Research Article

Existence and Uniqueness of Periodic Solutions for a Second-Order Nonlinear Differential Equation with Piecewise Constant Argument

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Based on a continuation theorem of Mawhin, a unique periodic solution is found for a second-order nonlinear differential equation with piecewise constant argument.

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1. Introduction

Qualitative behaviors of first-order delay differential equations with piecewise constant arguments are the subject of many investigations (see, e.g., [1–19]), while those of higher-order equations are not.

However, there are reasons for studying higher-order equations with piecewise constant arguments. Indeed, as mentioned in [10], a potential application of these equations is in the stabilization of hybrid control systems with feedback delay, where a hybrid system is one with a continuous plant and with a discrete (sampled) controller. As an example, suppose that a moving particle with time variable mass \( r(t) \) is subjected to a restoring controller \( -\phi(x[t]) \) which acts at sampled time \([t]\). Then Newton’s second law asserts that

\[
(r(t)x'(t))'' = -\phi(x[t]).
\]  

(1.1)

Since this equation is “similar” to the harmonic oscillator equation

\[
(r(t)x'(t))'' + \kappa x(t) = 0,
\]  

(1.2)
we expect that the well-known qualitative behavior of the later equation may also be found in the former equation, provided appropriate conditions on \(r(t)\) and \(\phi(x)\) are imposed.

In this paper we study a slightly more general second-order delay differential equation with piecewise constant argument:

\[
(r(t)x'(t))' + f(t, x([t])) = p(t),
\]

where \(f(t, x)\) is a real continuous function defined on \(\mathbb{R}^2\) with positive integer period \(\omega\) for \(t\); \(r(t)\) and \(p(t)\) are continuous function defined on \(\mathbb{R}\) with period \(\omega\), \(r(t) > 0\) for \(t \in \mathbb{R}\) and \(\int_0^\infty p(t)dt = 0\).

By a solution of (1.3) we mean a function \(x(t)\) which is defined on \(\mathbb{R}\) and which satisfies the following conditions: (i) \(x'(t)\) is continuous on \(\mathbb{R}\), (ii) \(r(t)x'(t)\) is differentiable at each point \(t \in \mathbb{R}\), with the possible exception of the points \([t] \in \mathbb{R}\) where one-sided derivatives exist, and (iii) substitution of \(x(t)\) into (1.3) leads to an identity on each interval \([n, n+1) \subset \mathbb{R}\) with integral endpoints.

In this note, existence and uniqueness criteria for periodic solutions of (1.3) will be established. For this purpose, we will make use of a continuation theorem of Mawhin. Let \(X\) and \(Y\) be two Banach spaces and \(L : \text{Dom} \ L \subset X \rightarrow Y\) be a Fredholm mapping of index zero, \(\text{dim Ker } L = \text{codim Im } L < +\infty\), and \(\text{Im } L\) is closed in \(Y\). If \(L\) is a Fredholm mapping of index zero, there exist continuous projectors \(P : X \rightarrow X\) and \(Q : Y \rightarrow Y\) such that \(\text{Im } P = \text{Ker } L\) and \(\text{Im } L = \text{Ker } Q = \text{Im } (I - Q)\). It follows that \(L|_{\text{Dom } L \cap \text{Ker } P : (I - P)X \rightarrow \text{Im } L}\) has an inverse which will be denoted by \(K_p\). If \(\Omega\) is an open and bounded subset of \(X\), the mapping \(N\) will be called \(L\)-compact on \(\overline{\Omega}\) if \(QN(\overline{\Omega})\) is bounded and \(K_p(I - Q)N : \overline{\Omega} \rightarrow X\) is compact. Since \(\text{Im } Q\) is isomorphic to \(\text{Ker } L\), there exists an isomorphism \(J : \text{Im } Q \rightarrow \text{Ker } L\).

**Theorem A** (Mawhin’s continuation theorem [18]). Let \(L\) be a Fredholm mapping of index zero, and let \(N\) be \(L\)-compact on \(\overline{\Omega}\). Suppose that

(i) for each \(\lambda \in (0, 1)\), \(x \in \partial \Omega\), \(Lx = \lambda Nx\);

(ii) for each \(x \in \partial \Omega \cap \text{Ker } L\), \(QNx \neq 0\) and \(\text{deg}(JQN, \Omega \cap \text{Ker } 0, 0) \neq 0\).

