Research Article

Universal Verma Modules and the Misra-Miwa Fock Space

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Received 7 March 2010; Accepted 5 November 2010

Academic Editor: Alistair Savage

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The Misra-Miwa v -deformed Fock space is a representation of the quantized affine algebra $U_v(\widehat{\mathfrak{sl}}_\ell)$. It has a standard basis indexed by partitions, and the nonzero matrix entries of the action of the Chevalley generators with respect to this basis are powers of v . Partitions also index the polynomial Weyl modules for $U_q(\mathfrak{gl}_N)$ as N tends to infinity. We explain how the powers of v which appear in the Misra-Miwa Fock space also appear naturally in the context of Weyl modules. The main tool we use is the Shapovalov determinant for a universal Verma module.

1. Introduction

Fock space is an infinite dimensional vector space which is a representation of several important algebras, as described in, for example, [1, Chapter 14]. Here we consider the charge zero part of Fock space, which we denote by F , and its v -deformation F_v . The space F has a standard $\mathbb{Q}$ -basis $\{|\mu\rangle \mid \lambda \text{ is a partition}\}$ and $F_v := F \otimes_{\mathbb{Q}} \mathbb{Q}(v)$. Following Hayashi [2], Misra and Miwa [3] define an action of the quantized universal enveloping algebra $U_v(\widehat{\mathfrak{sl}}_\ell)$ on F_v . The only nonzero matrix elements $\langle \mu | F_{\vec{i}} | \lambda \rangle$ of the Chevalley generators $F_{\vec{i}}$ in terms of the standard basis occur when μ is obtained by adding a single $\vec{i}$ -colored box to λ , and these are powers of v .

We show that these powers of v also appear naturally in the following context: partitions with at most N parts index polynomial Weyl modules $\Delta(\lambda)$ for the integral quantum group $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$. Let V be the standard N dimensional representation of $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$. If the matrix element $\langle \mu | F_{\vec{i}} | \lambda \rangle$ is nonzero then, for sufficiently large N , $(\Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} V) \otimes_{\mathcal{A}} \mathbb{Q}(q)$ contains the highest weight vector of weight μ . There is a unique such highest weight vector v_{μ} which satisfies a certain triangularity condition with respect to an integral basis of $\Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} V$. We show that the matrix element $\langle \mu | F_{\vec{i}} | \lambda \rangle$ is equal to $v^{\text{val}_{\phi_{2\ell}}(v_{\mu}, v_{\mu})}$, where $(\cdot, \cdot)$ is the Shapovalov form and $\text{val}_{\phi_{2\ell}}$ is the valuation at the cyclotomic polynomial $\phi_{2\ell}$.

Our proof is computational, making use of the Shapovalov determinant [4–6]. This is a formula for the determinant of the Shapovalov form on a weight space of a Verma module. The necessary computation is most easily done in terms of the universal Verma modules introduced in the classical case by Kashiwara [7] and studied in the quantum case by Kamita [8]. The statement for Weyl modules is then a straightforward consequence.

Before beginning, let us discuss some related work. In [9], Kleshchev carefully analyzed the $\mathfrak{gl}_{N-1}$ highest weight vectors in a Weyl module for $\mathfrak{gl}_N$ and used this information to give modular branching rules for symmetric group representations. Brundan and Kleshchev [10] have explained that highest weight vectors in the restriction of a Weyl module to $\mathfrak{gl}_{N-1}$ give information about highest weight vectors in a tensor product $\Delta(\lambda) \otimes V$ of a Weyl module with the standard N -dimensional representation of $\mathfrak{gl}_N$. Our computations put a new twist on the analysis of the highest weight vectors in $\Delta(\lambda) \otimes V$, as we study them in their “universal” versions and by the use of the Shapovalov determinant. Our techniques can be viewed as an application of the theory of Jantzen [11] as extended to the quantum case by Wiesner [12].

Brundan [13] generalized Kleshchev’s [9] techniques and used this information to give modular branching rules for Hecke algebras. As discussed in [14, 15], these branching rules are reflected in the fundamental representation of $\widehat{\mathfrak{sl}}_p$ and its crystal graph, recovering much of the structure of the Misra-Miwa Fock space. Using Hecke algebras at a root of unity, Ryom-Hansen [16] recovered the full $U_v(\widehat{\mathfrak{sl}}_\ell)$ action on Fock space. To complete the picture, one should construct a graded category, where multiplication by v in the $\widehat{\mathfrak{sl}}_\ell$ representation corresponds to a grading shift. Recent work of Brundan-Kleshchev [17] and Ariki [18] explains that one solution to this problem is through the representation theory of Khovanov-Lauda-Rouquier algebras [19, 20]. It would be interesting to explicitly describe the relationship between their category and the present work. Another related construction due to Brundan-Stroppel considers the case when the Fock space is replaced by $\wedge^m V \otimes \wedge^n V$, where V is the natural $\mathfrak{gl}_\infty$ module and m, n are fixed natural numbers.

