Research Article

On $(N, p, q)(E, 1)$ Summability of Fourier Series

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A new theorem on $(N, p, q)(E, 1)$ summability of Fourier series has been established.

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1. Introduction

Let $\{p_n\}$ and $\{q_n\}$ be the sequences of constants, real or complex, such that

$$\begin{aligned} P_n &= p_0 + p_1 + p_2 + \cdots + p_n = \sum_{v=0}^n p_v \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \\ Q_n &= q_0 + q_1 + q_2 + \cdots + q_n = \sum_{v=0}^n q_v \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \\ R_n &= p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0 = \sum_{v=0}^n p_v q_{n-v} \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{1.1}$$

Given two sequences $\{p_n\}$ and $\{q_n\}$ convolution $(p * q)$ is defined as

$$R_n = (p * q)_n = \sum_{k=0}^n p_{n-k} q_k. \tag{1.2}$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of its n th partial sums $\{s_n\}$.

We write

$$t_n^{p,q} = \frac{1}{R_n} \sum_{k=0}^n p_{n-k} q_k s_k. \quad (1.3)$$

If $R_n \neq 0$, for all n , the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$.

If $t_n^{p,q} \rightarrow S$, as $n \rightarrow \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or sequence $\{s_n\}$ is summable to S by generalized Nörlund method (Borwein [1]) and is denoted by

$$S_n \longrightarrow S(N, p, q). \quad (1.4)$$

The necessary and sufficient conditions for (N, p, q) method to be regular are

$$\sum_{k=0}^n |p_{n-k} q_k| = O(|R_n|), \quad (1.5)$$

and $p_{n-k} = o(|R_n|)$, as $n \rightarrow \infty$ for every fixed $k \geq 0$, for which $q_k \neq 0$.

Now

$$E_n^1 = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} s_k. \quad (1.6)$$

If $E_n^1 \rightarrow s$, as $n \rightarrow \infty$, then the series $\sum_{n=0}^{\infty} u_n$ is said to be $(E, 1)$ summable to s (Hardy [2]):

$$\begin{aligned} t_n^{p,q,E} &= \frac{1}{R_n} \sum_{k=0}^n p_{n-k} q_k E_k^1 \\ &= \frac{1}{R_n} \sum_{k=0}^n p_{n-k} q_k \cdot \frac{1}{2^k} \sum_{v=0}^k \binom{k}{v} s_v. \end{aligned} \quad (1.7)$$

If $t_n^{p,q,E} \rightarrow \infty$, as $n \rightarrow \infty$, then we say that the series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is summable to S by $(N, p, q)(E, 1)$ summability method.

Particular Cases

- (1) $(N, p, q)(E, 1)$ mean reduces to $(N, p_n)(E, 1)$ summability mean if $q_n = 1$, $\forall n$.
- (2) $(N, p, q)(E, 1)$ mean reduces to $(N, 1/(n+1))(E, 1)$ mean if $p_n = 1/(n+1)$ and $q_n = 1$, $\forall n$.
- (3) $(N, p, q)(E, 1)$ method reduces to $(\overline{N}, q_n)(E, 1)$ if $p_n = 1$, $\forall n$.
- (4) $(N, p, q)(E, 1)$ method reduces to $(C, \alpha)(E, 1)$ if $p_n = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$, and $q_n = 1$, $\forall n$.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$.

Let its Fourier series be given by

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \quad (1.8)$$

We use the following notations:

$$\begin{aligned} \phi(t) &= f(x+t) - f(x-t) - 2f(x), \\ \Phi(t) &= \int_0^t |\phi(u)| du, \\ \tau = \left[\frac{1}{t} \right] &= \text{the integral part of } \frac{1}{t}, \\ R\left(\frac{1}{t}\right) &= R_\tau, \quad R_n = R(n), \\ K_n(t) &= \frac{1}{2\pi R_n} \sum_{k=0}^n p_{n-k} q_k \frac{\cos^k(t/2) \cos(k+1)(t/2)}{\sin(t/2)}. \end{aligned} \quad (1.9)$$

2. Theorem

A quite good amount of work is known for Fourier series by ordinary summability method. The purpose of this paper is to study Fourier series by $(N, p, q)(E, 1)$ summability method in the following form.

Theorem 2.1. Let $\{p_n\}$ and $\{q_n\}$ be positive monotonic, nonincreasing sequences of real numbers such that

$$R_n = \sum_{k=0}^n p_k q_{n-k} \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \quad (2.1)$$

Let $\alpha(t)$ be a positive, nondecreasing function of t . If

$$\Phi(t) = \int_0^t |\phi(u)| du = o\left(\frac{t}{\alpha(1/t)}\right), \quad \text{as } t \longrightarrow +0, \quad (2.2)$$

$$\alpha(n) \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \quad (2.3)$$

then a sufficient condition that the Fourier Series (1.8) be summable $(N, p, q)(E, 1)$ to $f(x)$ at the point $t = x$ is

$$\int_1^n \frac{R(u)}{u\alpha(u)} du = O(R_n), \quad \text{as } n \longrightarrow \infty. \quad (2.4)$$

3. Lemmas

Proof of the theorem needs some lemmas.

