in the case of particle physiscs in Section 1. In Section 2 we discuss the electric-magnetic duality. In Section 3, we start this job by considering the embedding of the conplexified Minkowski space $M_{\mathbf{C}}$ into the twistor space $\mathbf{T} = \mathbf{C}^4$. Section 4 is devoted to the construction of representations starting by reduction and finished by the final construction of representations in Fock spaces. Section 5 is to show the corresponding construction by the Geometric Langlands Correspondence.

2. Electric-Magnetic GNO Duality

Let us discuss first about the Langlands duality or electric-magnetic GNO duality.

Theorem 2.1. *Let ${}^L\nabla$ be a ${}^LG$-connection of the target space X. Then there exists a unique dual connection on the Langlands dual G-bundle on the same base X.*

Proof. This theorem is a direct consequence of the electric-magnetic dualty. □

3. Kaluza-Klein Model

Following physical ideas, the only-nontrival quantum field theories that are believed exist have dimension $d \leq 6$ and the most standard ones have $d \leq 4$. We can pass to different quantum field theories from each other by the operation of so called *Kaluza-Klein Reduction*. It means that we can consider the case when the target space M is decomposed into a Cartesian product

$$M = N \times K, \tag{3.1}$$

where the action may be very large for the field that are not constant over K and therefore the correlation functions are localized along the fields that are constant along K. The mirror symmetry theory says that one needs only to concentrate in the case of dimension $1 + 1$ or 2.

Following Kapustin and Witten [4] we reduce the theory to the case of complex curve C.

Theorem 3.1. *The connection ∇ on $X = C$ is uniquely defined by a representation of the fundamental groups $\pi_1(C) \to G$.*

Proof. Over the universal covering $\widetilde{C}$, there is a unique trivial connection. Passing on the manifold C we have some fixed representation of the fundamential group. □

4. General Momentum Maps

4.1. Momentum Maps

Theorem 4.1. *There is local diffeomorphisms mapping the G-orbits in the principal bundle total space P and the coadjoint G-orbits of G in the space* $\mathfrak{g}^*$ *dual to the Lie algebra* $\mathfrak{g} = \mathrm{Lie}(G)$ *of G.*

Proof. One looks at each point of the manifold as some functional on the space of functions. This dual gives us the necessary momentum map; see [1] or [2] for more detail. □

The general scheme of σ model consists of the following.

(i) Choose a Langangian submanifold M.

(ii) Choose the ground state space $\mathscr{H}_{(0)} = H_{DR}(M)$.

(iii) Identify the vertex algebra.

(iv) Define the Verma modules over the vertex algebra.

(v) The resulting modules can be considered as some induced modules.

4.2. Polarization and Ground States

Theorem 4.2. *Suppose that* Ω *is an coadjoint orbit of Lie group of symmetry G in the dual space of its Lie algebra,* $(\mathfrak{p}, F, H, M, \rho)$ *is a polarization at* $F \in \Omega$*, then the Lie algebra* $\mathfrak{g}$ *is acting on the de Rham cohomology* $\mathscr{H}^*_{DR}(\Gamma(\mathcal{E}_{\rho,F}(V)))$ *with coefficients in the sheaf of partially holomorphic and partially invariant sections of the vector bundle* $\mathcal{E}_{\rho,F}(V) = G\times_{M,\rho}V$ *as differential operators with coefficients in the Lie algebra.*

Proof. Following the Orbit method theory [1, 2] we have a representation of the Lie algebra $\mathfrak{g}$ by the differential operators in the space of partially invariant and partially holomorphic sections of the induced vector bundle. Therefore by the universal property of the enveloping algebra $U(\mathfrak{g})$ we have a corresponding homomorphism of associative algebras

$$U(\mathfrak{g}) \longrightarrow \mathrm{PSDO}(\mathcal{E}_{\rho,F}(V)). \tag{4.1}$$

□

4.3. Fock Space Construction

Theorem 4.3. *The Fock space construction gives a realization of the weight modules of the Virassoro algebras on the Fock space as subspaces of the tensor product of of standard action obtained from the sigma models.*

Proof. Taking polarizations of the coadjoint orbits of the Lie group G one obtains the natural action of the Lie agebra $\mathfrak{g}$ on the representation space obtained from the orbit method. Taking the tensor product of these action, one have the corresponding Verma modules. □

The rest of the paper is devoted to the corresponding Kapustin-Witten theory of Geometric Langlands Correspondence for the brane model.

