Research Article

Two Fixed-Point Theorems for Mappings Satisfying a General Contractive Condition of Integral Type in the Modular Space

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1. Introduction

In [1], Branciari established that a function \(f\) defined on a complete metric space satisfying a contraction condition of the form

\[
\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt
\]

has a unique attractive fixed point where \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) is a Lebesgue-integrable mapping and \(c \in [0,1)\).

In [2], Rhoades extended this result to a quasicontraction function \(f\). The purpose of this paper is to extend these theorems in modular space.

First, we introduce the notion of modular space.

Definition 1.1. Let \(X\) be an arbitrary vector space over \(K(=\mathbb{R} \text{ or } \mathbb{C})\). A functional \(\rho : X \to [0, +\infty)\) is called modular if
(1) \( \rho(x) = 0 \) if and only if \( x = 0 \);
(2) \( \rho(\alpha x) = \rho(x) \) for \( \alpha \in K \) with \( |\alpha| = 1 \), for all \( x, y \in X \);
(3) \( \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \) if \( \alpha, \beta \geq 0 \), \( \alpha + \beta = 1 \), for all \( x, y \in X \).

If (2.14) in Definition 1.1 is replaced by

\[
\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y) \tag{1.2}
\]

for \( \alpha, \beta \geq 0 \), \( \alpha^s + \beta^s = 1 \) with an \( s \in (0, 1] \), then the modular \( \rho \) is called an \( s \)-convex modular; and if \( s = 1 \), \( \rho \) is called a convex modular.

**Definition 1.2.** A modular \( \rho \) defines a corresponding modular space, that is, the space \( X_\rho \) is given by

\[
X_\rho = \{ x \in X | \rho(\lambda x) \to 0 \text{ as } \lambda \to 0 \}. \tag{1.3}
\]

**Definition 1.3.** Let \( X_\rho \) be a modular space.

(1) A sequence \( \{x_n\} \) in \( X_\rho \) is said to be

(a) \( \rho \)-convergent to \( x \) if \( \rho(x_n - x) \to 0 \) as \( n \to +\infty \),

(b) \( \rho \)-Cauchy if \( \rho(x_n - x_m) \to 0 \) as \( n, m \to +\infty \).

(2) \( X_\rho \) is \( \rho \)-complete if any \( \rho \)-Cauchy sequence is \( \rho \)-convergent.

(3) A subset \( B \subset X_\rho \) is said to be \( \rho \)-closed if for any sequence \( \{x_n\}_n \subset B \) with \( x_n \to x \) then \( x \in B. \overline{B}^\rho \) denotes the closure of \( B \) in the sense of \( \rho \).

(4) A subset \( B \subset X_\rho \) is called \( \rho \)-bounded if

\[
\delta_\rho(B) = \sup_{x, y \in B} \rho(x - y) < +\infty, \tag{1.4}
\]

where \( \delta_\rho(B) \) is called the \( \rho \)-diameter of \( B \).

(5) We say that \( \rho \) has Fatou property if

\[
\rho(x - y) \leq \liminf_{n} \rho(x_n - y_n) \tag{1.5}
\]

whenever

\[
x_n \xrightarrow{\rho} x, \quad y_n \xrightarrow{\rho} y. \tag{1.6}
\]

(6) \( \rho \) is said to satisfy the \( \Delta_2 \)-condition if: \( \rho(2x_n) \to 0 \) as \( n \to +\infty \) whenever \( \rho(x_n) \to 0 \)

as \( n \to +\infty \).