Then the equation \(Lx = Nx\) has at least one solution in \(\overline{\Omega} \cap \text{dom } L\).

2. Existence and Uniqueness Criteria

Our main results of this paper are as follows.

**Theorem 2.1.** Suppose that there exist constants \(D > 0\) and \(\delta \geq 0\) such that

(i) \(f(t, x) \text{sgn } x > 0\) for \(t \in \mathbb{R}\) and \(|x| > D\),

(ii) \(\lim_{x \rightarrow -\infty} \text{max}_{0 \leq s \leq \omega} (f(t, x)/x) \leq \delta\) (or \(\lim_{x \rightarrow +\infty} \text{max}_{0 \leq s \leq \omega} (f(t, x)/x) \leq \delta\)).

If \(\omega^2 \delta(\text{max}_{0 \leq s \leq \omega} (1/r(t))) < 1\), then (1.3) has an \(\omega\)-periodic solution. Furthermore, the \(\omega\)-periodic solution is unique if in addition one has the following.

(iii) \(f(t, x)\) is strictly monotonous in \(x\) and there exists nonnegative constant \(b < (4/\omega^2)\min_{0 \leq s \leq \omega} r(t)\) such that

\[
|f(t, x_1) - f(t, x_2)| \leq b|x_1 - x_2|, \quad (t, x_1), (t, x_2) \in \mathbb{R}^2.
\]

(2.1)
Theorem 2.2. Suppose that there exist constants \( D > 0 \) and \( \delta \geq 0 \) such that

(i') \( f(t, x) \sgn x < 0 \) for \( t \in \mathbb{R} \) and \( |x| > D \),

(ii') \( \lim_{x \to +\infty} \max_{0 \leq t \leq \omega} (f(t, x)/x) \geq \delta \) (or \( \lim_{x \to -\infty} \max_{0 \leq t \leq \omega} (f(t, x)/x) \geq -\delta \)).

If \( \omega^2 \delta (\max_{0 \leq t \leq \omega} (1/r(t))) < 1 \), then (1.3) has an \( \omega \)-periodic solution. Furthermore, the \( \omega \)-periodic solution is unique if in addition one has the following.

(iii) \( f(t, x) \) is strictly monotonous in \( x \) and there exists nonnegative constant \( b < (4/\omega^2) \min_{0 \leq t \leq \omega} r(t) \) such that (2.1) holds.

We only give the proof of Theorem 2.1, as Theorem 2.2 can be proved similarly.

First we make the simple observation that \( x(t) \) is an \( \omega \)-periodic solution of the following equation:

\[
 r(t)x'(t) = r(0)x'(0) - \int_0^t (f(s, x([s])) - p(s))ds,
\]

if, and only if, \( x(t) \) is an \( \omega \)-periodic solution of (1.3). Next, let \( X_\omega \) be the Banach space of all real \( \omega \)-periodic continuously differentiable functions of the form \( x = x(t) \) which is defined on \( \mathbb{R} \) and endowed with the usual linear structure as well as the norm \( \|x\|_1 = \sum_{i=0}^1 \max_{0 \leq t \leq \omega} |x^{(i)}(t)| \). Let \( Y_\omega \) be the Banach space of all real continuous functions of the form \( y = at + h(t) \) such that \( y(0) = 0 \), where \( a \in \mathbb{R} \) and \( h(t) \in X_\omega \), and endowed with the usual linear structure as well as the norm \( \|y\|_2 = |a| + \|h\|_1 \). Let the zero element of \( X_\omega \) and \( Y_\omega \) be denoted by \( 0_1 \) and \( 0_2 \) respectively.

Define the mappings \( L : X_\omega \to Y_\omega \) and \( N : X_\omega \to Y_\omega \), respectively, by

\[
 Lx(t) = r(t)x'(t) - r(0)x'(0),
\]

\[
 Nx(t) = \int_0^t (f(s, x([s])) - p(s))ds.
\]

Let

\[
 \overline{h}(t) = -\int_0^t (f(s, x([s])) - p(s))ds + \frac{t}{\omega} \int_0^\omega f(s, x([s]))ds.
\]

