Research Article

Connectedness Degrees in L -Fuzzy Topological Spaces

Fu-Gui Shi

Department of Mathematics, School of Science, Beijing Institute of Technology, Beijing 100081, China

Correspondence should be addressed to Fu-Gui Shi, fuguishi@bit.edu.cn

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The notion of separatedness degrees of L -fuzzy subsets is introduced in L -fuzzy topological spaces by means of L -fuzzy closure operators. Furthermore, the notion of connectedness degrees of L -fuzzy subsets is introduced. Many properties of connectedness in general topology are generalized to L -fuzzy topological spaces.

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1. Introduction

Since Chang [1] introduced fuzzy theory into topology, many authors have discussed various aspects of fuzzy topology. In a Chang I -topology, the open sets are fuzzy, but the topology comprising those open sets is a crisp subset of I^X . However, in a completely different direction, Höhle [2] presented a notion of fuzzy topology being viewed as an I -fuzzy subset of 2^X . Then Kubiak [3] and Šostak [4] independently extended Höhle's fuzzy topology to L -subsets of L^X , which is called L -fuzzy topology (see [5, 6]). From a logical point of view, Ying [7] studied Höhle's topology and called it fuzzifying topology.

Connectivity is one of the most important notions in general topology. It has been generalized to L -topology in terms of many forms (see [8–17], etc.). In a fuzzifying topological space, Ying [18] introduced a definition of connectivity and Fang [19] proved Fan's theorem. In a $[0, 1]$ -fuzzy topological space $(X, \mathcal{T})$, Šostak introduced a notion of connectedness degree by means of the level $[0, 1]$ -topological spaces $(X, \mathcal{T}_\alpha)$ [20, 21], that is, it can be viewed as connectivity in a $[0, 1]$ -topological space. Although a definition of connectivity was also presented by Yue and Fang [22] in $[0, 1]$ -fuzzy topological spaces, it was defined for whole L -fuzzy topological space not for arbitrary L -fuzzy subset.

In this paper, we first introduce the notion of separatedness degrees in L -fuzzy topological spaces by means of L -fuzzy closure operators. Furthermore, we present the notion of connectedness degrees of L -fuzzy subsets, which is a generalization of Yue and

Fang's connectedness degree. Many properties of connectedness in general topology can be generalized to L -fuzzy topological spaces.

2. Preliminaries

Throughout this paper, $(L, \vee, \wedge, ')$ denotes a completely distributive DeMorgan algebra. The smallest element and the largest element in L are denoted by $\perp$ and $\top$, respectively. The set of all nonzero co-prime elements of L is denoted by $J(L)$.

We say that a is wedge below b in L , denoted by $a < b$, if for every subset $D \subseteq L$, $\vee D \geq b$ implies $d \geq a$ for some $d \in D$. A complete lattice L is completely distributive if and only if $b = \vee \{a \in L : a < b\}$ for each $b \in L$. For any $b \in L$, define $\beta(b) = \{a \in L : a < b\}$. Some properties of β can be found in [23].

For a nonempty set X , the set of all nonzero coprime elements of L^X is denoted by $J(L^X)$. It is easy to see that $J(L^X)$ is exactly the set of all fuzzy points x_λ ($\lambda \in J(L)$). The smallest element and the largest element in L^X are denoted by $\underline{\perp}$ and $\underline{\top}$, respectively.

For any L -fuzzy set $A \in L^X$ and any $a \in L$, we use the following notations:

$$\begin{aligned} A_{[a]} &= \{x \in X : A(x) \geq a\}, \\ \underline{a}(x) &= a, \quad \forall x \in X. \end{aligned} \tag{2.1}$$

Definition 2.1 (see [3–5]). An L -fuzzy topology on a set X is a map $\tau : L^X \rightarrow L$ such that

- (LFT1) $\tau(\underline{\top}) = \tau(\underline{\perp}) = \top$;
- (LFT2) for all $U, V \in L^X$, $\tau(U \wedge V) \geq \tau(U) \wedge \tau(V)$;
- (LFT3) for all $U_j \in L^X$, $j \in J$, $\tau(\bigvee_{j \in J} U_j) \geq \bigwedge_{j \in J} \tau(U_j)$.