5. Branes and GLC

The main ideas of the Geometric Langlands Correspondence is now summerized and compaired with our construction in order to show that the same automorphic representations are obtained by the both methods. The *Geometric Langlands Correspondence* [4] can be formulated as follows.

(i) One begins with the Langlands dual group ${}^LG$ and a semistable homomorphism $\theta : \pi_1(C) \to {}^LG_{\mathbf{C}}$. The space of such homomorphisms is the Hitchin moduli space $\mathcal{M}_H({}^LG, C)$.

(ii) If θ is irreducible the corresponding point $x(\theta) \in \mathcal{M}_H({}^LG, C)$ is a smooth point and a zerobrane $\mathcal{B}_{x(\theta)}$ supported at $x(\theta)$ is an *electric eigenbrane* in the sigma model of the target space $\mathcal{M}_H({}^LG, C)$.

(iii) Applying the S duality to this electric eigenbrane will give a *magnetic eigenbrane* in the sigma-model of the target space $\mathcal{M}_H(G, C)$, whose support is a fiber of the Hitchin fibration, endowed with a Chan-Paton bundle of rank 1.

(iv) *The main claim of the Langlands correspondence* is that a homomorphism $\theta : \pi_1(C) \to {}^LG_{\mathbf{C}}$ is associated in a natural way to a sheaf on $\mathcal{M}_H(G, C)$, that is, a Hecke eigensheaf and also a holonomic $\mathfrak{D}$-module.

What follows is an entry into details.

5.1. Reduction to a Theory on Curve

Consider the four-manifold $M = \Sigma \times C$, where C is a compact Riemann surface of genus greater than one, Σ is either a complete but noncompact two-manifold such as $\mathbf{R}^2$, or a second compact Riemann surface.

To find an effective physics on Σ, we must find the configuration on M that minimize or nearly minimize the action in Euclidean signature or the energy in the Lorentz signature. In either case the four-dimensional twisted $\mathcal{N} = 4$ supersymmetric gauge theory reduces on Σ to a sigma-model of maps $\Phi : \Sigma \to \mathcal{M}_H(G, C)$, where $\mathcal{M}_H(G, C)$ is the moduli space of the solutions on C of Hitchin's equation with gauge group.

The minimum is obtained if

$$\mathcal{F} = D^*\Phi = 0. \tag{5.1}$$

It is equivalent to the following system:

$$\begin{gathered} F - \Phi \wedge \Phi = 0, \\ D\Phi = D^*\Phi = 0, \\ \dim \mathcal{M}_H(G, C) = (2g - 2)\dim G, \end{gathered} \tag{5.2}$$

where g is the genus of the surface C.

5.2. 't Hooft Operators and Operator Product Expansion

5.2.1. Wilson Loop and Line Operators

Let $E \to M$ be the G-bundle, associated with a representation R. Let A be a connection with curvature F, Φ a scalar field with values in the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$, $\mathcal{A} = A + i\Phi$, and $\overline{\mathcal{A}} = A - i\Phi$, S an *oriented* loop. The *Wilson operator* is defined as the trace in the representation R of the holonomy

$$W_0(R,S) = \operatorname{Tr}_R \operatorname{Hol}_S(A), \tag{5.3}$$

that is,

$$\begin{aligned} W_0(R,S) &= \operatorname{Tr}_R P \exp\left(-\oint_S \mathcal{A}\right) := \operatorname{Tr}_R P \exp\left(-\oint_S (A + i\Phi)\right), \\ W_0(R,S) &= \operatorname{Tr}_R P \exp\left(-\oint_S \overline{\mathcal{A}}\right) := \operatorname{Tr}_R P \exp\left(-\oint_S (A - i\Phi)\right). \end{aligned} \tag{5.4}$$

If S is a line with endpoints p and q at infinity, we can define $W(R,S)$ as a matrix of parallel transport (of the connection $\mathcal{A}$ or $\overline{\mathcal{A}}$, from the fiber at the point p, taken in the representation R, to the point q.

The dual of a Wilson operator for G and R is a *'t Hooft operator* $T({}^L R, S)$ for the dual group ${}^L G$ and the corresponding representation ${}^L R$.