We would also like to mention very recent work of Peng Shan [21] which independently develops a similar story to the one presented here, but using representations of a quantum Schur algebra where we use representations of $U_\varepsilon(\mathfrak{gl}_N)$. The approach taken there is somewhat different and in particular relies on localization techniques of Beilinson and Bernstein [22].

This paper is arranged as follows. Sections 2 and 3 are background on the quantum group $U_q(\mathfrak{gl}_N)$ and the Fock space F_v . Sections 4 and 5 explain universal Verma modules and the Shapovalov determinant. Section 6 contains the statement and proof of our main result relating Fock space and Weyl modules.

2. The Quantum Group $U_q(\mathfrak{gl}_N)$ and Its Integral Form $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$

This is a very brief review, intended mainly to fix notation. With slight modifications, the construction in this section works in the generality of symmetrizable Kac-Moody algebras. See [23, Chapters 6 and 9] for details.

2.1. The Rational Quantum Group

$U_q(\mathfrak{gl}_N)$ is the associative algebra over the field of rational functions $\mathbb{Q}(q)$ generated by

$$X_1, \dots, X_{N-1}, \quad Y_1, \dots, Y_{N-1}, \quad L_1^{\pm 1}, \dots, L_N^{\pm 1}, \quad (2.1)$$

with relations

$$\begin{aligned}
 L_i L_j &= L_j L_i, & L_i L_i^{-1} &= L_i^{-1} L_i = 1, & X_i Y_j - Y_j X_i &= \delta_{i,j} \frac{L_i L_{i+1}^{-1} - L_{i+1} L_i^{-1}}{q - q^{-1}}, \\
 L_i X_j L_i^{-1} &= \begin{cases} q X_j, & \text{if } i = j, \\ q^{-1} X_j, & \text{if } i = j + 1, \\ X_j, & \text{otherwise,} \end{cases} & L_i Y_j L_i^{-1} &= \begin{cases} q^{-1} Y_j, & \text{if } i = j, \\ q Y_j, & \text{if } i = j + 1, \\ Y_j, & \text{otherwise,} \end{cases} \\
 X_i X_j &= X_j X_i, & Y_i Y_j &= Y_j Y_i, & \text{if } |i - j| &\geq 2, \\
 X_i^2 X_j - (q + q^{-1}) X_i X_j X_i + X_j X_i^2 &= Y_i^2 Y_j - (q + q^{-1}) Y_i Y_j Y_i + Y_j Y_i^2 = 0, & \text{if } |i - j| &= 1.
 \end{aligned} \tag{2.2}$$

The algebra $U_q(\mathfrak{gl}_N)$ is a Hopf algebra with coproduct and antipode given by

$$\begin{aligned}
 \Delta(L_i) &= L_i \otimes L_i, & S(L_i) &= L_i^{-1}, \\
 \Delta(X_i) &= X_i \otimes L_i L_{i+1}^{-1} + 1 \otimes X_i, & S(X_i) &= -X_i L_i^{-1} L_{i+1}, \\
 \Delta(Y_i) &= Y_i \otimes 1 + L_i^{-1} L_{i+1} \otimes Y_i, & S(Y_i) &= -L_i L_{i+1}^{-1} Y_i,
 \end{aligned} \tag{2.3}$$

respectively, (see [23, Section 9.1]).

As a $\mathbb{Q}(q)$ -vector space, $U_q(\mathfrak{gl}_N)$ has a triangular decomposition

$$U_q(\mathfrak{gl}_N) \cong U_q(\mathfrak{gl}_N)^{<0} \otimes U_q(\mathfrak{gl}_N)^0 \otimes U_q(\mathfrak{gl}_N)^{>0}, \tag{2.4}$$

where the inverse isomorphism is given by multiplication (see [23, Proposition 9.1.3]). Here $U_q(\mathfrak{gl}_N)^{<0}$ is the subalgebra generated by the Y_i for $i = 1, \dots, N-1$, $U_q(\mathfrak{gl}_N)^0$ is the subalgebra generated by the X_i for $i = 1, \dots, N-1$, and $U_q(\mathfrak{gl}_N)^{>0}$ is the subalgebra generated by the $L_i^{\pm 1}$ for $i = 1, \dots, N$.