Since \( \overline{h} \in X_\omega \) and \( \overline{h}(0) = 0 \), \( N \) is a well-defined operator from \( X_\omega \) to \( Y_\omega \). Let us define \( P : X_\omega \to X_\omega \) and \( Q : Y_\omega \to Y_\omega \), respectively, by

\[
 Px(t) = x(0), \quad n \in \mathbb{Z}
\]

for \( x = x(t) \in X_\omega \) and

\[
 Qy(t) = at
\]

for \( y(t) = at + h(t) \in Y_\omega \).
Lemma 2.3. Let the mapping $L$ be defined by (2.3). Then

\[ \text{Ker } L = R. \] (2.8)

Proof. It suffices to show that if $x(t)$ is a real $\omega$-periodic continuously differentiable function which satisfies

\[ r(t)x'(t) = r(0)x'(0), \quad t \in R, \] (2.9)

then $x(t)$ is a constant function. To see this, note that for such a function $x = x(t)$,

\[ x'(t) = \frac{r(0)x'(0)}{r(t)}, \quad t \in R. \] (2.10)

Hence by integrating both sides of the above equality from 0 to $t$, we see that

\[ x(t) = x(0) + r(0)x'(0) \int_0^t \frac{ds}{r(s)}, \quad t \in R. \] (2.11)

Since $r(t)$ is positive, continuous, and periodic,

\[ \int_0^\infty \frac{ds}{r(s)} = \infty. \] (2.12)

Since $x(t)$ is bounded, we may infer from (2.11) that $x'(0) = 0$. But then (2.9) implies $x'(t) = 0$ for $t \in R$. The proof is complete. \hfill \Box

Lemma 2.4. Let the mapping $L$ be defined by (2.3). Then

\[ \text{Im } L = \{ y \in X_\omega \mid y(0) = 0 \} \subset Y_\omega. \] (2.13)

Proof. It suffices to show that for each $y = y(t) \in X_\omega$ that satisfies $y(0) = 0$, there is a $x = x(t) \in X_\omega$ such that

\[ y(t) = r(t)x'(t) - r(0)x'(0), \quad t \geq 0. \] (2.14)

But this is relatively easy, since we may let

\[ a = \frac{1}{\int_0^\infty (ds/r(s))}, \] (2.15)

\[ x(t) = \int_0^t \frac{y(s)}{r(s)} ds - a \int_0^\infty \frac{y(s)}{r(s)} ds \int_0^t \frac{ds}{r(s)}, \quad t \geq 0. \] (2.16)

Then it may easily be checked that (2.14) holds. The proof is complete. \hfill \Box
Lemma 2.5. The mapping $L$ defined by (2.3) is a Fredholm mapping of index zero.

Indeed, from Lemmas 2.3 and 2.4 and the definition of $Y_\omega$, $\dim \ker L = \text{codim } \text{im } L = 1 < +\infty$. From (2.13), we see that $\text{im } L$ is closed in $Y_\omega$. Hence $L$ is a Fredholm mapping of index zero.

Lemma 2.6. Let the mapping $L$, $P$, and $Q$ be defined by (2.3), (2.6), and (2.7), respectively. Then $\text{im } P = \ker L$ and $\text{im } L = \ker Q$.

Indeed, from Lemmas 2.3 and 2.4 and defining conditions (2.6) and (2.7), it is easy to see that $\text{im } P = \ker L$ and $\text{im } L = \ker Q$.

Lemma 2.7. Let $L$ and $N$ be defined by (2.3) and (2.4), respectively. Suppose that $\Omega$ is an open and bounded subset of $X_\omega$. Then $N$ is $L$-compact on $\overline{\Omega}$.

Proof. It is easy to see that for any $x \in \overline{\Omega}$,

$$QN x(t) = -\frac{t}{\omega} \int_0^\omega f(s, x([s])) ds,$$

so that

$$\|QN x\|_2 = \left| \frac{1}{\omega} \int_0^\omega f(s, x([s])) ds \right|,$$  \hspace{1cm} (2.18)

$$\quad (I - Q) x(t) = -\int_0^t (f(s, x([s])) - p(s)) ds + \frac{t}{\omega} \int_0^\omega f(s, x([s])) ds, \quad t \geq 0. \hspace{1cm} (2.19)$$