$\tau(U)$ can be interpreted as the degree to which U is an open set. $\tau^*(U) = \tau(U')$ will be called the degree of closedness of U . The pair (X, τ) is called an L -fuzzy topological space.

A mapping $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is said to be continuous with respect to L -fuzzy topologies τ_1 and τ_2 if $\tau_1(f_L^{\leftarrow}(U)) \geq \tau_2(U)$ holds for all $U \in L^Y$, where $f_L^{\leftarrow}$ is defined by $f_L^{\leftarrow}(U)(x) = U(f(x))$ [24].

Definition 2.2 (see [25]). An L -fuzzy closure operator on X is a mapping $\text{Cl} : L^X \rightarrow L^{J(L^X)}$ satisfying the following conditions:

- (LFC1) $\text{Cl}(A)(x_\lambda) = \bigwedge_{\mu < \lambda} \text{Cl}(A)(x_\mu)$, for all $x_\lambda \in J(L^X)$;
- (LFC2) $\text{Cl}(\underline{\perp})(x_\lambda) = \perp$ for any $x_\lambda \in J(L^X)$;
- (LFC3) $\text{Cl}(A)(x_\lambda) = \top$ for any $x_\lambda \leq A$;
- (LFC4) $\text{Cl}(A \vee B) = \text{Cl}(A) \vee \text{Cl}(B)$;
- (LFC5) for all $a \in L \setminus \{\perp\}$, $(\text{Cl}(\bigvee(\text{Cl}(A))_{[a]}))_{[a]} \subseteq (\text{Cl}(A))_{[a]}$.

$\text{Cl}(A)(x_\lambda)$ is called the degree to which x_λ belongs to the closure of A .

Lemma 2.3 (see [25]). Let $(X, \mathcal{T})$ be an L -fuzzy topological space and let Cl be the L -fuzzy closure operator induced by $\mathcal{T}$. Then for all $x_\lambda \in J(L^X)$, for all $A \in L^X$,

$$\begin{aligned} \text{Cl}(A)(x_\lambda) &= \bigwedge \left\{ (\mathcal{T}(D'))' : D \in L^X, x_\lambda \not\leq D \geq A \right\} \\ &= \bigwedge_{x_\lambda \not\leq D \geq A} (\mathcal{T}(D'))'. \end{aligned} \quad (2.2)$$

Definition 2.4 (see [17, 23]). In an L -topological space (X, τ) , two L -fuzzy sets A, B are called separated if $A^- \wedge B = A \wedge B^- = \perp$, where A^- denotes the closure of A .

Definition 2.5 (see [17, 23]). In an L -topological space (X, τ) , an L -fuzzy set D is called connected if D can not be represented as a union of two separated non-null L -fuzzy sets.

3. Separatedness Degrees in L -Fuzzy Topological Spaces

In this section, in order to generalize Definition 2.5 to L -fuzzy topological spaces, we will introduce the concept of separatedness degrees in L -fuzzy topological spaces by means of L -fuzzy closure operators.

Definition 3.1. Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $A, B \in L^X$. Define

$$\begin{aligned} \text{Sep}(A, B) &= \left(\bigwedge \{ (\text{Cl}(B)(x_\lambda))' : x_\lambda \leq A \} \right) \wedge \left(\bigwedge \{ (\text{Cl}(A)(y_\mu))' : y_\mu \leq B \} \right) \\ &= \left(\bigwedge_{x_\lambda \leq A} (\text{Cl}(B)(x_\lambda))' \right) \wedge \left(\bigwedge_{y_\mu \leq B} (\text{Cl}(A)(y_\mu))' \right). \end{aligned} \quad (3.1)$$

Then $\text{Sep}(A, B)$ is said to be the separatedness degree of A and B .