2.2. The Integral Quantum Group

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. For $n, k \in \mathbb{Z}_{>0}$ and $c \in \mathbb{Z}$, let

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad x^{(k)} := \frac{x^k}{[k][k-1] \cdots [2][1]}, \quad \begin{bmatrix} x; c \\ k \end{bmatrix} := \prod_{s=1}^k \frac{x q^{c+1-s} - x^{-1} q^{s-1-c}}{q^s - q^{-s}}, \tag{2.5}$$

in $\mathbb{Q}(q, x)$. The *restricted integral form* $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ is the $\mathcal{A}$ -subalgebra of $U_q(\mathfrak{gl}_N)$ generated by $X_i^{(k)}, Y_i^{(k)}, L_i^{\pm 1}$ and $\left[\begin{smallmatrix} L_{i;c} \\ k \end{smallmatrix} \right]$ for $1 \leq i \leq N, c \in \mathbb{Z}, k \in \mathbb{Z}_{>0}$. As discussed in [24, Section 6], this is an integral form in the sense that

$$U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathbb{Q}(q) = U_q(\mathfrak{gl}_N). \quad (2.6)$$

As with $U_q(\mathfrak{gl}_N)$, the algebra $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ has a triangular decomposition

$$U_q^{\mathcal{A}}(\mathfrak{gl}_N) \cong U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \otimes U_q^{\mathcal{A}}(\mathfrak{gl}_N)^0 \otimes U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0}, \quad (2.7)$$

where the isomorphism is given by multiplication (see [23, Proposition 9.3.3]). In this case, $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0}$ is the subalgebra generated by the $Y_i^{(k)}$, $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0}$ is the subalgebra generated by the $X_i^{(k)}$, and $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^0$ is generated by $L_i^{\pm 1}$ and $\left[\begin{smallmatrix} L_{i;c} \\ k \end{smallmatrix} \right]$ for $1 \leq i \leq N, c \in \mathbb{Z}$, and $k \in \mathbb{Z}_{>0}$.

2.3. Rational Representations

The Lie algebra $\mathfrak{gl}_N = M_N(\mathbb{C})$ of $N \times N$ matrices has standard basis $\{E_{ij} \mid 1 \leq i, j \leq N\}$, where E_{ij} is the matrix with 1 in position (i, j) and 0 everywhere else. Let $\mathfrak{h} = \text{span}\{E_{11}, E_{22}, \dots, E_{NN}\}$. Let $\varepsilon_i \in \mathfrak{h}^*$ be the weight of $\mathfrak{gl}_N$ given by $\varepsilon_i(E_{jj}) = \delta_{i,j}$. Define

$$\begin{aligned} \mathfrak{h}_{\mathbb{Z}}^* &:= \{\lambda = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \dots + \lambda_N \varepsilon_N \in \mathfrak{h}^* \mid \lambda_1, \dots, \lambda_N \in \mathbb{Z}\}, \\ (\mathfrak{h}_{\mathbb{Z}}^*)^+ &:= \{\lambda = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \dots + \lambda_N \varepsilon_N \in \mathfrak{h}_{\mathbb{Z}}^* \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N\}, \\ P^+ &:= \left\{ \lambda = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \dots + \lambda_N \varepsilon_N \in (\mathfrak{h}_{\mathbb{Z}}^*)^+ \mid \lambda_N \geq 0 \right\}, \\ R^+ &:= \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq N\}, \\ Q &:= \text{span}_{\mathbb{Z}}(R^+), \quad Q^+ := \text{span}_{\mathbb{Z}_{\geq 0}}(R^+), \quad Q^- := \text{span}_{\mathbb{Z}_{\leq 0}}(R^+) \end{aligned} \quad (2.8)$$

to be the set of *integral weights*, the set of *dominant integral weights*, the set of *dominant polynomial weights*, the set of *positive roots*, the *root lattice*, the *positive part of the root lattice*, and the *negative part of the root lattice*, respectively.

For an integral weight $\lambda = \lambda_1 \varepsilon_1 + \dots + \lambda_N \varepsilon_N$, the *Verma module* $M(\lambda)$ for $U_q(\mathfrak{gl}_N)$ of the highest weight λ is

$$M(\lambda) := U_q(\mathfrak{gl}_N) \otimes_{U_q(\mathfrak{gl}_N)^{\geq 0}} \mathbb{Q}(q)_{\lambda}, \quad (2.9)$$

where $\mathbb{Q}(q)_{\lambda} = \text{span}_{\mathbb{Q}(q)}\{v_{\lambda}\}$ is the one dimensional vector space over $\mathbb{Q}(q)$ with $U_q(\mathfrak{gl}_N)^{\geq 0}$ action given by

$$X_i \cdot v_{\lambda} = 0, \quad L_j \cdot v_{\lambda} = q^{\lambda_j} v_{\lambda}, \quad \text{for } 1 \leq i \leq N-1, 1 \leq j \leq N. \quad (2.10)$$

Theorem 2.1 (see [23, Chapter 10.1]). *If $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^+$ then $M(\lambda)$ has a unique finite dimensional quotient $\Delta(\lambda)$ and the map $\lambda \mapsto \Delta(\lambda)$ is a bijection between $(\mathfrak{h}_{\mathbb{Z}}^*)^+$, and the set of irreducible finite dimensional $U_q(\mathfrak{gl}_N)$ -modules.*

A *singular vector* in a representation of $U_q(\mathfrak{gl}_N)$ is a vector v such that $X_i \cdot v = 0$ for all i .