These lead us to

$$K_P (I - Q) x(t) = -\int_0^t \frac{1}{r(v)} dv \int_0^\omega (f(s, x([s])) - p(s)) ds$$

$$\quad + a \left( \int_0^\omega \frac{dv}{r(v)} \int_0^\omega (f(s, x([s])) - p(s)) ds \right) \int_0^t \frac{1}{r(v)} dv$$

$$\quad + \frac{1}{\omega} \int_0^t \frac{v}{r(v)} dv \int_0^\omega f(s, x([s])) ds$$

$$\quad - \frac{a}{\omega} \left( \int_0^\omega \frac{v dv}{r(v)} \int_0^\omega f(s, x([s])) ds \right) \int_0^t \frac{1}{r(v)} dv,$$  \hspace{1cm} (2.20)

where $a$ is defined by (2.15). By (2.18), we see that $QN(\overline{\Omega})$ is bounded. Noting that (2.7) holds and $N$ is a completely continuous mapping, by means of the Arzela-Ascoli theorem we know that $K_P (I - Q) N(\overline{\Omega})$ is relatively compact. Thus $N$ is $L$-compact on $\overline{\Omega}$. The proof is complete.  \hfill \Box
Lemma 2.8. Suppose that \( g(t) \) is a real, bounded and continuous function on \([a, b]\) and \( \lim_{x \to b^-} g(t) \) exists. Then there is a point \( \xi \in (a, b) \) such that

\[
\int_a^b g(s) ds = g(\xi)(b - a). \tag{2.21}
\]

The above result is only a slight extension of the integral mean value theorem and is easily proved.

Lemma 2.9. Suppose that condition (i) in Theorem 2.1 holds. Suppose further that \( x(t) \in X_\omega \) satisfies

\[
\int_0^{\omega^*} f(s, x([s])) ds = 0. \tag{2.22}
\]

Then there is \( t_1 \in [0, \omega] \) such that \( |x(t_1)| \leq D \).

Proof. From (2.22) and Lemma 2.8, we have \( \xi_i \in (i - 1, i) \) for \( i = 1, \ldots, \omega \) such that

\[
\sum_{i=1}^{\omega} f(\xi_i, x(i - 1)) = \sum_{i=1}^{\omega} \int_{i-1}^{i} f(s, x([s])) ds \int_0^{\omega^*} f(s, x([s])) ds = 0. \tag{2.23}
\]

In case \( \omega = 1 \), from the condition (i) in Theorem 2.1 and (2.23), we know that \(|x(0)| \leq D\). Suppose \( \omega \geq 2 \). Our assertion is true if one of \( x(0), x(1), \ldots, x(\omega - 1) \) has absolute value less than or equal to \( D \). Otherwise, there should be \( x(\eta_1) \) and \( x(\eta_2) \) among \( x(0), x(1), \ldots, x(\omega - 1) \) such that \( x(\eta_1) > D \) and \( x(\eta_2) < -D \). Since \( x(t) \) is continuous, in view of the intermediate value theorem, there is \( x(\eta_3) \) such that \(-D \leq x(\eta_3) \leq D\) (here \( \eta_1 > \eta_3 > \eta_2 \) or \( \eta_2 > \eta_3 > \eta_1 \)). Since \( x(t) \) is periodic, there is \( t_1 \in [0, \omega] \) such that \(|x(t_1)| = |x(\eta_3)| \leq D\). The proof is complete. \( \square \)

Now, we consider the following equation:

\[
r(t)x'(t) - r(0)x'(0) = -\lambda \int_0^{\omega^*} (f(s, x([s])) - p(s)) ds, \tag{2.24}
\]

where \( \lambda \in (0, 1) \).

Lemma 2.10. Suppose that conditions (i) and (ii) of Theorem 2.1 hold. If \( \omega^2 \delta(\max_{0 \leq t < t_0}(1/r(t))) < 1 \), then there are positive constants \( D_0 \) and \( D_1 \) such that for any \( \omega \)-periodic solution \( x(t) \) of (2.24),

\[
\left| x^{(i)}(t) \right| \leq D_i, \quad t \in [0, \omega]; \quad i = 0, 1. \tag{2.25}
\]