The following result is obvious.

Proposition 3.2. Let $\mathcal{T} : L^X \rightarrow \{\perp, \top\}$ be an L -topology on X and $A, B \in L^X$. Then $\text{Sep}(A, B) = \top$ if and only if A and B are separated in $(X, \mathcal{T})$.

Lemma 3.3. Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $A, B \in L^X$. If $A \wedge B \neq \perp$, then $\text{Sep}(A, B) = \perp$.

Proof. From $A \wedge B \neq \perp$, we can take $z_\gamma \in J(L^X)$ such that $z_\gamma \leq A \wedge B$. Thus we have

$$\begin{aligned} \text{Sep}(A, B) &= \left(\bigwedge_{x_\lambda \leq A} (\text{Cl}(B)(x_\lambda))' \right) \wedge \left(\bigwedge_{x_\lambda \leq B} (\text{Cl}(A)(x_\lambda))' \right) \\ &\leq (\text{Cl}(B)(z_\gamma))' \wedge (\text{Cl}(A)(z_\gamma))' = \top' \wedge \top' = \perp. \end{aligned} \quad (3.2)$$

□

Lemma 3.4. Let $(X, \mathcal{T})$ be an L -fuzzy topological space, and $A, B, C, D \in L^X$. If $C \leq A$ and $D \leq B$, then $\text{Sep}(A, B) \leq \text{Sep}(C, D)$.

Proof. If $C \leq A$ and $D \leq B$, then $\text{Cl}(C) \leq \text{Cl}(A)$ and $\text{Cl}(D) \leq \text{Cl}(B)$. Hence we have

$$\begin{aligned} \text{Sep}(A, B) &= \left(\bigwedge_{x_\lambda \leq A} (\text{Cl}(B)(x_\lambda))' \right) \wedge \left(\bigwedge_{y_\mu \leq B} (\text{Cl}(A)(y_\mu))' \right) \\ &\leq \left(\bigwedge_{x_\lambda \leq A} (\text{Cl}(D)(x_\lambda))' \right) \wedge \left(\bigwedge_{y_\mu \leq B} (\text{Cl}(C)(y_\mu))' \right) \\ &\leq \left(\bigwedge_{x_\lambda \leq C} (\text{Cl}(D)(x_\lambda))' \right) \wedge \left(\bigwedge_{y_\mu \leq D} (\text{Cl}(C)(y_\mu))' \right) \\ &= \text{Sep}(C, D). \end{aligned} \quad (3.3)$$

□

Lemma 3.5. Let $(X, \mathcal{T})$ be an L -fuzzy topological space, $A, B \in L^X$ and $a \in J(L)$. Then $(\text{Sep}(A, B))' \not\geq a$ if and only if there exist $D, E \in L^X$ such that

$$D \geq A, \quad E \geq B, \quad D \wedge B = E \wedge A = \perp, \quad (\mathcal{T}(D'))' \vee (\mathcal{T}(E'))' \not\geq a. \quad (3.4)$$

Proof. Suppose that $(\text{Sep}(A, B))' \not\geq a$. Then $(\text{Sep}(A, B))' \not\geq b$ for some $b \in \beta^*(a)$. This implies

$$\bigvee_{x_\lambda \leq A} \text{Cl}(B)(x_\lambda) \vee \bigvee_{y_\mu \leq B} \text{Cl}(A)(y_\mu) \not\geq b. \quad (3.5)$$

Further more, we have

$$\bigvee_{x_\lambda \leq A} \bigwedge_{x_\lambda \not\geq E \geq B} (\mathcal{T}(E'))' \vee \bigvee_{y_\mu \leq B} \bigwedge_{y_\mu \not\geq D \geq A} (\mathcal{T}(D'))' \not\geq b. \quad (3.6)$$