To define a Wilson loop operator associated with a loop $S \subset M$, S must be oriented. The 't Hooft operator is labelled by the representations ${}^L R$ of the L-group ${}^L G$ and instead requires an orientation of the normal bundle to S. A small neighborhood of S can be identified with $S \times \mathbf{R}^3$. Once M is oriented we can ask for the orientation ds of S and the orientation $\mathbf{e}_3$ are compatible in the sense $\mathbf{e}_4 = ds \wedge \mathbf{e}_3$ along S

$$\mathbf{e}_3 = ds \wedge dr \wedge d\mathrm{Vol} = -dr \wedge ds \wedge d\mathrm{Vol}. \tag{5.5}$$

The line operators that preserve the topological symmetry at rational values of Ψ are called *mixed Wilson-'t Hooft operators*.

Combined Wilson-'t Hooft operators: Abelian Case $G = U(1)$

To the action of the gauge field, add a term

$$I_\theta = i\frac{\Psi}{4\pi}\int_M \operatorname{Tr} F \wedge F, \tag{5.6}$$

$A = A_0 + \widehat{A}$, where A_0 is gauge invariant that has singularity at S, but $\widehat{A}$ is smooth near S,

$$\widehat{I}_\theta = i\frac{im\Psi}{2\pi}\int_M \operatorname{Tr} F \wedge \widehat{F} \tag{5.7}$$

with δ_T being the generator of the topological symmetry $\delta_T A_0 = 0$, because A_0 is gauge invariant $\delta_T \widehat{A} = \delta_T A$, $\delta_T \widehat{F} = d(\delta_T \widehat{A})$

$$\begin{gathered}
\delta_T \widehat{I}_\theta = i\frac{im\Psi}{2\pi}\int_M d\left(F_0 \wedge \delta_T \widehat{A}\right) = \frac{im\Psi}{2\pi}\int_{\partial V_\varepsilon} F_0 \wedge \delta_T \widehat{A}, \\
\delta I_\theta = m\Psi \int_S \delta_T \widehat{A}, \\
\delta_T\left(\exp(-I_\theta)\right), \\
\delta_T\left(\exp\left(m\Psi\int_S \widehat{A}\right)\exp(-I_\theta)\right) = 0.
\end{gathered} \tag{5.8}$$

This means that we can restore the topological symmetry if we include a Wilson operator $\exp(m\Psi\int_S \widehat{A})$ as an additional factor in the path integral. The expression $\exp(-n\Psi\int_S \widehat{A})$ is gauge invariant if and only if $n = m\Psi$ an integer; that is, Ψ must be rational number $-n/m$ Rational transformation

$$(m,n) \longmapsto (m,n)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{5.9}$$

Combined Wilson-'t Hooft Operators: Nonabelian Case

Let $\rho : U(1) \to G$ be a homomorphism, $A = \rho(A_0) + \widehat{A}$, where A_0 is the singular $U(1)$ gauge field with Dirac singularity along the curve S. The required conditions are

$$\begin{gathered}
\left[\rho(A_0), \widehat{A}\right] = 0, \\
\delta_T I_\theta = \Psi \int_S \mathrm{Tr}\rho(1)\delta_T \widehat{A}, \\
\delta_T\left(\exp\left(\Psi\int_S \mathrm{Tr}\rho(1)\widehat{A}\right)\exp(-I_\theta)\right) = 0.
\end{gathered} \tag{5.10}$$

5.2.2. 't Hooft Operators and Hecke Operators

The main result is to show that the 't Hooft operators correspond to the Hecke operators of the Geometric Langlands Correspondence. More precisely, the line operators provide an algebra isomorphic to the affine Hecke algebra.

Consider $M = \mathbf{R} \times W$ for some 3-manifold W. The 't Hooft operators are supported on 1-manifold of the form $\mathbf{R} \times p$, $p \in W$, *two-dimensional A-model with target* X: a supersymmetric

classical field $\Phi : \Sigma \to X$; the first approximation: $\Sigma = \mathbf{R} \times I$, I being an interval; the space of supersymmetric states: the cohomology of the space of constant maps to X that obey the boundary conditions.

