Research Article

Abnormal Curves on the Goursat Systems of $\mathbb{R}^n$

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Received 9 May 2010; Accepted 27 December 2010

Academic Editor: Frédéric Robert

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We study the abnormal and the rigid curves of the 2-distributions of $\mathbb{R}^n$ satisfying everywhere the Goursat condition. We give the directions for the rigid and the abnormal curves when the systems satisfy the strong Goursat condition or when they have a singularity of order 2 in each dimension.

1. Introduction

Let $E$ be a 2-distribution on $\mathbb{R}^n$. We denote by

$$
E^1 = E_1 = E, \quad E^i = [E^{i-1}, E^{i-1}], \quad E_i = [E, E_{i-1}].
$$

(1.1)

A small growth vector (sgv) of $E$, at a point $p \in \mathbb{R}^n$, is the sequence

$$
[r_1(p), r_2(p), \ldots]_S,
$$

(1.2)

where $r_i(p) = \dim E_i(p)$, for every $i \geq 1$.

The great growth vector, at $p$, is the sequence

$$
[m_1(p), m_2(p), \ldots]_G,
$$

(1.3)

where $m_j(p) = \dim E^j(p)$, for every $j \geq 1$.

If the dimensions of $E_i$ (resp., $E^i$) are independent of $p$, then the distribution is called regular (resp., totally regular).

If the great growth vector, at a point $p \in \mathbb{R}^n$, is $[2, 3, 4, \ldots, n]_G$, then the distribution is called distribution satisfying the Goursat condition at $p$. Moreover, if $E$ satisfies, on a
neighborhood of \( p \), the Goursat condition, then its annihilator, \( E^\perp \), is called \textit{Goursat system} and denoted by (GS).

The classification of the distributions, with respect to the small and great growth vectors, was the object of many articles. The beginning was by Engel \([1]\), where he gave the normal form of the (GS) in dimension 4.

In an article written in 1910, Cartan \([2]\) studied the case of dimension 5. In 1978 Giaro et al. completed the work of Cartan about the systems of dimension 5 \([3]\). In such a case 2 nonequivalent models are presented. In 1981, Kumpera and Ruiz \([4]\) gave the different normal forms in dimension \( n \leq 6 \).

The classification, of models, in dimensions 7 and 8 are given by \([5]\). The study of the models in dimension \( n \) is also open. We say that \([6]\), when the small and the great growth vector are the same, we have the system \( \text{GNF} \).

Zhitomirskiĭ \([7]\) gave the asymptotic normal forms of the regular distributions and the generic case studied in many articles, for example \([8]\).

The normal form of the model, satisfying at a neighborhood of a point the small growth vector \([2,3,4,4,5,5,\ldots,n-1,n-1,n]_S\), is given in \([9]\).

2. Rigid and Abnormal Line Subdistributions of the Goursat Systems Satisfying the Strong Condition of Goursat

The \textit{Goursat systems} are given by the following theorem.

**Theorem 2.1** (see \([4,5]\)). Let \( E \) be a 2-distribution on \( \mathbb{R}^n \), satisfying in each point, the condition of Goursat, then

\[
E^\perp = \begin{cases} 
\omega_1 = dx_2 + x_3dx_1, \\
\omega_2 = dx_3 + x_4dx_1, \\
\omega_3 = dx_{i_3} + x_3dx_{j_3}, \quad (i_3,j_3) \in \{(4,1),(1,4)\}, \\
\omega_4 = dx_{i_4} + x_6dx_{j_4}, \quad (i_4,j_4) \in \{(5,3),(3,5)\}, \\
\vdots \\
\omega_{n-2} = dx_{i_{n-2}} + x_{n}dx_{j_{n-2}}, \quad (i_{n-2},j_{n-2}) \in \{(n-1,n-3),(n-3,n-1)\},
\end{cases}
\]

where

\[
X_l = \begin{cases} 
x_l, & \text{if } (i_l,j_l) = (j_{l-3},l-1), \\
x_l + c_l, & \text{if } (i_l,j_l) = (l-1,j_{l-3}),
\end{cases}
\]

for \( 6 \leq l \leq n \) and \( c_6, c_7, \ldots, c_{n-2} \) are real arbitrary constants.