Hence for any $x_\lambda \leq A$ and for any $y_\mu \leq B$, there are $D_{y_\mu}, E_{x_\lambda} \in L^X$ such that $x_\lambda \not\geq E_{x_\lambda} \geq B$, $y_\mu \not\geq D_{y_\mu} \geq A$ and $(\mathcal{T}(D'_{y_\mu}))' \vee (\mathcal{T}(E'_{x_\lambda}))' \not\geq b$. Let $E = \bigwedge_{x_\lambda \leq A} E_{x_\lambda}$ and $D = \bigwedge_{y_\mu \leq B} D_{y_\mu}$. Then, obviously, we have that $D \geq A$, $E \geq B$, $D \wedge B = E \wedge A = \perp$ and

$$\begin{aligned} (\mathcal{T}(D'))' \vee (\mathcal{T}(E'))' &= \left(\mathcal{T} \left(\bigvee_{y_\mu \leq B} D'_{y_\mu} \right) \right)' \vee \left(\mathcal{T} \left(\bigvee_{x_\lambda \leq A} E'_{x_\lambda} \right) \right)' \\ &\leq \bigvee_{y_\mu \leq B} (\mathcal{T}(D'_{y_\mu}))' \vee \bigvee_{x_\lambda \leq A} (\mathcal{T}(E'_{x_\lambda}))' \not\geq a. \end{aligned} \quad (3.7)$$

Conversely if there exist $D, E \in L^X$ such that

$$D \geq A, \quad E \geq B, \quad D \wedge B = E \wedge A = \perp, \quad (\tau(D'))' \vee (\tau(E'))' \not\geq a. \quad (3.8)$$

Then by

$$\begin{aligned} (\text{Sep}(A, B))' &= \bigvee_{x_\lambda \leq A} \text{Cl}(B)(x_\lambda) \vee \bigvee_{y_\mu \leq B} \text{Cl}(A)(y_\mu) \\ &= \bigvee_{x_\lambda \leq A} \bigwedge_{x_\lambda \not\leq G \geq B} (\tau(G'))' \vee \bigvee_{y_\mu \leq B} \bigwedge_{y_\mu \not\leq H \geq A} (\tau(H'))' \\ &\leq (\tau(D'))' \vee (\tau(E'))' \end{aligned} \quad (3.9)$$

we can obtain that $(\text{Sep}(A, B))' \not\geq a$. □

4. Connectedness Degrees in L -Fuzzy Topological Spaces

Definition 4.1. Let (X, τ) be an L -fuzzy topological space and $G \in L^X$. Define

$$\begin{aligned} \text{Con}(G) &= \bigwedge \left\{ (\text{Sep}(A, B))' : A, B \in L^X \setminus \{\perp\}, G = A \vee B \right\} \\ &= \bigwedge \left\{ \bigvee_{x_\lambda \leq A} \text{Cl}(B)(x_\lambda) \vee \bigvee_{y_\mu \leq B} \text{Cl}(A)(y_\mu) : A, B \in L^X \setminus \{\perp\}, G = A \vee B \right\}. \end{aligned} \quad (4.1)$$

Then $\text{Con}(G)$ is said to be the connectedness degree of G .

The following proposition shows that Definition 4.1 is a generalization of Definition 2.5.

Proposition 4.2. Let $\tau : L^X \rightarrow \{\perp, \top\}$ be an L -topology on X and $G \in L^X$. Then $\text{Con}(G) = \top$ if and only if G is connected in (X, τ) .