Proof. Let \( x(t) \) be a \( \omega \)-periodic solution of (2.24). By (2.24) and our assumption that \( \int_0^{\omega^*} p(s) ds = 0 \), we have

\[
\int_0^{\omega^*} f(s, x([s])) ds = 0. \tag{2.26}
\]
By Lemma 2.9, there is $t_1 \in [0, \omega]$ such that

$$|x(t_1)| \leq D. \quad (2.27)$$

Since $x(t)$ and $x'(t)$ are with period $\omega$, thus for any $t \in [t_1, t_1 + \omega]$, we have

$$x(t) = x(t_1) + \int_{t_1}^{t} x'(s)ds, \quad (2.28)$$

$$x(t) = x(t_1 + \omega) + \int_{t_1}^{t_1 + \omega} x'(s)ds = x(t_1) + \int_{t_1}^{t_1 + \omega} x'(s)ds. \quad (2.29)$$

From (2.28), we see that for any $t \in [t_1, t_1 + \omega],$

$$|x(t)| \leq |x(t_1)| + \frac{1}{2} \int_{t_1}^{t_1 + \omega} |x'(s)|ds = |x(t_1)| + \frac{1}{2} \int_{0}^{\omega} |x'(s)|ds. \quad (2.30)$$

It is easy to see from (2.27) and (2.29) that for any $t \in [0, \omega]$

$$|x(t)| \leq |x(t_1)| + \frac{1}{2} \int_{0}^{\omega} |x'(s)|ds \leq D + \frac{1}{2} \int_{0}^{\omega} |x'(s)|ds. \quad (2.31)$$

In view of the condition $\omega^2 \delta (\max_{0 \leq t \leq \omega} (1/r(t))) < 1$, we know that there is a positive number $\epsilon$ such that

$$\eta_1 := \omega^2 (\delta + \epsilon) \left( \max_{0 \leq t \leq \omega} \frac{1}{r(t)} \right) < 1. \quad (2.32)$$

From condition (ii), we see that there is a $\rho > D$ such that for $t \in R$ and $x < -\rho,$

$$\frac{f(t, x)}{x} < \delta + \epsilon. \quad (2.33)$$

Let

$$E_1 = \{ t \mid t \in [0, \omega], x([t]) < -\rho \}, \quad (2.34)$$

$$E_2 = \{ t \mid t \in [0, \omega], |x([t])| \leq \rho \}, \quad (2.35)$$

$$E_3 = [0, \omega] \setminus (E_1 \cup E_2), \quad (2.36)$$

$$M_0 = \max_{0 \leq s \leq \omega} |f(t, x)|. \quad (2.37)$$

By (2.32) and (2.33), we have

$$\int_{E_1} |f(s, x([s]))|ds \leq (\delta + \epsilon) \int_{E_1} |x([s])|ds \leq (\delta + \epsilon) \omega \max_{0 \leq s \leq \omega} |x(t)|. \quad (2.38)$$
From (2.34) and (2.36), we have

$$\int_{E_3} |f(s, x([s]))| \, ds \leq \omega M_0. \tag{2.38}$$

In view of condition (i), (2.26), (2.37), and (2.38), we get

$$\int_{E_3} |f(s, x([s]))| \, ds = \int_{E_3} f(s, x([s])) \, ds
- \int_{E_1} f(s, x([s])) \, ds - \int_{E_2} f(s, x([s])) \, ds
\leq \int_{E_3} |f(s, x([s]))| \, ds + \int_{E_2} |f(s, x([s]))| \, ds
\leq (\delta + \varepsilon) \omega \max_{0 \leq s \leq \omega} |x(t)| + \omega M_0. \tag{2.39}$$

It follows from (2.37), (2.38), and (2.39) that

$$\int_{0}^{\omega} |f(s, x([s]))| \, ds = \int_{E_1} |f(s, x([s]))| \, ds + \int_{E_2} |f(s, x([s]))| \, ds + \int_{E_3} |f(s, x([s]))| \, ds
\leq 2(\delta + \varepsilon) \omega \max_{0 \leq s \leq \omega} |x(t)| + 2\omega M_0. \tag{2.40}$$