**Remark 1.4.** Note that since \( \rho \) does not satisfy a priori the triangle inequality, we cannot expect that if \( \{x_n\} \) and \( \{y_n\} \) are \( \rho \)-convergent, respectively, to \( x \) and \( y \) then \( \{x_n + y_n\} \) is \( \rho \)-convergent to \( x + y \), neither that a \( \rho \)-convergent sequence is \( \rho \)-Cauchy.
2. Main Result

Theorem 2.1. Let $X_{\rho}$ be a complete modular space, where $\rho$ satisfies the $\Delta_2$-condition. Assume that $\varphi : \mathbb{R}^+ \rightarrow [0, \infty)$ is an increasing and upper semicontinuous function satisfying

$$\varphi(t) < t, \quad \forall t > 0. \tag{2.1}$$

Let $\varphi : [0, +\infty) \rightarrow [0, +\infty]$ be a nonnegative Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty]$ and such that for $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$ and let $f : X_{\rho} \rightarrow X_{\rho}$ be a mapping such that there are $c, l \in \mathbb{R}^+$ where $l < c$,

$$\int_0^{\rho(c(f(x-y)))} \varphi(t)dt \leq \varphi\left(\int_0^{\rho(0(x-y))} \varphi(t)dt\right), \tag{2.2}$$

for each $x, y \in X_{\rho}$. Then $f$ has a unique fixed point in $X_{\rho}$.

Proof. First, we show that for $x \in X_{\rho}$, the sequence $\{\rho(c(f^n x - f^{n-1} x))\}$ converges to 0. For $n \in \mathbb{N}$, we have

$$\int_0^{\rho(c(f^n x - f^{n-1} x))} \varphi(t)dt \leq \varphi\left(\int_0^{\rho(0(f^{n-1} x - f^{n-2} x))} \varphi(t)dt\right)$$

$$< \int_0^{\rho(0(f^{n-1} x - f^{n-2} x))} \varphi(t)dt$$

$$< \int_0^{\rho(c(f^{n-1} x - f^{n-2} x))} \varphi(t)dt. \tag{2.3}$$

Consequently, $\{\int_0^{\rho(c(f^n x - f^{n-1} x))} \varphi(t)dt\}$ is decreasing and bounded from below. Therefore $\int_0^{\rho(c(f^n x - f^{n-1} x))} \varphi(t)dt$ converges to a nonnegative point $a$.

Now, if $a \neq 0$,

$$a = \lim_{n \to \infty} \int_0^{\rho(c(f^n x - f^{n-1} x))} \varphi(t)dt$$

$$\leq \lim_{n \to \infty} \varphi\left(\int_0^{\rho(0(f^{n-1} x - f^{n-2} x))} \varphi(t)dt\right) \tag{2.4}$$

$$\leq \lim_{n \to \infty} \varphi\left(\int_0^{\rho(c(f^{n-1} x - f^{n-2} x))} \varphi(t)dt\right),$$

then

$$a \leq \varphi(a), \tag{2.5}$$
which is a contradiction, so \( a = 0 \) and

\[
\int_0^{\rho(c(f^n x - f^{n+1} x))} \varphi(t) \, dt \to 0^+ \quad \text{as } n \to +\infty. \tag{2.6}
\]

This concludes \( \rho(c(f^n x - f^{n+1} x)) \to 0 \). Suppose that

\[
\lim_{n \to \infty} \sup \rho(c(f^n x - f^{n+1} x)) = \varepsilon > 0 \tag{2.7}
\]

then there exist a \( \nu_\varepsilon \in \mathbb{N} \) and a sequence \( (f^n x)_{\nu\ge\nu_\varepsilon} \) such that

\[
\rho(c(f^n x - f^{n+1} x)) \to \varepsilon > 0, \quad \nu \to \infty, \\
\rho(c(f^n x - f^{n+1} x)) \geq \frac{\varepsilon}{2}, \quad \forall \nu \geq \nu_\varepsilon. \tag{2.8}
\]

then we get the following contradiction:

\[
0 = \lim_{\nu \to \infty} \int_0^{\rho(c(f^n x - f^{n+1} x))} \varphi(t) \, dt \geq \int_0^{\varepsilon/2} \varphi(t) \, dt > 0. \tag{2.9}
\]

Now, we prove for each \( x \in X_\rho \) the sequence \( \{f^n x\}_{n\in\mathbb{N}} \) is a \( \rho \)-Cauchy sequence.