Theorem 4.3. Let (X, τ) be an L -fuzzy topological space and $G \in L^X$. Then

$$\text{Con}(G) = \bigwedge \left\{ (\tau(A'))' \vee (\tau(B'))' : G \wedge A \neq \perp, G \wedge B \neq \perp, G \wedge A \wedge B = \perp, G \leq A \vee B \right\}. \quad (4.2)$$

Proof. On one hand, we have the following inequality:

$$\begin{aligned}
 & \text{Con}(G) \\
 &= \bigwedge \left\{ \bigvee_{x_\lambda \leq A} \text{Cl}(B)(x_\lambda) \vee \bigvee_{y_\mu \leq B} \text{Cl}(A)(y_\mu) : A, B \in L^X \setminus \{\perp\}, G = A \vee B \right\} \\
 &= \bigwedge \left\{ \bigvee_{x_\lambda \leq A} \bigwedge_{x_\lambda \not\geq D \geq B} (\tau(D'))' \vee \bigvee_{y_\mu \leq B} \bigwedge_{y_\mu \not\geq E \geq A} (\tau(E'))' : A, B \in L^X \setminus \{\perp\}, G = A \vee B \right\} \\
 &= \bigwedge \left\{ \bigvee_{x_\lambda \leq G \wedge A} \bigwedge_{x_\lambda \not\geq D \geq G \wedge B} (\tau(D'))' \vee \bigvee_{y_\mu \leq G \wedge B} \bigwedge_{y_\mu \not\geq E \geq G \wedge A} (\tau(E'))' : G \wedge A \neq \perp, G \wedge B \neq \perp, G \leq A \vee B \right\} \\
 &\leq \bigwedge \left\{ \bigvee_{x_\lambda \leq G \wedge A} (\tau(B'))' \vee \bigvee_{y_\mu \leq G \wedge B} (\tau(A'))' : G \wedge A \neq \perp, G \wedge B \neq \perp, G \wedge A \wedge B = \perp, G \leq A \vee B \right\} \\
 &= \bigwedge \left\{ (\tau(B'))' \vee (\tau(A'))' : G \wedge A \neq \perp, G \wedge B \neq \perp, G \wedge A \wedge B = \perp, G \leq A \vee B \right\};
 \end{aligned} \tag{4.3}$$

on the other hand, in order to prove the inverse, we suppose that $\text{Con}(G) \not\geq a$ ($a \in J(L)$). Then there exist $A, B \in L^X \setminus \{\perp\}$ such that $G = A \vee B$ and $(\text{Sep}(A, B))' \not\geq a$. By Lemma 3.5 we know that there exists $D, E \in L^X$ such that

$$D \geq A, \quad E \geq B, \quad D \wedge B = E \wedge A = \perp, \quad (\tau(D'))' \vee (\tau(E'))' \not\geq a. \tag{4.4}$$

Obviously $G \wedge D \neq \perp, G \wedge E \neq \perp, G \wedge D \wedge E = \perp, G \leq D \vee E$. Hence we have

$$\bigwedge \left\{ (\tau(B'))' \vee (\tau(A'))' : G \wedge A \neq \perp, G \wedge B \neq \perp, G \wedge A \wedge B = \perp, G \leq A \vee B \right\} \not\geq a. \tag{4.5}$$

Therefore,

$$\text{Con}(G) \geq \bigwedge \left\{ (\tau(B'))' \vee (\tau(A'))' : G \wedge A \neq \perp, G \wedge B \neq \perp, G \wedge A \wedge B = \perp, G \leq A \vee B \right\}. \tag{4.6}$$

The proof is completed. $\square$

Example 4.4. Let $X = \{x, y\}$ and $L = [0, 1]$. Define $B \in [0, 1]^X$ by $B(x) = 0.5$ and $B(y) = 0$, and define $C \in [0, 1]^X$ by $C(y) = 0.5$ and $C(x) = 0$, respectively. Let $\tau : [0, 1]^X \rightarrow [0, 1]$ be defined as follows:

$$\tau(A) = \begin{cases} 1, & A \in \{\top, \perp, 0.5\}, \\ 0.5, & A \in \{B', C'\}, \\ 0, & \text{others.} \end{cases} \quad (4.7)$$

Then τ is an L -fuzzy topology on X . It is easy to verify that $\text{Con}(\underline{a}) = 0.5$ for any $a \in (0, 0.5]$ and $\text{Con}(\underline{b}) = 1$ for any $b \in (0.5, 1]$.