This theorem gives the different Goursat systems denoted by (GS).
Definition 2.2. Let $E$ be a 2-distribution on $\mathbb{R}^n$, $p \in \mathbb{R}^n$. $E$ satisfies the strong condition of Goursat, at $p$, if the small and the big growth vectors, at this point, are $[2,3,\ldots,n]_S$ and $[2,3,\ldots,n]_B$.

Theorem 2.3 (see [6]). Let $E$ be a 2-distribution on $\mathbb{R}^n$ satisfying, in each point, the condition of Goursat. Suppose that $E$ satisfies the strong condition of Goursat, at a point $p \in \mathbb{R}^n$, then there exists a local coordinate system $(x,U)$, around $p$, such that

$$E^\perp = \begin{cases} 
\omega_1 = dx_2 + x_3dx_1, \\
\omega_2 = dx_3 + x_4dx_1, \\
\omega_3 = dx_4 + x_5dx_1, \\
\omega_4 = dx_5 + x_6dx_1, \\
\vdots \\
\omega_{n-2} = dx_{n-1} + x_n dx_1,
\end{cases}$$

(2.3)

it means that $E$ is spanned by $v_1 = \partial/\partial x_n$ and

$$v_2 = \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} - \cdots - x_n \frac{\partial}{\partial x_{n-1}}.$$  

(2.4)  

Remark that, in this theorem, $E$ satisfies the strong condition of Goursat, at a point $p$. Such property can be extended without difficulty to a neighborhood of $p$. For the definitions of abnormal and rigid curves, see [10].

Definition 2.4. Let $E$ be a 2-distribution on $M$; a $C^1$-curve $\gamma : [\alpha,\beta] \to M$ is said to be horizontal (or $E$-curve) if $\gamma(t) \in E(\gamma(t))$, for any $t \in [\alpha,\beta]$.

The set of horizontal curves connecting two points $a$ and $b$ of $M$, will be denoted by $\Omega_{a,b}([\alpha,\beta])$. The theorem of Chow [11] certified that $\Omega_{a,b}([\alpha,\beta]) \neq \emptyset$, for any $a,b \in M$.

Definition 2.5. Let $E$ be a 2-distribution on $M$, a $C^1$-curve $\gamma : [\alpha,\beta] \to M$ is said to be rigid, if $\gamma$ is an isolated point of $\Omega_{a,b}([\alpha,\beta])$ for the $C^1$-topology.

Definition 2.6. Let $E$ be a 2-distribution on $M$. A line subdistribution (i.e., distribution of dimension one) of $L$ is said to be rigid, if any $L$-curve is rigid. $L$ is said to be local rigid, if for any $p \in M$, there exists a neighborhood $U$ of $p$, such that any $LU$-curve is rigid.

If $E$ is a 2-distribution on $M$, we denote $\Omega_{a}([\alpha,\beta])$ the set of $E$-curves $\gamma : [\alpha,\beta] \to M$, starting from the point $a$.

Definition 2.7. A curve $\gamma \in \Omega_{a}([\alpha,\beta])$, is said to be abnormal, if the mapping end: $\Omega_{a}([\alpha,\beta]) \to M$, defined by end $(\gamma) = \gamma(\beta)$, is not a submersion at $\gamma$.  

Proposition 2.8 (see [10]). Let $E$ be a $k$-distribution on $M$. If $v_1, v_2, \ldots, v_k$ form a basis of $E$ and if $\gamma \in \Omega_2([\alpha, \beta])$, such that $\gamma'(t) = u_1(t)v_1 + \cdots + u_k(t)v_k|_{1(t)}$, then the following propositions are equivalent.

1. $\gamma$ is abnormal.
2. There exists a lift curve $\Gamma : [\alpha, \beta] \to T^*M$, absolutely continuous, of coordinates $(q_1, q_2, \ldots, q_n)$, such that
   
   (a) $\Gamma(t) \neq 0$, for any $t \in [\alpha, \beta]$,
   
   (b) $\Gamma(t) \in E^1$,
   
   (c) $\Gamma$ satisfies the equation $(q_1', q_2', \ldots, q_n') = u_1(t)(q_1, q_2, \ldots, q_n)dv_1 + \cdots + u_k(t)(q_1, q_2, \ldots, q_n)dv_k|_{1(t)}$.