Reduction to Two Dimensions

We have $M = \mathbf{R} \times W$, $W = I \times C$, with C being a Riemann surface, I an interval, at whose ends one takes the boundary conditions. Then the BPS equations are

$$\begin{gathered}(F - \Phi \wedge \Phi + tD\Phi)^{+} = 0,\\ \left(F - \Phi \wedge \Phi - t^{-1}D\Phi\right)^{-} = 0,\\ D^{*}\Phi = 0.\end{gathered} \tag{5.11}$$

In the term of complex connection $\mathcal{A} = A + i\Phi$, curvarture $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$:

$$\mathcal{F} + i \star \overline{\mathcal{F}} = 0. \tag{5.12}$$

5.2.3. 't Hooft Operators and Eigenbranes

The Case of Group $G = U(1)$

A 't Hooft operator $T(m; p_0)$ at the point p_0 is classified by an integer m. The operator $T(m)$ is defined by saying that near the point $p = p_0 \times y_0$, the curvature has the singular behaviour

$$F \sim \star d\left(\frac{im}{2}\frac{1}{|\mathbf{x} - p|}\right), \tag{5.13}$$

where $|\mathbf{x} - p|$ is the distance from the point p to the nearby point $\mathbf{x} \in \mathbf{R} \times C$. This implies that if S is a small enclosing the point p, then

$$\int_S c_1(\mathcal{L}) = m, \tag{5.14}$$

and the 't Hooft operator is acting by twisting with $\mathcal{O}(p_0)^m$ and

$$T(m, p_0) = T(1; p_0)^m. \tag{5.15}$$

Nonabelian Case of $SU(N)$

The singular part of the curvature near p is diagonal

$$F \sim \star d\left(\frac{i}{2}\frac{1}{|\mathbf{x}-p|}\right)\begin{pmatrix} m_1 & & & & \\ & m_2 & & & \\ & & m_3 & & \\ & & & \ddots & \\ & & & & m_N \end{pmatrix}. \tag{5.16}$$

Near p, the Bogomolny equation reduces to equation in some maximal torus $\mathbf{T} = \mathrm{U}(1)^N$ of $\mathrm{U}(N)$. The corresponding vector bundle $\mathcal{L}_y$ near $p_0 \times y \in C_y$ splits up to a sum $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_N$ of the line bundles $\mathcal{L}_i$. The effect of the 't Hooft operator on $\mathcal{L}_i$ is

$$\mathcal{L}_i \longmapsto \mathcal{L}_i \otimes \mathcal{O}(p_0)^{m_i}. \tag{5.17}$$

Therefore we have some action of the 't Hooft operators $T({}^L w)$ what is the same as the action of the Hecke operators on bunldes for $\mathrm{SU}(N)$. This proves that the eigensheaves of these 't Hooft operators are the same as Hecke eigensheaves. Remark that $U(N)_{\mathbf{C}} = \mathrm{GL}(N, \mathbf{C})$.

5.3. The Extended Bogomolny Equation

We consider only the supersymmetric time-independent and time-reversal invariant. Then the BPS equations are

$$\begin{gathered} (F - \Phi \wedge \Phi + tD\Phi)^+ = 0, \\ \left(F - \Phi \wedge \Phi - t^{-1}D\Phi\right)^- = 0, \\ D^*\Phi = 0, \end{gathered} \tag{5.18}$$

reducing to the ordinary Bogomolny equations, and the 't Hooft operators reduce to the usual geometric Hecke operators.

On the four-manifold $M = \mathbf{R} \times W$ we write the Higgs field Φ as $\Phi_0 dx^0 + \pi^*\widetilde{\Phi}$, where $\pi : M \to W$ is the projection.

The gauge fields $A = A_0 dx^0 + \widetilde{A}$, where $\widetilde{A}$ is a 3-dimensional connection with curvature $\widetilde{F}$.

The time-independent BPS equation for $t = 1$ is

$$\begin{gathered} \widetilde{F} - \widetilde{\Phi} \wedge \widetilde{\Phi} = \star\left(D\Phi_0 - \left[A_0, \widetilde{\Phi}\right]\right), \\ *D\widetilde{\Phi} = \left[\Phi_0, \widetilde{\Phi}\right] + DA_0, \\ D^*\widetilde{\Phi} + [A_0, \Phi_0] = 0, \end{gathered} \tag{5.19}$$

where D is the exterior derivative, the Hodge operator $\star$, and the operator $D^* = \star D \star$.