Lemma 3.1. For $0 \leq t \leq 1/n$,

$$|K_n(t)| = O(n). \quad (3.1)$$

Proof.

$$\begin{aligned} |K_n(t)| &= \frac{1}{2\pi R_n} \left| \sum_{k=0}^n p_{n-k} q_k \frac{\cos^k(t/2) \sin(k+1)(t/2)}{\sin(t/2)} \right| \\ &\leq \frac{1}{2\pi R_n} \sum_{k=0}^n p_{n-k} q_k \frac{(k+1)|\sin(t/2)|}{|\sin(t/2)|} = O(n) \frac{1}{R_n} \sum_{k=0}^n p_{n-k} q_k = O(n). \end{aligned} \quad (3.2)$$

□

Lemma 3.2. If $\{p_n\}$ and $\{q_n\}$ are nonnegative and nonincreasing, then for $0 \leq a \leq b < \infty$, $0 \leq t \leq \pi$, and any n we have

$$\frac{1}{2\pi R_n} \left| \sum_{k=a}^b p_{n-k} q_k \frac{\cos^k(t/2) \sin(k+1)(t/2)}{\sin(t/2)} \right| = O\left(\frac{R_\tau}{tR_n}\right). \quad (3.3)$$

Proof.

$$\begin{aligned} &\frac{1}{2\pi R_n} \left| \sum_{k=a}^b p_{n-k} q_k \frac{\cos^k(t/2) \sin(k+1)(t/2)}{\sin(t/2)} \right| \\ &\leq \frac{1}{t\pi R_n} \left| \sum_{k=a}^b p_{n-k} q_k \cos^k\left(\frac{t}{2}\right) \sin(k+1) \frac{t}{2} \right| \\ &= \frac{1}{t\pi R_n} \left| \operatorname{Im} \left\{ \sum_{k=a}^b p_{n-k} q_k \cos^k\left(\frac{t}{2}\right) e^{i(k+1)(t/2)} \right\} \right| \\ &\leq \frac{1}{t\pi R_n} \left| \sum_{k=a}^b p_{n-k} q_k \cos^k\left(\frac{t}{2}\right) e^{ikt/2} \right| \left| e^{it/2} \right| \\ &\leq \frac{1}{t\pi R_n} \left| \sum_{k=a}^b p_{n-k} q_k \cos^k\left(\frac{t}{2}\right) e^{ikt/2} \right| \\ &\leq \frac{1}{t\pi R_n} \left\{ \left| \sum_{k=a}^{\tau-1} p_{n-k} q_k \cos^k\left(\frac{t}{2}\right) e^{\frac{ikt}{2}} \right| + \left| \sum_{k=\tau}^b p_{n-k} q_k \cos^k\left(\frac{t}{2}\right) e^{ik\left(\frac{t}{2}\right)} \right| \right\}. \end{aligned} \quad (3.4)$$

Now considering first term of (3.4), we have

$$\begin{aligned} \frac{1}{t\pi R_n} \left| \sum_{k=a}^{\tau-1} p_{n-k} q_k \cos^k \left(\frac{t}{2} \right) e^{ik(t/2)} \right| &\leq \frac{1}{t\pi R_n} \sum_{k=a}^{\tau-1} p_{n-k} q_k \left| e^{ik(t/2)} \right| \leq \frac{1}{t\pi R_n} \sum_{k=a}^{\tau-1} p_{n-k} q_k \\ &\leq \frac{1}{t\pi R_n} \sum_{k=a}^{\tau-1} p_{\tau-k} q_k \leq \frac{1}{t\pi R_n} (R_\tau) = O\left(\frac{R_\tau}{tR_n}\right). \end{aligned} \quad (3.5)$$