2.4. Integral Representations

The *integral Verma module* $M^{\mathcal{A}}(\lambda)$ is the $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ -submodule of $M(\lambda)$ generated by v_{λ} . The *integral Weyl module* $\Delta^{\mathcal{A}}(\lambda)$ is the $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ -submodule of $\Delta(\lambda)$ generated by v_{λ} . Using (2.6) and (2.4),

$$M^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{Q}(q) = M(\lambda), \quad \Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{Q}(q) = \Delta(\lambda). \quad (2.11)$$

In general, $\Delta^{\mathcal{A}}(\lambda)$ is not irreducible as a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ module.

3. Partitions and Fock Space

We now describe the v -deformed Fock space representation of $U_v(\widehat{\mathfrak{sl}}_{\ell})$ constructed by Misra and Miwa [3] following work of Hayashi [2]. Our presentation largely follows [25, Chapter 10].

3.1. Partitions

A partition λ is a finite length nonincreasing sequence of positive integers. Associated to a partition is its Ferrers diagram. We draw these diagrams as in Figure 1 so that, if $\lambda = (\lambda_1, \dots, \lambda_N)$, then λ_i is the number of boxes in row i (rows run southeast to northwest $\nearrow$). Say that λ is contained in μ if the diagram for λ fits inside the diagram for μ and let μ/λ be the collection of boxes of μ that are not in λ . For each box $b \in \lambda$, the *content* $c(b)$ is the horizontal position of b and the *color* $\bar{c}(b)$ is the residue of $c(b)$ modulo ℓ . In Figure 1, the numbers $c(b)$ are listed below the diagram. The *size* $|\lambda|$ of a partition λ is the total number of boxes in its Ferrers diagram.

The set P^+ of dominant polynomial weights from Section 2.3 is naturally identified with partitions with at most N parts. If $\lambda \in P^+$, then

$$\Delta(\lambda) \otimes \Delta(\varepsilon_1) \cong \bigoplus_{\substack{1 \leq k \leq N \\ \lambda + \varepsilon_k \in P^+}} \Delta(\lambda + \varepsilon_k) \quad (3.1)$$

as $U_q(\mathfrak{gl}_N)$ -modules. The diagram of $\lambda + \varepsilon_k$ is obtained from the diagram of λ by adding a box on row k , and $\Delta(\lambda + \varepsilon_k)$ appears in the sum on the right side of (3.1) if and only if $\lambda + \varepsilon_k$ is a partition. See, for example, [26, Section 6.1, Formula 6.8] for the classical statement and [23, Proposition 10.1.16] for the quantum case.

Figure 1: The partition $(7, 6, 6, 5, 5, 3, 3, 1)$ with each box containing its color for $\ell = 3$. The content $c(b)$ of a box b is the horizontal position of b reading right to left. The contents of boxes are listed beneath the diagram so that $c(b)$ is aligned with all boxes b of that content.

3.2. The Quantum Affine Algebra

Let $U'_v(\widehat{\mathfrak{sl}}_\ell)$ be the quantized universal enveloping algebra corresponding to the ℓ -node Dynkin diagram

More precisely, $U'_v(\widehat{\mathfrak{sl}}_\ell)$ is the algebra generated by $E_{\vec{i}}, F_{\vec{i}}, K_{\vec{i}}^{\pm 1}$, for $\vec{i} \in \mathbb{Z}/\ell\mathbb{Z}$, with relations

$$\begin{aligned}
 K_{\vec{i}}K_{\vec{j}} &= K_{\vec{j}}K_{\vec{i}}, & K_{\vec{i}}K_{\vec{i}}^{-1} &= K_{\vec{i}}^{-1}K_{\vec{i}} = 1, & E_{\vec{i}}F_{\vec{j}} - F_{\vec{j}}E_{\vec{i}} &= \delta_{\vec{i},\vec{j}} \frac{K_{\vec{i}} - K_{\vec{i}}^{-1}}{v - v^{-1}}, \\
 K_{\vec{i}}E_{\vec{j}}K_{\vec{i}}^{-1} &= \begin{cases} v^2 E_{\vec{j}}, & \text{if } \vec{i} = \vec{j}, \\ v^{-1} E_{\vec{j}}, & \text{if } \vec{i} = \vec{j} \pm 1, \\ E_{\vec{j}}, & \text{otherwise,} \end{cases} & K_{\vec{i}}F_{\vec{j}}K_{\vec{i}}^{-1} &= \begin{cases} v^{-2} F_{\vec{j}}, & \text{if } \vec{i} = \vec{j}, \\ v F_{\vec{j}}, & \text{if } \vec{i} = \vec{j} \pm 1, \\ F_{\vec{j}}, & \text{otherwise,} \end{cases} \\
 E_{\vec{i}}E_{\vec{j}} &= E_{\vec{j}}E_{\vec{i}}, & F_{\vec{i}}F_{\vec{j}} &= F_{\vec{j}}F_{\vec{i}}, & \text{if } |\vec{i} - \vec{j}| &\geq 2, \\
 E_{\vec{i}}^2 E_{\vec{j}} - (v + v^{-1}) E_{\vec{i}} E_{\vec{j}} E_{\vec{i}} + E_{\vec{j}} E_{\vec{i}}^2 &= F_{\vec{i}}^2 F_{\vec{j}} - (v + v^{-1}) F_{\vec{i}} F_{\vec{j}} F_{\vec{i}} + F_{\vec{j}} F_{\vec{i}}^2 = 0, & \text{if } |\vec{i} - \vec{j}| &= 1.
 \end{aligned} \tag{3.2}$$