Corollary 4.5. *Let (X, τ) be an L -fuzzy topological space. Then*

$$\begin{aligned} \text{Con}(\top) &= \bigwedge \left\{ (\tau(A'))' \vee (\tau(B'))' : A \neq \perp, B \neq \perp, A \wedge B = \perp, \top = A \vee B \right\} \\ &= \bigwedge \left\{ (\tau(A))' \vee (\tau(B))' : A \neq \perp, B \neq \perp, A \wedge B = \perp, \top = A \vee B \right\}. \end{aligned} \quad (4.8)$$

Remark 4.6. Yue and Fang [22] introduced a definition of connectivity in a $[0, 1]$ -fuzzy topological space. It is easy to see that Yue and Fang's definition is a special case of our definition from Corollary 4.5.

Theorem 4.7. *For any $e \in J(L^X)$, it follows that $\text{Con}(e) = \top$.*

Proof. From Theorem 4.3 we have

$$\begin{aligned} \text{Con}(e) &= \bigwedge \left\{ (\tau(A'))' \vee (\tau(B'))' : e \wedge A \neq \perp, e \wedge B \neq \perp, e \wedge A \wedge B = \perp, e \leq A \vee B \right\} \\ &= \bigwedge \emptyset = \top. \end{aligned} \quad (4.9)$$

□

Theorem 4.8. *For any $G \in L^X$, one has*

$$\bigvee_{r \in J(L)} \text{Con} \left(\bigvee (\text{Cl}(G))_{[r]} \right) \geq \text{Con}(G). \quad (4.10)$$

Proof. Let $a \in J(L)$ and $a \leq \text{Con}(G)$. Now we prove $\bigvee_{r \in J(L)} \text{Con}(\bigvee (\text{Cl}(G))_{[r]}) \geq a$. Suppose that $\bigvee_{r \in J(L)} \text{Con}(\bigvee (\text{Cl}(G))_{[r]}) \not\geq a$. Then $\text{Con}(\bigvee (\text{Cl}(G))_{[a]}) \not\geq a$. By Theorem 4.3 we know that there exists $A, B \in L^X$ such that

$$\begin{aligned} \left(\bigvee (\text{Cl}(G))_{[a]} \right) \wedge A \neq \perp, \quad \left(\bigvee (\text{Cl}(G))_{[a]} \right) \wedge B \neq \perp, \quad \left(\bigvee (\text{Cl}(G))_{[a]} \right) \wedge A \wedge B = \perp, \\ \bigvee (\text{Cl}(G))_{[a]} \leq A \vee B, \quad (\tau(B'))' \vee (\tau(A'))' \not\geq a. \end{aligned} \quad (4.11)$$

By $(\bigvee(\text{Cl}(G))_{[a]}) \wedge A \neq \perp$ we know that there exists $x_\lambda \leq A$ such that $\text{Cl}(G)(x_\lambda) \geq a$. Furthermore by $(\bigvee(\text{Cl}(G))_{[a]}) \wedge A \wedge B = \perp$ we obtain $x_\lambda \not\leq B$.

Now we prove $G \wedge A \neq \perp$. In fact, if $G \wedge A = \perp$, then by $G \leq \bigvee(\text{Cl}(G))_{[a]} \leq A \vee B$ we have $G \leq B$, hence it follows that

$$a \leq \text{Cl}(G)(x_\lambda) = \bigwedge_{x_\lambda \not\leq E \geq G} (\tau(E'))' \leq (\tau(B'))', \quad (4.12)$$

contradicting $a \not\leq (\tau(B'))'$. Analogously, we can prove $G \wedge B \neq \perp$. Thus by