Now considering second term of (3.4) and using Abel's lemma, we have

$$\begin{aligned} \frac{1}{t\pi R_n} \left| \sum_{k=\tau}^b p_{n-k} q_k \cos^k \left(\frac{t}{2} \right) e^{ik(t/2)} \right| &\leq \frac{1}{t\pi R_n} \left| \sum_{k=\tau}^b p_{n-k} q_k e^{ik(t/2)} \right| \\ &\leq \frac{2p_{n-b} q_\tau}{t\pi R_n} \max_{\tau+1 \leq k \leq b} \left| \frac{1 - e^{i(k+1)(t/2)}}{1 - e^{it/2}} \right| \\ &\leq \frac{4p_{n-b} q_\tau}{t\pi R_n} \left| \frac{e^{-it/4}}{e^{it/4} - e^{-it/4}} \right| \\ &\leq \frac{2q_\tau}{t\pi R_n} \left(\frac{p_{n-b}}{P_\tau} \right) P_\tau \left| \frac{1}{\sin(t/4)} \right| \quad \left(\text{where } P_\tau = \sum_{k=0}^{\tau} p_{\tau-k} \right) \\ &\leq \frac{8q_\tau}{t\pi R_n} \left(\frac{p_{n-b}}{P_\tau} \right) P_\tau \left| \frac{1}{t} \right| \\ &\leq \frac{8q_\tau P_\tau}{t\pi R_n} \\ &\leq \frac{8R_\tau}{t\pi R_n} \quad \left(\text{since } R_\tau = \sum_{k=0}^{\tau} p_{\tau-k} q_k \geq P_\tau q_\tau \right) \\ &= O\left(\frac{R_\tau}{tR_n}\right). \end{aligned} \quad (3.6)$$

Using (3.5) and (3.6), we get the required result of Lemma 3.2. $\square$

4. Proof of Theorem

Following Zygmund [3], the n th partial sum $s_n(x)$ of the series (1.8) at $t = x$ is given by

$$s_n(x) = f(x) + \frac{1}{2\pi} \int_0^\pi \phi_x(t) \frac{\sin(n+1/2)t}{\sin(t/2)} dt. \quad (4.1)$$

So the $(E, 1)$ mean of the series (1.8) at $t = x$ is given by

$$\begin{aligned}
 E_n^1(x) &= \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} s_k(x) \\
 &= f(x) + \frac{1}{2^{n+1}\pi} \int_0^\pi \frac{\phi_x(t)}{\sin(t/2)} \left\{ \sum_{k=0}^n \binom{n}{k} \sin\left(k + \frac{1}{2}\right)t \right\} dt \\
 &= f(x) + \frac{1}{2^{n+1}\pi} \int_0^\pi \frac{\phi_x(t)}{\sin(t/2)} \operatorname{Im} \left\{ e^{it/2} (1 + e^{it})^n \right\} dt \\
 &= f(x) + \frac{1}{2^{n+1}\pi} \int_0^\pi \frac{\phi_x(t)}{\sin(t/2)} \operatorname{Im} \left\{ e^{it/2} (1 + \cos t + i \sin t)^n \right\} dt \quad (4.2) \\
 &= f(x) + \frac{1}{2^{n+1}\pi} \int_0^\pi \frac{\phi_x(t)}{\sin(t/2)} \operatorname{Im} \left\{ e^{it/2} 2^n \cos^n\left(\frac{t}{2}\right) \left(\cos \frac{t}{2} + i \sin \frac{t}{2}\right)^n \right\} dt \\
 &= f(x) + \frac{1}{2^{n+1}\pi} \int_0^\pi \frac{\phi_x(t)}{\sin(t/2)} \operatorname{Im} \left\{ e^{it/2} 2^n \cos^n\left(\frac{t}{2}\right) \left\{ \cos \frac{nt}{2} + i \sin \frac{nt}{2} \right\} \right\} dt \\
 &= f(x) + \frac{1}{2\pi} \int_0^\pi \phi_x(t) \frac{\cos^n(t/2) \sin(n+1)(t/2)}{\sin(t/2)} dt.
 \end{aligned}$$

Therefore (N, p, q) transform of $\{E_n^1(x)\}$ is given by

$$\begin{aligned}
 t_n^{p,q,E}(x) &= \frac{1}{R_n} \sum_{k=0}^n p_{n-k} q_k E_k^1(x) = f(x) + \frac{1}{2\pi} \int_0^\pi \frac{1}{R_n} \sum_{k=0}^n p_{n-k} q_k \phi_x(t) \frac{\cos^k(t) \sin(k+1)(t/2)}{\sin(t/2)} \\
 &= f(x) + \int_0^\pi K_n(t) \phi_x(t) dt, \\
 t_n^{p,q,E}(x) - f(x) &= \left[\int_0^{1/n} + \int_{1/n}^\delta + \int_\delta^\pi \right] K_n(t) \phi_x(t) dt = I_1 + I_2 + I_3 \quad (\text{say}). \quad (4.3)
 \end{aligned}$$