Definition 2.9. Let $E$ be a 2-distribution on $M$; a line subdistribution $L$ is said to be abnormal, if any $L$-curve is abnormal. $L$ is said to be local abnormal, if for any $p \in M$, there exists a neighborhood $U$ of $p$, such that any $L_U$-curve is abnormal.

Definition 2.10. Let $E$ be a 2-distribution on $M$; a distribution $D$ on $M$ is said to be nice with respect to $E$ if $D$ is an involutive distribution of codimension 2 such that $E_p \notin D_p$ and $\dim(E_p \cap D_p) = 2$, for any point $p \in M$.

Proposition 2.11 (see [10]). Let $E$ be a 2-distribution on $\mathbb{R}^n$ and $L$ be a line subdistribution on $E$. Consider the following properties.

(a) $L$ is locally rigid.

(b) $L$ is locally abnormal.

(c) Locally $L$ is the intersection of $E$ and a nice distribution.

(d) $\dim(ad_L)|_p < n$, for every $p \in \mathbb{R}^n$.

Then, one has the following implication:

\[(a) \quad (c) \quad \Rightarrow \quad (b) \quad \Rightarrow \quad (d)\]  \hspace{1cm} (2.5)

Zhitomirskii, in [10], conjectured that $(d) \Rightarrow (b)$, and he proved that $(a)$, $(b)$, $(c)$, and $(d)$ are not equivalent in general. Now we prove that, The properties are equivalent if the distribution satisfies the strong condition of Goursat.

Theorem 2.12. Let $E$ be a 2-distribution on $\mathbb{R}^n$, $n \geq 4$, satisfying in each point the strong condition of Goursat, then the properties (a), (b), (c), and (d) are equivalent.

Proof. By Theorem 2.3, $E$ is spanned, on a neighborhood $U$, by

\[v_1 = \frac{\partial}{\partial x_n'}, \quad v_2 = \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} - \cdots - x_n \frac{\partial}{\partial x_{n-1}}.\]  \hspace{1cm} (2.6)
Let $L$ be a line subdistribution satisfying (d), and let $u = av_1 + bv_2$ be a generator of $L$. We have

$$[v_2, v_1] = \frac{\partial}{\partial x_{n-1}}, \quad [v_2, [v_2, v_1]] = \frac{\partial}{\partial x_{n-2}}. \quad (2.7)$$

Easily, by induction, we say that

$$[ad^i_{v_2}, v_1] = \frac{\partial}{\partial x_{n-i}}, \quad [v_1, [ad^i_{v_2}, v_1]] = \left[ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-i}} \right] = 0, \quad (2.8)$$

for every $i = 1, 2, \ldots, n - 2$.

A simple induction shows that

$$ad^i_u(v_1) = a_1 v_1 + a_2 v_2 \sum_{j=1}^{i+1} a_j ad^j_{v_2}(v_1) + b_i ad^i_{v_2}(v_1),$$

where $a^j$, for $j = 1, 2, \ldots, i$, are $C^\infty$ functions on $U$ to $\mathbb{R}$. Because $\dim(ad^i_u)e < n$, for every $p \in \mathbb{R}^n$, we have necessarily $b = 0$ and by consequently $L$ is spanned by $v_1$.

Prove now (d) $\Rightarrow$ (c). Let $Z = \ker(dx_1 \wedge dx_2)$, we say easily $Z$ is a nice distribution (see [10]). In fact: $v_1(x_1) = v_1(x_2) = 0$, then $v_1 \in E \cap Z$. Otherwise $[v_1, v_2] = -\partial/\partial x_{n-1}$, then $[v_1, v_2](x_1) = [v_1, v_2](x_2) = 0$, we deduce that $E^2 \cap Z = \{v_1, [v_1, v_2]\}$ and consequently $\dim(E^2 \cap Z)_p = 2$.

Now cod$(Z) = 2$ and $Z$ is integrable. Because $v_2(x_1) = 0$, we obtain $E_p$ is not a subset of $Z_p$ for every $p \in \mathbb{R}^n$, then $Z$ is a nice distribution. Moreover $L = E \cap Z$, then (d) $\Rightarrow$ (c), by [10].