Because of the time independence one deduces that $\Phi_1 = A_0 = 0$. Choose a local coordinated $z = x^2 + ix^3$. $y = x^1$ and $\widetilde{A}_y = 0$. Because the metric on W is $ds^2 = h|dz|^2 + dy^2$, then the extended Bogomolny equations are

$$\begin{aligned} D_{\overline{z}}\Phi_z &= 0, \\ D_y A_{\overline{z}} &= -iD_{\overline{z}}\Phi_0, \\ D_y\Phi_z &= -i[\Phi_z, \Phi_0], \\ F_{z\overline{z}} - [\Phi_z, \Phi_{\overline{z}}] &= \frac{ih}{2}\partial_z\Phi_0. \end{aligned} \tag{5.20}$$

The first equation means that $\varphi = \Phi_z dz$ reatricted on $C_y = \{y\} \times C$ is a holomorphic section of $\mathrm{End}(E) \otimes K_C$, E is the holommorphic bundle over C_y defined by the $\overline{\partial}$ operator

$$\overline{D} = d\overline{z}(\partial_{\overline{z}} + A_{\overline{z}}). \tag{5.21}$$

The pair (E, φ) is a *Higgs bundle* or *Hecke modification* for any y.

6. Proof of the Main Theorem

First we remark that following the previous results, under mirror symmetry, the category of $\mathcal{A}$-branes invariant under the 't Hooft operators is equivalent to the corresponding category of $\mathcal{B}$-branes invariant under the Wilson loop operators. And the both categories are equivalent to the category of $\mathfrak{D}$-modules invariant under the Hecke operators. More precisely, the second category is equivalent to the third category under the Geometric Langlands correspondence and the first to the third under the Fukaya category.

Next, it is known that the Fukaya category has the analytic version as the Batalin-Vilkovski and B. Tsygan quantization following the deformation quantization formula

$$\begin{aligned} \widehat{f}g &= \exp\left(\frac{\hbar}{\sqrt{-1}}\omega(\partial_y, \partial_z)\right) f(y)g(z)\Big|_{y=z} \\ &= \sum_{i_1,\ldots,i_n=0}^{\infty} \frac{1}{i_1!\cdots i_n!}\left(\frac{\hbar}{\sqrt{-1}}\right)^{|(i_1,\ldots,i_n)|} \omega^{i_1 j_1}\cdots\omega^{i_n j_n}\partial_{y_{i_1}}\cdots\partial_{y_{i_n}} f(y)\partial_{z_{j_1}}\cdots\partial_{z_{j_n}} g(z)\Big|_{y=z}, \end{aligned} \tag{6.1}$$

where $|(i_1, \ldots, i_n)| = i_1 + \cdots + i_n$ and the multi-index notation is used.

The main difficulty is that the formulas of the deformation quantization are formal and cannot be convergent.

Lemma 6.1. *The first-order jet of the deformation quantization is equivalent to geometric quantization.*

Proof. The first-order component is

$$f + \frac{\hbar}{\sqrt{-1}}\omega^{ij}\partial_i f \partial_j = f + \frac{\hbar}{\sqrt{-1}}\nabla_{\xi_f}, \tag{6.2}$$

which is the geometric quantization formula. □

Lemma 6.2. *The $(i_1, \ldots, i_n)$th component of of the series is the continued $(i, \ldots, i_n)$th component of the universal enveloping algebra.*

Proof. The linear differential operators are defined following the universal property of the universal enveloping algebra and therefore we have the identical components. □

Lemma 6.3. *The $(i_1, \ldots, i_n)$th component of of the series is linear operator from the tensor product of finite-dimensional Hilbert space $T_xM \otimes \Omega_x^n(M) = T_xM \otimes \Omega_x^1(M) \otimes \cdots \otimes \Omega_x^1(M)$ into the Hilbert space T_xM.*

Proof. It is clear from the formula for deformation. □

Lemma 6.4. *The formal series are elements of the tensor product of Hilbert space $T_xM \otimes \exp(\Omega_x^1(M))$.*

Proof. The exponential series formula provides the convergence. □

Now the proof of the main theorem is achieved.

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