We have

$$\begin{aligned}
 |I_1| &\leq \int_0^{1/n} |K_n(t)| |\phi_x(t)| dt \\
 &= O(n) \int_0^{1/n} |\phi_x(t)| dt \quad (\text{using Lemma 3.1}) \\
 &= O(n) o\left(\frac{1}{n\alpha(n)}\right) \quad (\text{by (2.2)}) \\
 &= o\left(\frac{1}{\alpha(n)}\right) = o(1) \quad \text{as } n \rightarrow \infty \quad (\text{by (2.3)}). \quad (4.4)
 \end{aligned}$$

Now we consider

$$\begin{aligned}
|I_2| &\leq \int_{1/n}^{\delta} |K_n(t)| |\phi_x(t)| dt \quad (\text{where } 0 < \delta < 1) \\
&= \int_{1/n}^{\delta} O\left(\frac{R(1/t)}{tR_n}\right) |\phi_x(t)| dt \quad (\text{using Lemma 3.2}) \\
&= O\left(\frac{1}{R_n}\right) \int_{1/n}^{\delta} \left(\frac{R(1/t)}{t}\right) |\phi_x(t)| dt \\
&= O\left(\frac{1}{R_n}\right) \left[\left\{ \left(\frac{R(1/t)}{t}\right) \phi_x(t) \right\}_{1/n}^{\delta} - \int_{1/n}^{\delta} d\left(\frac{R(1/t)}{t}\right) \phi_x(t) \right] \\
&= O\left(\frac{1}{R(n)}\right) \left[\left\{ o\left(\frac{R(1/t)}{\alpha(1/t)}\right) \right\}_{1/n}^{\delta} - \int_{1/n}^{\delta} \phi_x(t) d\left(\frac{R(1/t)}{t}\right) \right] \quad (\text{by (2.1)}) \\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{R(n)}\right) \left[\int_{1/n}^{\delta} \phi_x(t) \left\{ d\left(\frac{R(1/t)\alpha(1/t)}{t\alpha(1/t)}\right) \right\} \right] \\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{R(n)}\right) \\
&\quad \times \left[\int_{1/n}^{\delta} o\left(\frac{t}{\alpha(1/t)}\right) \left\{ d\alpha(1/t) \left(\frac{R(1/t)}{t\alpha(1/t)}\right) \right\} + \alpha\left(\frac{1}{t}\right) d\left(\frac{R(1/t)}{t\alpha(1/t)}\right) \right] \quad (4.5) \\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o(1) \left[\int_{1/n}^{\delta} \frac{d\alpha(1/t)}{\{\alpha(1/t)\}^2} + o\left(\frac{1}{R(n)}\right) \int_{1/n}^{\delta} t d\left(\frac{R(1/t)}{t\alpha(1/t)}\right) \right] \\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o(1) \left\{ \frac{1}{\alpha(1/t)} \right\}_{1/n}^{\delta} \\
&\quad + o\left(\frac{1}{R(n)}\right) \left[\left\{ \frac{tR(1/t)}{t\alpha(1/t)} \right\}_{1/n}^{\delta} - \int_{1/n}^{\delta} \left(\frac{R(1/t)}{t\alpha(1/t)}\right) dt \right] \\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{R(n)}\right) \int_{1/n}^1 \left(\frac{R(1/t)}{t\alpha(1/t)}\right) dt \\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{R(n)}\right) \int_1^n \left(\frac{R(u)}{u\alpha(u)}\right) du \quad \left\{ \because \frac{1}{t} = u \right\} \\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{R(n)}\right) O(R_n) \quad (\text{by (2.4)}) \\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o(1) \\
&= o(1), \quad \text{as } n \rightarrow \infty \quad (\text{by virtue of (2.1) and (2.2)}).
\end{aligned}$$

Now by Riemann-Lebesgue theorem and by regularity of the method of summability we have

$$\begin{aligned} I_3 &= \int_{\delta}^{\pi} |k_n(t)| |\phi_x(t)| dt \\ &= o(1), \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (4.6)$$

This completes the proof of the theorem.

5. Corollaries

Following corollaries can be derived from our main theorem.

Corollary 5.1. *If*

$$\Phi(t) = o\left[\frac{t}{\log(1/t)}\right], \quad \text{as } t \rightarrow +0, \quad (5.1)$$

then the Fourier series (1.8) is $(C, 1)(E, 1)$ summable to $f(x)$ at the point $t = x$.

Corollary 5.2. *If*

$$\Phi(t) = o(t), \quad \text{as } t \rightarrow +0, \quad (5.2)$$

then the Fourier series (1.8) is $(N, p_n)(E, 1)$ summable to $f(x)$ at the point $t = x$, provided that $\{p_n\}$ be a positive, monotonic, and nonincreasing sequence of real numbers such that

$$p_n = p_0 + p_1 + \cdots + p_n \rightarrow \infty, \quad \text{as } n \rightarrow \infty. \quad (5.3)$$

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