Assume that there is an \( \varepsilon > 0 \) such that for each \( \nu \in \mathbb{N} \) there exist \( m_\nu, n_\nu \in \mathbb{N} \) that \( m_\nu > n_\nu > \nu \),

\[
\rho(l(f^{m_\nu} x - f^{n_\nu} x)) \geq \varepsilon. \tag{2.10}
\]

Then we choose the sequence \( (m_\nu)_{\nu\in\mathbb{N}} \) and \( (n_\nu)_{\nu\in\mathbb{N}} \) such that for each \( \nu \in \mathbb{N}, m_\nu \) is minimal in the sense that

\[
\rho(l(f^{m_\nu} x - f^{n_\nu} x)) \geq \varepsilon. \tag{2.11}
\]

But

\[
\rho(l(f^h x - f^{n_\nu} x)) < \varepsilon, \tag{2.12}
\]

for each \( h \in \{n_\nu + 1, \ldots, m_\nu - 1\} \).
Now, let \( \alpha \in \mathbb{R}^+ \) be such that \( l/c + 1/\alpha = 1 \), then we have

\[
\int_0^\varepsilon \varphi(t) \, dt \leq \int_0^{\rho(l(f_{n^2}x - f_{n^2}x))} \varphi(t) \, dt \\
\leq \int_0^{\rho(c(f_{n^2}x - f_{n^2+1}x))} \varphi(t) \, dt + \int_0^{\rho(\rho_{n}x - f_{n}x))} \varphi(t) \, dt \\
\leq \varphi \left( \int_0^{\rho(l(f_{n^2}x - f_{n^2}x))} \varphi(t) \, dt \right) + \int_0^{\rho(\rho_{n}x - f_{n}x))} \varphi(t) \, dt \tag{2.13}
\]

Thus, as \( \nu \to \infty \), by \( \Delta_2 \)-condition, \( \int_0^{\rho(\rho_{n}x - f_{n}x))} \varphi(t) \, dt \to 0 \). Therefore

\[
\int_0^{\rho(l(f_{n^2}x - f_{n^2}x))} \varphi(t) \, dt \to \varepsilon^+, \quad \nu \to \infty. \tag{2.14}
\]

Now,

\[
\int_0^{\rho(l(f_{n^2}x - f_{n^2}x))} \varphi(t) \, dt \leq \int_0^{\rho(c(f_{n^2+1}x - f_{n^2+1}x))} \varphi(t) \, dt + \int_0^{\rho(\rho_{n+1}x - f_{n+1}x))} \varphi(t) \, dt \\
+ \int_0^{\rho(\rho_{n+1}x - f_{n+1}x))} \varphi(t) \, dt \\
\leq \varphi \left( \int_0^{\rho(l(f_{n^2}x - f_{n^2}x))} \varphi(t) \, dt \right) + \int_0^{\rho(\rho_{n+1}x - f_{n+1}x))} \varphi(t) \, dt \tag{2.15}
\]

If \( \nu \to \infty \) we get

\[
\int_0^\varepsilon \varphi(t) \, dt \leq \varphi \left( \int_0^\varepsilon \varphi(t) \, dt \right), \tag{2.16}
\]

which is a contradiction for \( \varepsilon > 0 \). Therefore \( \{lf^n x\} \) is a \( \rho \)-Cauchy sequence and by \( \Delta_2 \)-condition \( \{f^n x\} \) is \( \rho \)-Cauchy. By the fact that \( X_\rho \) is \( \rho \)-complete, there is a \( z \in X_\rho \) such that \( \rho(f^n z - z) \to 0 \) as \( n \to \infty \). Furthermore, \( z \) is the fixed point for \( f \). In fact

\[
\rho \left( \frac{z}{2} - f z \right) \leq \rho \left( c(z - f^n z) \right) + \rho \left( c(f^n z - f z) \right) \to 0, \quad n \to \infty \tag{2.17}
\]

then \( \rho((c/2)(z - f z)) = 0 \) and \( f z = z \).
Now, assume that we have more than one fixed point for \( f \). Let \( z \) and \( u \) be two distinct fixed points, then

\[
\int_0^{\rho(c(z-u))} \varphi(t) dt = \int_0^{\rho(c(fz-fu))} \varphi(t) dt \leq \varphi\left(\int_0^{\rho(l(z-u))} \varphi(t) dt\right)
\]

\[
< \int_0^{\rho(l(z-u))} \varphi(t) dt \leq \int_0^{\rho(c(z-u))} \varphi(t) dt,
\]

which is a contradiction. So \( z = u \) and the proof is complete.