Since $x(0) = x(\omega)$, thus there is a $t_1 \in (0, \omega)$ such that $x'(t_1) = 0$. In view of (2.24) and the fact that $x'(t_1) = 0$, we conclude that for any $t \in [t_1, t_1 + \omega]$,

$$|r(t)x'(t)| = \left| r(t_1)x'(t_1) - \lambda \int_{t_1}^{t} (f(s, x([s])) - p(s)) \, ds \right|
= \left| -\lambda \int_{t_1}^{t} (f(s, x([s])) - p(s)) \, ds \right|
\leq \int_{t_1}^{t} |f(s, x([s])) - p(s)| \, ds
\leq \int_{t_1}^{t_1 + \omega} |f(s, x([s]))| \, ds + \int_{t_1}^{t_1 + \omega} |p(s)| \, ds
\leq \int_{0}^{\omega} |f(s, x([s]))| \, ds + \int_{0}^{\omega} |p(s)| \, ds. \tag{2.41}$$
Suppose that condition (iii) of Theorem 2.1 is satisfied. Then

\[ \text{Lemma 2.11.} \]

From (2.30), (2.31), and (2.42) that

\[
\max_{0 \leq t \leq \omega} |x(t)| \leq 2 \int_0^{\omega} |x(s)| ds
\]

\[
\leq \omega^2 \left( \max_{0 \leq t \leq \omega} \frac{1}{r(t)} \right) (\delta + \varepsilon) \max_{0 \leq t \leq \omega} |x(t)| + M_1
\]

\[
= \eta_1 \max_{0 \leq t \leq \omega} |x(t)| + M_1,
\]

where

\[
M_1 = D + \left( \max_{0 \leq t \leq \omega} \frac{1}{r(t)} \right) \left( 2\omega M_0 + \max_{0 \leq t \leq \omega} |p(t)| \right).
\]

Let \( D_0 = M_1 / (1 - \eta_1) \), then from (2.43) we have

\[ \max_{0 \leq t \leq \omega} |x(t)| \leq D_0. \]  

(2.45)

From (2.42) and (2.45), for any \( t \in [0, \omega] \), we have

\[ \max_{0 \leq t \leq \omega} |x'(t)| \leq D_1, \]

(2.46)

where

\[
D_1 = \left( \max_{0 \leq t \leq \omega} \frac{1}{r(t)} \right) \left( 2(\delta + \varepsilon)\omega D_0 + 2\omega M_0 + \max_{0 \leq t \leq \omega} |p(t)| \right).
\]

(2.47)

The proof is complete.

\[ \square \]

**Lemma 2.11.** Suppose that condition (iii) of Theorem 2.1 is satisfied. Then (1.3) has at most one \( \omega \)-periodic solution.

**Proof.** Suppose that \( x_1(t) \) and \( x_2(t) \) are two \( \omega \)-periodic solutions of (1.3). Set \( z(t) = x_1(t) - x_2(t) \). Then we have

\[ (r(t)z'(t))' + f(t, x_1([t])) - f(t, x_2([t])) = 0. \]

(2.48)
Case (i). For all \( t \in [0, \omega] \), \( z(t) \neq 0 \). Without loss of generality, we assume that \( z(t) > 0 \), that is, \( x_1(t) > x_2(t) \) for \( t \in [0, \omega] \). Integrating (2.48) from 0 to \( \omega \), we have
\[
\int_0^\omega \left[ f(t, x_1([t])) - f(t, x_2([t])) \right] dt = 0. \tag{2.49}
\]
Combining condition (iii) and \( x_1(t) > x_2(t) \), either
\[
f(t, x_1([t])) - f(t, x_2([t])) > 0, \quad t \in [0, \omega] \tag{2.50}
\]
or
\[
f(t, x_1([t])) - f(t, x_2([t])) < 0, \quad t \in [0, \omega] \tag{2.51}
\]
holds. This is contrary to (2.49).