$$\begin{aligned} G \wedge A \neq \perp, \quad G \wedge B \neq \perp, \quad G \wedge A \wedge B = \perp, \\ G \leq A \vee B, \quad (\tau(B'))' \vee (\tau(A'))' \not\leq a, \end{aligned} \quad (4.13)$$

and Theorem 4.3, we know that $\text{Con}(G) \not\geq a$, contradicting $\text{Con}(G) \geq a$. It is proved that $\bigvee_{r \in J(L)} \text{Con}(\bigvee(\text{Cl}(G))_{[r]}) \geq \text{Con}(G)$. $\square$

Theorem 4.9. For any $G, H \in L^X$, one has

$$\text{Con}(G \vee H) \geq (\text{Sep}(G, H))' \wedge \text{Con}(G) \wedge \text{Con}(H). \quad (4.14)$$

Proof. Let $a \in J(L)$ and $a \leq (\text{Sep}(G, H))' \wedge \text{Con}(G) \wedge \text{Con}(H)$. Now we prove $\text{Con}(G \vee H) \geq a$. Suppose that $\text{Con}(G \vee H) \not\geq a$. Then by Theorem 4.3 we know that there exist $A, B \in L^X$ such that

$$\begin{aligned} (G \vee H) \wedge A \neq \perp, \quad (G \vee H) \wedge B \neq \perp, \quad (G \vee H) \wedge A \wedge B = \perp, \\ G \vee H \leq A \vee B, \quad (\tau(B'))' \vee (\tau(A'))' \not\leq a. \end{aligned} \quad (4.15)$$

By $(G \vee H) \wedge A \neq \perp$ we know that one of $G \wedge A \neq \perp$ and $H \wedge A \neq \perp$ must be true.

Suppose that $G \wedge A \neq \perp$ (the case of $H \wedge A \neq \perp$ is analogous). Then we must have $G \wedge B = \perp$, otherwise if $G \wedge B \neq \perp$, then by

$$G \wedge A \neq \perp, \quad G \wedge B \neq \perp, \quad G \wedge A \wedge B = \perp, \quad G \leq A \vee B, \quad (\tau(B'))' \vee (\tau(A'))' \not\leq a \quad (4.16)$$

we know that $\text{Con}(G) \not\geq a$, contradicting $\text{Con}(G) \geq a$. In this case by $(G \vee H) \wedge B \neq \perp$ we know that $H \wedge B \neq \perp$. Analogously we can prove $H \wedge A = \perp$. Thus by $G \vee H \leq A \vee B$ we can obtain that $G \leq A$ and $H \leq B$. Hence by

$$G \leq A, \quad H \leq B, \quad G \wedge B = H \wedge A = \perp, \quad (\tau(B'))' \vee (\tau(A'))' \not\leq a \quad (4.17)$$

and Lemma 3.5 we know that $(\text{Sep}(G, H))' \not\geq a$, contradicting $(\text{Sep}(G, H))' \geq a$. This shows that $\text{Con}(G \vee H) \geq a$. It is proved that $\text{Con}(G \vee H) \geq (\text{Sep}(G, H))' \wedge \text{Con}(G) \wedge \text{Con}(H)$. $\square$

By Lemma 3.3 we can obtain the following result.

Corollary 4.10. Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $G, H \in L^X$. If $A \wedge B \neq \perp$, then $\text{Con}(G \vee H) \geq \text{Con}(G) \wedge \text{Con}(H)$.