**Corollary 2.2** (see [1]). Let \( X_\rho \) be a complete modular space, where \( \rho \) satisfies the \( \Delta_2 \)-condition. Let \( f : X_\rho \to X_\rho \) be a mapping such that there exists an \( \lambda \in (0, 1) \) and \( c, l \in \mathbb{R}^+ \) where \( l < c \) and for each \( x, y \in X_\rho \),

\[
\int_0^{\rho(c(fx-fy))} \varphi(t) dt \leq \lambda \left(\int_0^{\rho(l(x-y))} \varphi(t) dt\right),
\]

then \( f \) has a unique fixed point.

**Corollary 2.3** (see [3]). Let \( X_\rho \) be a complete modular space, where \( \rho \) satisfies the \( \Delta_2 \)-condition. Assume that \( \varphi : \mathbb{R}^+ \to [0, \infty) \) is an increasing and upper semicontinuous function satisfying

\[
\varphi(t) < t, \quad \forall t > 0.
\]

Let \( B \) be a \( \rho \)-closed subset of \( X_\rho \) and \( T : B \to B \) be a mapping such that there exist \( c, l \in \mathbb{R}^+ \) with \( c > l \),

\[
\rho(c(Tx-Ty)) \leq \varphi(\rho(l(x-y)))
\]

for all \( x, y \in B \). Then \( T \) has a fixed point.

In the next theorem we use the following notation:

\[
m(x, y) = \max\left\{ \rho(x-y), \rho(x-Tx), \rho(y-Ty), \rho(1/2(x-y)) + \rho(1/2(y-Tx)) \right\}.
\]

(2.22)
Theorem 2.4. Let \((X_\rho, \rho)\) be a \(\rho\)-complete modular space that \(\rho\) satisfies the \(\Delta_2\)-condition and let \(T : X_\rho \rightarrow X_\rho\) be a mapping such that for each \(x, y \in X_\rho\),

\[
\int_0^{\rho(T^x - Ty)} \phi(t) \, dt \leq \varphi \left( \int_0^{m(x,y)} \phi(t) \, dt \right),
\]

(2.23)

where \(\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) and \(\varphi : \mathbb{R}^+ \rightarrow [0, \infty)\) are as in Theorem 2.1. Then \(T\) has a unique fixed point.

Proof. Let \(x \in X_\rho\), we will show that \(\{T^n x\}\) is a Cauchy sequence. First, we prove that \(\{\rho(T^n x - T^{n-1} x)\}\) converges to 0. From (2.23),

\[
\int_0^{\rho(T^n x - T^{n-1} x)} \phi(t) \, dt \leq \varphi \left( \int_0^{m(T^n x, T^{n-1} x)} \phi(t) \, dt \right).
\]

(2.24)

By the definition of \(m(x, y)\),

\[
m(T^{n-1} x, T^{n-2} x) = \max \left\{ \rho(T^n x - T^{n-1} x), \rho(T^{n-1} - T^{n-2} x) \right\}, \rho(1/2(T^n x - T^{n-2} x)) \leq \rho(T^n x - T^{n-1} x) + \rho(T^{n-1} - T^{n-2} x)
\]

\[
\leq \max \left\{ \rho(T^n x - T^{n-1} x), \rho(T^{n-1} - T^{n-2} x) \right\}.
\]

(2.25)

Hence,

\[
m(T^{n-1} x, T^{n-2} x) = \max \left\{ \rho(T^n x - T^{n-1} x), \rho(T^{n-1} - T^{n-2} x) \right\}
\]