Case (ii). There exist \( \xi \in [0, \omega] \) such that \( z(\xi) = 0 \). As in the proof of (2.30) in Lemma 2.10, we have
\[
\max_{0 \leq s \leq \omega} |z(t)| \leq |z(\xi)| + \frac{1}{2} \int_0^\omega |z'(s)| ds = \frac{1}{2} \int_0^\omega |z'(s)| ds. \tag{2.52}
\]
On the other hand, since \( z(0) = z(\omega) \), thus there is a \( t_1 \in (0, \omega) \) such that \( z'(t_1) = 0 \). In view of (2.48), we conclude that for any \( t \in [t_1, t_1 + \omega] \),
\[
r(t)z'(t) = r(t_1)z'(t_1) - \int_{t_1}^t \left( f(s, x_1([s])) - f(s, x_2([s])) \right) ds,
\]
\[
r(t)z'(t) = r(t_1 + \omega)z'(t_1 + \omega) - \int_{t_1 + \omega}^t \left( f(s, x_1([s])) - f(s, x_2([s])) \right) ds
\]
\[
= r(t_1)z'(t_1) - \int_{t_1}^{t_1 + \omega} \left( f(s, x_1([s])) - f(s, x_2([s])) \right) ds. \tag{2.53}
\]
By (2.53) and the fact that \( z'(t_1) = 0 \), we have for any \( t \in [t_1, t_1 + \omega] \),
\[
r(t)z'(t) = r(t_1)z'(t_1) - \frac{1}{2} \int_{t_1}^t \left( f(s, x_1([s])) - f(s, x_2([s])) \right) ds
\]
\[
+ \frac{1}{2} \int_{t}^{t_1 + \omega} \left( f(s, x_1([s])) - f(s, x_2([s])) \right) ds.
\]
\[
= -\frac{1}{2} \int_{t_1}^t \left( f(s, x_1([s])) - f(s, x_2([s])) \right) ds
\]
\[
+ \frac{1}{2} \int_{t}^{t_1 + \omega} \left( f(s, x_1([s])) - f(s, x_2([s])) \right) ds. \tag{2.54}
\]
It follows that for any \( t \in [t_1, t_1 + \omega] \),

\[
|r(t)z'(t)| \leq \frac{1}{2} \int_{t_1}^{t_1+\omega} |f(s, (x_1([s]))) - f(s, x_2([s]))| ds
\]

\[
\leq \frac{1}{2} \int_{0}^{\omega} |f(s, (x_1([s]))) - f(s, x_2([s]))| ds \tag{2.55}
\]

\[
\leq \frac{1}{2} b\omega \max_{0 \leq s \leq \omega} |z(t)|.
\]

We know that for any \( t \in [0, \omega] \),

\[
|r(t)z'(t)| \leq \frac{1}{2} b\omega \max_{0 \leq s \leq \omega} |z(t)|. \tag{2.56}
\]

From (2.56), we have

\[
\max_{0 \leq s \leq \omega} |z'(t)| \leq \frac{b\omega}{2} \left( \max_{0 \leq s \leq \omega} r(t) \right) \max_{0 \leq s \leq \omega} |z(t)|. \tag{2.57}
\]

By (2.52), we get

\[
\max_{0 \leq s \leq \omega} |z(t)| \leq \frac{\omega}{2} \max_{0 \leq s \leq \omega} |z'(t)|. \tag{2.58}
\]

It is easy to see from (2.57) and (2.58) that

\[
\max_{0 \leq s \leq \omega} |z(t)| \leq \frac{b\omega^2}{4} \left( \max_{0 \leq s \leq \omega} r(t) \right) \max_{0 \leq s \leq \omega} |z(t)|. \tag{2.59}
\]

By condition (iii) of Theorem 2.1, we see that \((b\omega^2/4)(\max_{0 \leq s \leq \omega} (1/r(t))) < 1\). Thus (2.58) leads us to \( \max_{0 \leq s \leq \omega} |z(t)| = 0 \), which is contrary to \( x_1 \neq x_2 \). So (1.3) has at most one \( \omega \)-periodic solution. The proof is complete. \( \Box \)

We now turn to the proof of Theorem 2.1. Suppose \( \omega^2 \delta(\max_{0 \leq s \leq \omega} (1/r(t))) < 1 \). Let \( L, N, P, \) and \( Q \) be defined by (2.3), (2.4), (2.6), and (2.7), respectively. By Lemma 2.10, there are positive constants \( D_0 \) and \( D_1 \) such that for any \( \omega \)-periodic solution \( x(t) \) of (2.24) such that (2.25) holds. Set

\[
\Omega = \{ x \in X_\omega \mid \|x\|_1 < \overline{D} \}, \tag{2.60}
\]

where \( \overline{D} \) is a fixed number which satisfies \( \overline{D} > D + D_0 + D_1 \). It is easy to see that \( \Omega \) is an open and bounded subset of \( X_\omega \). Furthermore, in view of Lemmas 2.5 and 2.7, \( L \) is a Fredholm mapping of index zero and \( N \) is \( L \)-compact on \( \Omega \). Noting that \( \overline{D} > D_0 + D_1 \), by Lemma 2.10, for each \( \lambda \in (0, 1) \) and \( x \in \partial \Omega \), \( Lx \neq \lambda Nx \). Next note that a function \( x \in \partial \Omega \cap \text{Ker} L \) must be
constant: \( x(t) \equiv \overline{D} \) or \( x(t) \equiv -\overline{D} \). Hence by (i) and (2.17), \( x(t) \equiv -\overline{D} \). Hence by conditions (i), (iii) and (2.17),