See [23, Definition Proposition 9.1.1]. The algebra $U'_v(\widehat{\mathfrak{sl}}_\ell)$ is the quantum group corresponding to the nontrivial central extension $\widehat{\mathfrak{sl}}'_\ell = \mathfrak{sl}_\ell[t, t^{-1}] \oplus \mathbb{C}c$ of the algebra of polynomial loops in $\mathfrak{sl}_\ell$.

3.3. Fock Space

Define v -deformed Fock space to be the $\mathbb{Q}(v)$ vector space $\mathbf{F}_v$ with basis $\{|\mu\rangle \mid \lambda \text{ is a partition}\}$. Our $\mathbf{F}_v$ is only the charge 0 part of Fock space described in [27]. Fix $\bar{i} \in \mathbb{Z}/\ell\mathbb{Z}$ and partitions $\lambda \subseteq \mu$ such that μ/λ is a single box. Define

$$\begin{aligned} A_{\bar{i}}(\lambda) &:= \left\{ \text{boxes } b \mid b \notin \lambda, b \text{ has color } \bar{i} \text{ and } \lambda \cup b \text{ is a partition} \right\}, \\ R_{\bar{i}}(\lambda) &:= \left\{ \text{boxes } b \mid b \in \lambda, b \text{ has color } \bar{i} \text{ and } \lambda \setminus b \text{ is a partition} \right\}, \\ N_{\bar{i}}^l(\mu/\lambda) &:= \left| \{b \in R_{\bar{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\} \right| - \left| \{b \in A_{\bar{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\} \right|, \\ N_{\bar{i}}^r(\mu/\lambda) &:= \left| \{b \in R_{\bar{i}}(\lambda) \mid b \text{ is to the right of } \mu/\lambda\} \right| - \left| \{b \in A_{\bar{i}}(\lambda) \mid b \text{ is to the right of } \mu/\lambda\} \right|, \end{aligned} \quad (3.3)$$

to be the set of *addable boxes of color $\bar{i}$* , the set of *removable boxes of color $\bar{i}$* , the *left removable-addable difference*, and the *right removable-addable difference*, respectively.

Theorem 3.1 (see [25, Theorem 10.6]). *There is an action of $U'_v(\widehat{\mathfrak{sl}}_\ell)$ on $\mathbf{F}_v$ determined by*

$$E_{\bar{i}}|\lambda\rangle := \sum_{\bar{c}(\lambda/\mu)=\bar{i}} v^{-N_{\bar{i}}^r(\lambda/\mu)} |\mu\rangle, \quad F_{\bar{i}}|\lambda\rangle := \sum_{\bar{c}(\mu/\lambda)=\bar{i}} v^{N_{\bar{i}}^l(\mu/\lambda)} |\mu\rangle, \quad (3.4)$$

where $\bar{c}(\lambda/\mu)$ denotes the color of λ/μ and the sum is over partitions μ which differ from λ by removing (resp. adding) a single $\bar{i}$ -colored box.

As a $U'_v(\widehat{\mathfrak{sl}}_\ell)$ -module, $\mathbf{F}_v$ is isomorphic to an infinite direct sum of copies of the basic representation $V(\Lambda_0)$. Using the grading of $\mathbf{F}_v$ where $|\lambda\rangle$ has degree $|\lambda|$, the highest weight vectors in $\mathbf{F}_v$ occur in degrees divisible by ℓ , and the number of the highest weight vectors in degree ℓk is the number of partitions of k . Then, $\mathbf{F}_v \cong V(\Lambda_0) \otimes \mathbb{C}[x_1, x_2, \dots]$, where x_k has degree ℓk , and $U'_v(\widehat{\mathfrak{sl}}_\ell)$ acts trivially on the second factor (see [27, Proposition 2.3]). Note that we are working with the “derived” quantum group $U'_v(\widehat{\mathfrak{sl}}_\ell)$, not the “full” quantum group $U_v(\widehat{\mathfrak{sl}}_\ell)$, which is why there are no δ -shifts in the summands of $\mathbf{F}_v$.

Comment 1. Comparing with [25, Chapter 10], our $N_{\bar{i}}^l(\mu/\lambda)$ is equal to Ariki’s $-N_{\bar{i}}^a(\mu/\lambda)$ and our $N_{\bar{i}}^r(\mu/\lambda)$ is equal to Ariki’s $-N_{\bar{i}}^b(\mu/\lambda)$. However, these numbers play a slightly different role in Ariki’s work, which is explained by a different choice of conventions.