(2.26)

and therefore,

\[
\int_0^{\rho(T^n x - T^{n-1} x)} \phi(t) \, dt \leq \varphi \left( \int_0^{m(T^n x, T^{n-2} x)} \phi(t) \, dt \right)
\]

\[
\leq \int_0^{m(T^n x, T^{n-2} x)} \phi(t) \, dt
\]

\[
= \int_0^{\max \{\rho(T^n x - T^{n-1} x), \rho(T^{n-1} - T^{n-2} x)\}} \phi(t) \, dt
\]

\[
= \max \left\{ \int_0^{\rho(T^n x - T^{n-1} x)} \phi(t) \, dt, \int_0^{\rho(T^{n-1} - T^{n-2} x)} \phi(t) \, dt \right\}
\]

\[
= \int_0^{\rho(T^{n-1} - T^{n-2} x)} \phi(t) \, dt.
\]

(2.27)
This means that \( \{ \rho(T^n x - T^{n-1} x) \} \) is decreasing and since it is bounded from below, it is a convergent sequence. Similarly to Theorem 2.1, it is easy to show that

\[
\left\{ \rho\left( T^n x - T^{n-1} x \right) \right\} \rightarrow 0. \tag{2.28}
\]

Now, we show that \( \{ T^n x \} \) is Cauchy. If not, then there exist an \( \varepsilon > 0 \) and subsequences \( \{m(p)\} \) and \( \{n(p)\} \) such that \( m(p) < n(p) < m(p + 1) \) with

\[
\rho\left( T^{m(p)} x - T^{n(p)} x \right) \geq \varepsilon, \quad \rho\left( 2 \left( T^{m(p)} x - T^{n(p)-1} x \right) \right) < \varepsilon. \tag{2.29}
\]

From (2.22),

\[
m\left( T^{m(p)-1} x, T^{n(p)-1} x \right) = \max \left\{ \rho\left( T^{m(p)-1} x - T^{n(p)-1} x \right), \right.
\]

\[
\left. \rho\left( T^{m(p)} x - T^{n(p)-1} x \right), \rho\left( T^{n(p)} x - T^{n(p)-1} x \right), \right.
\]

\[
\left. \rho\left( 1/2(T^{m(p)} x - T^{n(p)-1} x) \right) + \rho\left( 1/2(T^{n(p)} x - T^{m(p)-1} x) \right) \right\}. \tag{2.30}
\]

By using (2.28), we get

\[
\lim_\rho \int_0^{\rho(T^{m(p)} x - T^{m(p)-1} x)} \phi(t) dt = \lim_\rho \int_0^{\rho(T^{n(p)} x - T^{n(p)-1} x)} \phi(t) dt = 0. \tag{2.31}
\]

On the other hand,

\[
\rho\left( T^{m(p)-1} x - T^{n(p)-1} x \right) \leq \rho\left( 2 \left( T^{m(p)-1} x - T^{m(p)} x \right) \right) + \rho\left( 2 \left( T^{m(p)} x - T^{n(p)-1} x \right) \right)
\]

\[
\leq \rho\left( 2 \left( T^{m(p)-1} x - T^{m(p)} x \right) \right) + \varepsilon, \tag{2.32}
\]

thus by the \( \Delta_2 \)-condition,

\[
\lim_\rho \int_0^{\rho(T^{m(p)-1} x - T^{n(p)-1} x)} \phi(t) dt \leq \int_0^{\varepsilon} \phi(t) dt. \tag{2.33}
\]
For the last term in \( m(T^{m(p)-1}x, T^{n(p)-1}x) \) by the fact that \( \rho(cx) \) is an increasing function of \( c \) we have

\[
v(m, n) := \frac{\rho(1/2(T^{m(p)}x - T^{n(p)}x)) + \rho(1/2(T^{m(p)}x - T^{n(p)}x))}{2} \leq \frac{\rho(T^{m(p)}x - T^{n(p)}x) + \rho(2(T^{m(p)}x - T^{n(p)}x))}{2} + \frac{\rho(2(T^{m(p)}x - T^{n(p)}x)) + \rho(1/2(T^{m(p)}x - T^{n(p)}x))}{2} \leq \varepsilon + \frac{\rho(T^{m(p)}x - T^{n(p)}x) + \rho(2(T^{m(p)}x - T^{n(p)}x))}{2}.
\]