\[
QN x(t) = -\frac{1}{\omega} \int_{0}^{\omega} f(s, x([s])) ds = -\frac{1}{\omega} \int_{0}^{\omega} f(s, x) ds,
\]

so \( QN x \neq \theta_2 \). The isomorphism \( J : \text{Im} Q \rightarrow \text{Ker} L \) is defined by \( J(t\alpha) = \alpha \) for \( \alpha \in \mathbb{R} \) and \( t \in \mathbb{R} \). Then

\[
JQN x = -\frac{1}{\omega} \int_{0}^{\omega} f(s, x) ds \frac{1}{\omega} \neq 0.
\]

In particular, we see that if \( x = \overline{D} \), then

\[
JQN x = -\frac{1}{\omega} \int_{0}^{\omega} f(s, \overline{D}) ds < 0,
\]

and if \( x = -\overline{D} \), then

\[
JQN x = -\frac{1}{\omega} \int_{0}^{\omega} f(s, -\overline{D}) ds > 0.
\]

Consider the mapping

\[
H(x, \mu) = \mu x + (1 - \mu) JQN x, \quad 0 \leq \mu \leq 1.
\]

From (2.63) and (2.65), for each \( \mu \in [0, 1] \) and \( x = \overline{D} \), we have

\[
H(x, \mu) = \mu \overline{D} + (1 - \mu) \frac{1}{\omega} \int_{0}^{\omega} f(s, \overline{D}) ds < 0.
\]

Similarly, from (2.64) and (2.65), for each \( \mu \in [0, 1] \) and \( x = -\overline{D} \), we have

\[
H(x, \mu) = \mu \overline{D} + (1 - \mu) \frac{1}{\omega} \int_{0}^{\omega} f(s, -\overline{D}) ds < 0.
\]

By (2.66) and (2.67), \( H(x, \mu) \) is a homotopy. This shows that

\[
\deg(JQN x, \Omega \cap \text{Ker} L, \theta_1) = \deg(-x, \Omega \cap \text{Ker} L, \theta_1) \neq 0.
\]

By Theorem A, we see that equation \( L x = N x \) has at least one solution in \( \overline{\Omega} \cap \text{Dom} L \). In other words, (1.3) has an \( \omega \)-periodic solution \( x(t) \). Furthermore, if (iii) is satisfied, from Lemma 2.11, we know that (1.3) has an \( \omega \)-periodic solution only. The proof is complete.
3. Example

Consider the equation

\[ \left( x'(t) \exp\left(-2 - \cos \frac{2\pi t}{5}\right) \right)' + \left( 3 - \sin \frac{2\pi t}{5} \right) \arctan x([t]) = \cos \frac{2\pi t}{5}, \]  

(3.1)

and we can show that it has a nontrivial 5-periodic solution. Indeed, take

\[ r(t) = \exp\left(2 - \cos \frac{2\pi t}{5}\right), \quad p(t) = \cos \frac{2\pi t}{5}, \]

\[ f(t, x) = \frac{1}{100} \left( 3 - \sin \frac{2\pi t}{5} \right) \arctan x. \]

(3.2)

We see that \( \min_{0 \leq t \leq 5} r(t) = e \). Let \( D > 0 \) and \( \delta = b = 1/25 \). Then condition (i) of Theorem 2.1 is satisfied:

\[ \lim_{x \to -\infty} \max_{0 \leq t \leq 5} \frac{f(t, x)}{x} = \frac{1}{25}. \]

(3.3)

Let \( D > 0 \) and \( \delta = b = 1/25 \). Then conditions (i), (ii) and (iii), of Theorem 2.1 are satisfied. Note further that \( 5^2 \delta (\max_{0 \leq t \leq 5} (1/r(t))) = e^{-1} < 1 \). Therefore (3.1) has exactly one 5-periodic solution. Furthermore, it is easy to see that any solution of (3.1) must be nontrivial. We have thus shown the existence of a unique nontrivial 5-periodic solution of (3.1).

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References


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