Prove now (d) $\Rightarrow$ (a). Consider the form $\omega_{n-2}$ of the system $E^\perp$. We have $(\omega_{n-2})_0 = (dx_{n-2})_0 \neq 0$ and

$$i_{v_1}d\omega_{n-2} = i_{v_1}(dx_{n-1} \wedge dx_1) = i_0/\partial x_n(dx_{n-1} \wedge dx_1) = 0. \quad (2.10)$$

Otherwise $[v_2, [v_1, v_2]] = -\partial/\partial x_{n-2}$ then $\omega([v_2, [v_1, v_2]]) = -1 \neq 0$ and $\omega_0$ is not in $E^3|_0$. By Theorem 5.7 of [10], $L$ is locally rigid. \hfill \Box

Let $E$ be a 2-distribution of $\mathbb{R}^n$, spanned by $v_1$ and $v_2$. $L_E$ is the line subdistribution spanned by a vector field in the form $av_1 + bv_2$, where $a$ and $b$ are such that $a[v_1, [v_1, v_2]] + b[v_2, [v_1, v_2]]$ is in $E^2$ and $(a^2 + b^2 \neq 0)$. We say that $L_E$ is independent of the choice of $v_1$ and $v_2$. Zhitomirskii [10] proved that $L_E$ is a line subdistribution locally rigid, also by a conjecture, it is unique, in the case where $E$ is regular and satisfying the condition

$$\dim E^2 = 3, \quad \dim E^3 = 4, \quad (2.11)$$

this is the case of (GS$_1$).
3. Rigid and Abnormal Line Subdistributions of the Goursat Systems Presenting in Each Dimension a Singularity of Order 2

Definition 3.1. Let $S$ be a Goursat system. $S$ is called presenting a transposition of order $l$, $l \in \{3,4,\ldots,n-2\}$ if

$$
\omega_{l-1} = dx_{l-1} + X_{l+1}dx_{l-1},
\omega_l = dx_{l+1} + x_{l+2}dx_{l+1}.
$$

(3.1)

Definition 3.2. If the small growth vector of a 2-distribution $E$ on $\mathbb{R}^n$, at a point $p$ of $\mathbb{R}^n$, has the form $[2,3,\ldots,s,s,\ldots,s,\ldots,n]$ (denoted by $[2,3,\ldots,s_k,\ldots,n]$), the distribution is called a distribution presenting, in the dimension $s$, a singularity of order $k$.

Remark 3.3. If the distribution satisfies the condition of Goursat the dimensions 2, 3 and $n$ are of order 1 at every point.

Notation. The system of Goursat satisfying, at every point $x \in \mathbb{R}^n$, the condition $[2,3,4_k,5_k,\ldots,(n-1)_k,n]_S$ is denoted by $(GS_k)$.

Theorem 3.4 (see [9]). Let $E$ be a 2-distribution on $\mathbb{R}^n$, satisfying at every point the Goursat condition, such that at $x_0 \in \mathbb{R}^n$, we have $[2,3,4_2,5_2,\ldots,(n-1)_2,n]_S$. Then there exists a local system of coordinates $(x,U)$, around $x_0$, such that

$$
E^+ = \begin{cases}
\omega_1 = dx_2 + x_3dx_1, \\
\omega_2 = dx_3 + x_4dx_1, \\
\omega_3 = dx_4 + x_5dx_1, \\
\omega_4 = dx_5 + x_6dx_1, \\
\vdots \\
\omega_{n-3} = dx_{n-2} + x_{n-1}dx_1, \\
\omega_{n-2} = dx_1 + x_ndx_{n-1},
\end{cases}
$$

(3.2)

it means that $E$ is spanned by

$$
v_1 = \frac{\partial}{\partial x_n}, \quad v_2 = -x_n\frac{\partial}{\partial x_1} + x_nx_3\frac{\partial}{\partial x_2} + x_nx_4\frac{\partial}{\partial x_3} + \cdots + x_nx_{n-1}\frac{\partial}{\partial x_{n-2}} + \frac{\partial}{\partial x_{n-1}}. 
$$

(3.3)

Now we want to study the rigid and the abnormal line subdistributions (directions) for the Goursat systems (GS$_2$).