Theorem 4.11. Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $G \in L^X$. Then

$$\text{Con}(G) = \bigwedge_{x_\lambda, y_\mu \leq G} \bigvee \left\{ \text{Con}(D_{x_\lambda y_\mu}) : x_\lambda, y_\mu \leq D_{x_\lambda y_\mu} \leq G \right\}. \quad (4.18)$$

Proof. It is obvious that

$$\text{Con}(G) \leq \bigwedge_{x_\lambda, y_\mu \leq G} \bigvee \left\{ \text{Con}(D_{x_\lambda y_\mu}) : x_\lambda, y_\mu \leq D_{x_\lambda y_\mu} \leq G \right\}. \quad (4.19)$$

Now we prove that

$$\text{Con}(G) \geq \bigwedge_{x_\lambda, y_\mu \leq G} \bigvee \left\{ \text{Con}(D_{x_\lambda y_\mu}) : x_\lambda, y_\mu \leq D_{x_\lambda y_\mu} \leq G \right\}. \quad (4.20)$$

Suppose that $\bigwedge_{x_\lambda, y_\mu \leq G} \bigvee \left\{ \text{Con}(D_{x_\lambda y_\mu}) : x_\lambda, y_\mu \leq D_{x_\lambda y_\mu} \leq G \right\} \geq a$ ($a \in J(L)$). Take a fixed $x_\lambda \leq G$. Then for any $y_\mu \leq G$, there exists a $D_{x_\lambda y_\mu} \in L^X$ such that $x_\lambda, y_\mu \leq D_{x_\lambda y_\mu} \leq G$ and $\text{Con}(D_{x_\lambda y_\mu}) \geq a$. Let $D_{x_\lambda} = \bigvee_{y_\mu \leq G} D_{x_\lambda y_\mu}$. Obviously $D_{x_\lambda} = G$ and $\bigwedge_{y_\mu \leq G} D_{x_\lambda y_\mu} \neq \perp$. By Corollary 4.10 we easily obtain $\text{Con}(G) = \text{Con}(D_{x_\lambda}) \geq \bigwedge_{y_\mu \leq G} \text{Con}(D_{x_\lambda y_\mu}) \geq a$. This shows

$$\text{Con}(G) \geq \bigwedge_{x_\lambda, y_\mu \leq G} \bigvee \left\{ \text{Con}(D_{x_\lambda y_\mu}) : x_\lambda, y_\mu \leq D_{x_\lambda y_\mu} \leq G \right\}. \quad (4.21)$$

□

Theorem 4.12. If $f_L^\rightarrow : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ is continuous, then $\text{Con}(f_L^\rightarrow(G)) \geq \text{Con}(G)$.

Proof. This can be proved from Theorem 4.3 and the following inequality:

$$\begin{aligned} \text{Con}(f_L^\rightarrow(G)) &= \bigwedge \left\{ (\mathcal{T}_2(A'))' \vee (\mathcal{T}_2(B'))' : f_L^\rightarrow(G) \wedge A \neq \perp, f_L^\rightarrow(G) \wedge B \neq \perp \right. \\ &\quad \left. f_L^\rightarrow(G) \wedge A \wedge B = \perp, f_L^\rightarrow(G) \leq A \vee B \right\} \\ &\geq \bigwedge \left\{ (\mathcal{T}_2(A'))' \vee (\mathcal{T}_2(B'))' : G \wedge f_L^\leftarrow(A) \neq \perp, G \wedge f_L^\leftarrow(B) \neq \perp \right. \\ &\quad \left. G \wedge f_L^\leftarrow(A) \wedge f_L^\leftarrow(B) = \perp, G \leq f_L^\leftarrow(A) \vee f_L^\leftarrow(B) \right\} \\ &\geq \bigwedge \left\{ (\mathcal{T}_1(f_L^\leftarrow(A)))' \vee (\mathcal{T}_1(f_L^\leftarrow(B)))' : G \wedge f_L^\leftarrow(A) \neq \perp, G \wedge f_L^\leftarrow(B) \neq \perp \right. \\ &\quad \left. G \wedge f_L^\leftarrow(A) \wedge f_L^\leftarrow(B) = \perp, G \leq f_L^\leftarrow(A) \vee f_L^\leftarrow(B) \right\} \\ &\geq \text{Con}(G). \end{aligned} \quad (4.22)$$

□

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