4. Universal Verma Modules

The purpose of this section is to construct a family of representations which are universal Verma modules in the sense that each can be “evaluated” to obtain any given Verma module. This notion was defined by Kashiwara [7] in the classical case and was studied in the quantum case by Kamita [8].

4.1. Rational Universal Verma Modules

Let $\mathbb{K} := \mathbb{Q}(q, z_1, z_2, \dots, z_N)$. This field is isomorphic to the field of fractions of $U_q(\mathfrak{gl}_N)^0$ via the map

$$\psi : U_q(\mathfrak{gl}_N)^0 \longrightarrow \mathbb{K}, \quad \text{defined by } \psi(L_i^{\pm 1}) = z_i^{\pm 1}. \quad (4.1)$$

For each $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$, define a $\mathbb{Q}(q)$ -linear automorphism $\sigma_\mu : \mathbb{K} \rightarrow \mathbb{K}$ by

$$\sigma_\mu(z_i) := q^{(\mu, \varepsilon_i)} z_i, \quad \text{for } 1 \leq i \leq N, \quad (4.2)$$

where $(\cdot, \cdot)$ is the inner product on $\mathfrak{h}_{\mathbb{Z}}^*$ defined by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$. Let $\mathbb{K}_\mu = \text{span}_{\mathbb{K}}\{v_{\mu+}\}$ be the one-dimensional vector space over $\mathbb{K}$ with basis vector $v_{\mu+}$ and $U_q(\mathfrak{gl}_N)^{\geq 0}$ action given by

$$X_i \cdot v_{\mu+} = 0, \quad \text{for } 1 \leq i \leq N-1, \quad a \cdot v_{\mu+} = \sigma_\mu(\psi(a))v_{\mu+}, \quad \text{for } a \in U_q(\mathfrak{gl}_N)^0. \quad (4.3)$$

The μ -shifted rational universal Verma module ${}^\mu \widetilde{M}$ is the $U_q(\mathfrak{gl}_N)$ -module

$${}^\mu \widetilde{M} := U_q(\mathfrak{gl}_N) \otimes_{U_q(\mathfrak{gl}_N)^{\geq 0}} \mathbb{K}_\mu. \quad (4.4)$$

The universal Verma module ${}^\mu \widetilde{M}$ is actually a module over $U_q(\mathfrak{gl}_N) \otimes_{U_q(\mathfrak{gl}_N)^0} \widetilde{U}_q(\mathfrak{gl}_N)^0$, where $\widetilde{U}_q(\mathfrak{gl}_N)^0$ is the field of fractions of $U_q(\mathfrak{gl}_N)^0$. However, if we identify $\widetilde{U}_q(\mathfrak{gl}_N)^0$ with $\mathbb{K}$ using the map ψ , the action of $\widetilde{U}_q(\mathfrak{gl}_N)^0$ on ${}^\mu \widetilde{M}$ is not by multiplication, but rather is twisted by the automorphism σ_μ . It is to keep track of the difference between the action of $U_q(\mathfrak{gl}_N)^0$ and multiplication that we use different notation for the generators of $\mathbb{K}$ and $U_q(\mathfrak{gl}_N)^0$ (i.e., z_i versus L_i).

4.2. Integral Universal Verma Modules

The field $\mathbb{K}$ contains an $\mathcal{A}$ -subalgebra

$$\mathcal{R} \text{ generated by } z_i^{\pm 1}, \quad \begin{bmatrix} z_i & c \\ & k \end{bmatrix}, \quad (1 \leq i \leq N, c \in \mathbb{Z}, k \in \mathbb{Z}_{>0}), \quad (4.5)$$

which is isomorphic to $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^0$ via the restriction of the map ψ in (4.1). The integral universal Verma module ${}^\mu \widetilde{M}^{\mathcal{R}}$ is the $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ -submodule of ${}^\mu \widetilde{M}$ generated by $v_{\mu+}$. By restricting (4.4),

$${}^\mu \widetilde{M}^{\mathcal{R}} = U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{\geq 0}} \mathcal{R}_\mu, \quad (4.6)$$

where $\mathcal{R}_\mu$ is the $\mathcal{R}$ -submodule of $\mathbb{K}_\mu$ spanned by $v_{\mu+}$. In particular, ${}^\mu \widetilde{M}^{\mathcal{R}}$ is a free $\mathcal{R}$ -module.