Definition 3.5. Let $E$ be a 2-distribution spanned by $v_1$ and $v_2$. The line subdistribution $L_E$, is the line subdistribution spanned by a vector field in the form $av_1 + bv_2$, where $a$ and $b$ are such that $a[v_1,[v_1,v_2]] + b[v_2,[v_1,v_2]] \in E^2$ and $a^2 + b^2 \neq 0$. 
Theorem 3.6. In the Goursat systems (GS₂), L_E is the unique direction of abnormal and rigid curves.

Proof. E is spanned by

\[ v_1 = \frac{\partial}{\partial x_1}, \quad v_2 = -x_n \frac{\partial}{\partial x_1} + x_n x_3 \frac{\partial}{\partial x_2} + x_n x_4 \frac{\partial}{\partial x_3} + \cdots + x_n x_{n-1} \frac{\partial}{\partial x_{n-2}} + \frac{\partial}{\partial x_{n-1}}, \]  

(3.4)

and E² is spanned by v₁, v₂, and [v₁, v₂], where [v₁, v₂] = −∂/∂x₁ + x₃(∂/∂x₂) + x₄(∂/∂x₃) + ⋯ + x_(n-1)(∂/∂x_n₋₂).

Prove now L_E is spanned by v₁. In fact [v₁, [v₁, v₂]] = 0 and [v₂, [v₁, v₂]] = ∂/∂x_n₋₂, then necessarily b = 0 and L_E = span{v₁}. Recall that L_E is a direction of rigid curves, then of abnormal curves.

Does exist another direction field of the abnormal curves?

Let L = Vect{αv₁ + βv₂} be an arbitrary line subdistribution of E. Let γ : I → Rⁿ be a horizontal curve of L, i.e., γ(t) ∈ L(γ(t)). Suppose that γ is an abnormal curve. There exists a lift curve Γ : I → T²Rⁿ satisfying the adjoint equation: (p₁, p₂, ..., pₙ) = −α(p₁, p₂, ..., pₙ)dv₁ − β(p₁, p₂, ..., pₙ)dv₂. In other hand:

\[
(p₁, ..., pₙ) = -\beta(p₁, ..., pₙ) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & x_n & 0 & 0 & \cdots & 0 & x₃ \\ 0 & 0 & 0 & x_n & 0 & \cdots & 0 & x₄ \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & x_n & x_{n₋₁} \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.
\]

(3.5)

We verify that E² = Vect{v₁, v₂, ∂/∂x_n₋₁}. But Γ ⊂ (E²)⊥, then we have

\[ pₙ = p_{n₋₁} = 0, \]  

(3.6)

\[ p₁ = x₃p₂ + x₄p₃ + \cdots + x_{n₋₁} p_{n₋₃}. \]  

(I)

Suppose that β ≠ 0. By the adjoint equation, we have p_{n₋₁} = −βx_n p_{n₋₂} = 0, but β ≠ 0, then p_{n₋₂} = 0. Similarly p_{n₋₂} = −βx_n p_{n₋₃} = 0, then p_{n₋₃} = 0.

Show that by induction p_{n₋₄} = 0, for every i = 1, 2, ..., n₋₂.

For i = 1, the property is true. Suppose that p_{n₋₁} = 0, prove that p_{n₋₁} = 0. By the adjoint equation

\[ p_i = -βx_n p_{i₋₁} = 0 \]  

for every i = 1, 2, ..., n₋₃. We deduce that p_i = 0. Finally, using (I) we have p₁ = 0. We deduce Γ = 0, impossible, then we obtain β = 0 and L is spanned by v₁, by consequently L = L_E and L is locally rigid. □
Corollary 3.7. With the same conditions of Theorem 2.3, the distribution $L_E$ is the unique line subdistribution locally rigid on $E$.

Proof. In fact, $[v_1, [v_1, v_2]] = 0$ and $a[v_1, [v_1, v_2]] + b[v_2, [v_1, v_2]] = b[v_2, [v_1, v_2]] = -b(\partial/\partial x_{n-2}) \in E^2 = \text{span}\{v_1, v_2, \partial/\partial x_{n-1}\}$ if $b = 0$, then $L_E = \text{span}\{v_1\}$, but the distribution spanned by $v_1$ is the unique locally rigid subdistribution, on $E$, of dimension 1. \qed

References