(2.34)

Hence, from (2.28) we get

\[
\lim_p \int_0^{v(m, n)} \phi(t) dt \leq \int_0^{\varepsilon} \phi(t) dt.
\]

(2.35)

Therefore from (2.31), (2.33), and (2.35) it can be concluded that

\[
\int_0^{\varepsilon} \phi(t) dt \leq \int_0^{\rho(T^{m(p)}x_{T^{n(p)}x})} \phi(t) dt \leq \phi \left( \int_0^{\rho(T^{m(p)-1}x, T^{n(p)-1}x)} \phi(t) dt \right)
\]

\[
< \int_0^{\rho(T^{m(p)}x_{T^{m(p)-1}x})} \phi(t) dt \leq \int_0^{\varepsilon} \phi(t) dt
\]

(2.36)

which is a contradiction, when \( p \) is large enough. Therefore, \( \{T^n x\} \) is Cauchy and since \( X_p \) is \( \rho \)-complete there is an \( z \in X_p \) that \( T^n x \rightarrow z \). Now, we should prove that \( z \) is the fixed point for \( T \). In fact,

\[
\int_0^{\rho(1/2(Tz-z))} \phi(t) dt \leq \int_0^{\rho(Tz-T^n z)} \phi(t) dt + \int_0^{\rho(T^n z-z)} \phi(t) dt
\]

\[
\leq \phi \left( \int_0^{\rho(z,T^n z)} \phi(t) dt \right) + \int_0^{\rho(T^n z-z)} \phi(t) dt \rightarrow 0 \quad \text{as} \ n \rightarrow \infty,
\]

(2.37)

by the definition of \( m \). It follows that \( Tz = z \).

Let \( w \in X_p \) be another fixed point of \( T \). Then,

\[
\int_0^{\rho(u-z)} \phi(t) dt = \int_0^{\rho(Tu-Tz)} \phi(t) dt \leq \phi \left( \int_0^{\rho(u,z)} \phi(t) dt \right)
\]

(2.38)

\[
< \int_0^{\rho(w,z)} \phi(t) dt = \int_0^{\rho(w-z)} \phi(t) dt.
\]
That is because
\[
m(w, z) = \max \left\{ \rho(z - w), \rho(z - w), \frac{\rho(1/2(z - w)) + \rho(1/2(w - z))}{2} \right\}
\]
(2.39)

Thus \( z = w \).

**Corollary 2.5** (see [2]). Let \((X, d)\) be complete metric space, \( k \in [0, 1) \), \( f : X \to X \) a mapping such that, for \( x, y \in X \),
\[
\int_0^{d(f(x), f(y))} \phi(t) dt \leq k \int_0^{m(x, y)} \phi(t) dt,
\]
(2.40)
where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue-integrable mapping which is summable, nonnegative, and such that
\[
\int_0^\varepsilon \phi(t) dt > 0 \quad \forall \varepsilon > 0,
\]
(2.41)
and where
\[
m(x, y) = \max \left\{ d(x, y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2} \right\}.
\]
(2.42)

Then \( f \) has a unique fixed point.

**Corollary 2.6** (see [4]). Let \((X, \rho)\) be a modular space such that \( \rho \) satisfies the Fatou property. Let \( C \) be a \( \rho \)-complete nonempty subset of \( X_{\rho} \) and \( T : C \to C \) be quasicontraction. Let \( x \in C \) such that \( \delta_{\rho}(x) < \infty \). Then \( \{T^n x\} \) \( \rho \)-converges to \( \omega \in C \). Here \( \delta_{\rho}(x) = \sup\{\rho(T^n x - T^m x) ; n, m \in \mathbb{N}\} \).

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