4.3. Evaluation

Let $\text{ev}_\lambda^{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{A}$ be the map defined by

$$\text{ev}_\lambda^{\mathcal{R}}(z_i) = q^{(\lambda, \varepsilon_i)}, \quad \text{ev}_\lambda^{\mathcal{R}} \begin{bmatrix} z_i; c \\ n \end{bmatrix} = \begin{bmatrix} q^{(\lambda, \varepsilon_i)}; c \\ n \end{bmatrix}, \quad (4.7)$$

where $(\cdot, \cdot)$ is the inner product on $\mathfrak{h}^*$ defined by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$. There is a surjective $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ -module homomorphism “evaluation at λ ”

$$\text{ev}_\lambda : {}^\mu \widetilde{M}^{\mathcal{R}} \longrightarrow M^{\mathcal{A}}(\mu + \lambda) \text{ defined by } \text{ev}_\lambda(a \cdot v_{\mu+}) := a \cdot v_{\mu+\lambda}, \quad \forall a \in U_q^{\mathcal{A}}(\mathfrak{gl}_N). \quad (4.8)$$

For fixed λ , the maps $\text{ev}_\lambda^{\mathcal{R}}$ and ev_λ extend to a map from the subspace of $\mathbb{K}$ and ${}^\mu \widetilde{M} = {}^\mu \widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{R}} \mathbb{K}$, respectively, where no denominators evaluate to 0. Where it is clear we denote both these extended maps by ev_λ .

Example 4.1. Computing the action of L_i on $v_{\mu+}$ and $v_{\mu+\lambda}$,

$$L_i \cdot v_{\mu+} = q^{(\mu, \varepsilon_i)} z_i v_{\mu+}, \quad L_i \cdot v_{\mu+\lambda} = \text{ev}_\lambda \left(q^{(\mu, \varepsilon_i)} z_i \right) v_{\mu+\lambda} = q^{(\mu, \varepsilon_i)} q^{(\lambda, \varepsilon_i)} v_{\mu+\lambda} = q^{(\mu+\lambda, \varepsilon_i)} v_{\mu+\lambda}. \quad (4.9)$$

4.4. Weight Decompositions

Let $\widetilde{V}$ be a $U_q(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$ -module. For each $\nu \in \mathfrak{h}_{\mathbb{Z}}^*$, we define the ν -weight space of $\widetilde{V}$ to be

$$\widetilde{V}_\nu := \left\{ v \in \widetilde{V} : L_i \cdot v = q^{(\nu, \varepsilon_i)} z_i v \right\}. \quad (4.10)$$

The universal Verma module ${}^\mu \widetilde{M}^{\mathcal{R}}$ is a $U_q(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$ -module, where the second factor acts as multiplication. The weight space ${}^\mu \widetilde{M}_\eta \neq 0$ if and only if $\eta = \mu - \nu$ with ν in the positive part Q^+ of the root lattice. These nonzero weight spaces and the weight decomposition of ${}^\mu \widetilde{M}$ can be described explicitly by

$${}^\mu \widetilde{M}_{\mu-\nu}^{\mathcal{R}} = U_q^{\mathcal{A}}(\mathfrak{gl}_N)_{-\nu}^{<0} \cdot \mathcal{R}_\mu, \quad {}^\mu \widetilde{M}^{\mathcal{R}} = \bigoplus_{\nu \in Q^+} {}^\mu \widetilde{M}_{\mu-\nu}^{\mathcal{R}}. \quad (4.11)$$

Here, $U_q^{\mathcal{A}}(\mathfrak{gl}_N)_{-\nu}^{<0}$ is defined using the grading of $U_q(\mathfrak{gl}_N)^{<0}$ with $F_i \in U_q(\mathfrak{gl}_N)_{-(\varepsilon_i - \varepsilon_{i+1})}^{<0}$.

4.5. Tensor Products

Let $\widetilde{V}$ be a $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$ -module and W a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ -module. The tensor product $\widetilde{V} \otimes_{\mathcal{A}} W$ is a $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$ -module, where the first factor acts via the usual coproduct and the second

factor acts by multiplication on $\tilde{V}$. In the case when $\tilde{V}$ and W both have weight space decompositions, the weight spaces of $\tilde{V} \otimes_{\mathcal{A}} W$ are

$$\left(\tilde{V} \otimes_{\mathcal{A}} W \right)_\nu = \bigoplus_{\gamma + \eta = \nu} \tilde{V}_\gamma \otimes_{\mathcal{A}} W_\eta. \quad (4.12)$$

We also need the following.

Proposition 4.2. *The tensor product of a universal Verma module with a Weyl module satisfies*

$$\left({}^\mu \widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} \Delta^{\mathcal{A}}(\nu) \right) \otimes_{\mathcal{R}} \mathbb{K} \cong \left(\bigoplus_{\gamma} ({}^{\mu+\gamma} \widetilde{M}^{\mathcal{R}})^{\oplus \dim \Delta^{\mathcal{A}}(\nu)_\gamma} \right) \otimes_{\mathcal{R}} \mathbb{K}. \quad (4.13)$$

Proof. Fix $\nu \in P^+$. In general, $M(\lambda + \mu) \otimes \Delta(\nu)$ has a Verma filtration (see, e.g., [28, Theorem 2.2]) and if $\lambda + \mu + \gamma$ is dominant for all γ such that $\Delta(\nu)_\gamma \neq 0$ then

$$M(\lambda + \mu) \otimes \Delta(\nu) \cong \bigoplus_{\gamma} M(\lambda + \mu + \gamma)^{\oplus \dim \Delta(\nu)_\gamma}, \quad (4.14)$$

which can be seen by, for instance, taking central characters. The proposition follows since this is true for a Zariski dense set of weights λ . $\square$

5. The Shapovalov Form and the Shapovalov Determinant

5.1. The Shapovalov Form

The Cartan involution $\omega : U_q(\mathfrak{gl}_N) \rightarrow U_q(\mathfrak{gl}_N)$ is the $\mathbb{Q}(q)$ -algebra anti-involution of $U_q(\mathfrak{gl}_N)$ defined by

$$\omega(L_i^{\pm 1}) = L_i^{\pm 1}, \quad \omega(X_i) = Y_i L_i L_{i+1}^{-1}, \quad \omega(Y_i) = L_i^{-1} L_{i+1} X_i. \quad (5.1)$$

The map ω is also a coalgebra involution. An ω -contravariant form on a $U_q(\mathfrak{gl}_N)$ -module V is a symmetric bilinear form $(\cdot, \cdot)$ such that

$$(u, a \cdot v) = (\omega(a) \cdot u, v), \quad \text{for } u, v \in V, a \in U_q(\mathfrak{gl}_N). \quad (5.2)$$

It follows by the same argument used in the classical case [4] that there is an ω -contravariant form (the Shapovalov form) on each Verma module $M(\lambda)$ and this is unique up to rescaling. The radical of $(\cdot, \cdot)$ is the maximal proper submodule of $M(\lambda)$, so $\Delta(\lambda) = M(\lambda)/\text{Rad}(\cdot, \cdot)$ for all $\lambda \in P^+$. In particular, $(\cdot, \cdot)$ descends to an ω -contravariant form on $\Delta(\lambda)$.

Since ω fixes $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \subseteq U_q(\mathfrak{gl}_N)$, there is a well-defined notion of an ω -contravariant form on a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ module. In particular, the restriction of the Shapovalov form on $\Delta(\lambda)$ to $\Delta^{\mathcal{A}}(\lambda)$ is ω -contravariant.

5.2. Universal Shapovalov Forms

There are surjective maps of $\mathcal{A}$ -algebras $p_- : U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \rightarrow \mathbb{Q}(q)$ and $p_+ : U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0} \rightarrow \mathbb{Q}(q)$ defined by $p_-(F_i) = 0$ and $p_+(E_i) = 0$, for $1 \leq i \leq N$. Using the triangular decomposition (2.7), there is an $\mathcal{A}$ -linear surjection

$$\pi_0 := p_- \otimes \text{Id} \otimes p_+ : U_q^{\mathcal{A}}(\mathfrak{gl}_N) \cong U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{gl}_N)^0 \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0} \longrightarrow U_q^{\mathcal{A}}(\mathfrak{gl}_N)^0. \quad (5.3)$$

The *standard universal Shapovalov form* is the $\mathcal{R}$ -bilinear form $(\cdot, \cdot)_{\mu \widetilde{M}^{\mathcal{R}}} : {}^{\mu} \widetilde{M}^{\mathcal{R}} \otimes {}^{\mu} \widetilde{M}^{\mathcal{R}} \rightarrow \mathcal{R}$ defined by

$$(a_1 \cdot v_{\mu+}, a_2 \cdot v_{\mu+})_{\mu \widetilde{M}^{\mathcal{R}}} = (\sigma_{\mu} \circ \psi \circ \pi_0)(\omega(a_2) a_1) \quad (5.4)$$

for all $a_1, a_2 \in U_q^{\mathcal{R}}(\mathfrak{gl}_N)^{<0}$. Here, ψ and σ_{μ} are as in (4.1) and (4.2). Since

$$(a_1 a_2 \cdot v_{\mu+}, a_3 \cdot v_{\mu+})_{\mu \widetilde{M}^{\mathcal{R}}} = (\sigma_{\mu} \circ \psi \circ \pi_0)(\omega(a_2) \omega(a_1) a_3) = (a_2 \cdot v_{\mu+}, \omega(a_1) a_3 \cdot v_{\mu+})_{\mu \widetilde{M}^{\mathcal{R}}} \quad (5.5)$$

for $a_1, a_2, a_3 \in U_q(\mathfrak{gl}_N)$, the form $(\cdot, \cdot)_{\mu \widetilde{M}^{\mathcal{R}}}$ is ω -contravariant. As with the usual Shapovalov form, distinct weight spaces are orthogonal, where weight spaces are defined as in Section 4.4.

Evaluation at λ gives an $\mathcal{A}$ -valued ω -contravariant form $(\cdot, \cdot)_{M^{\mathcal{A}}(\mu+\lambda)}$ on $M^{\mathcal{A}}(\mu+\lambda)$ by

$$(\text{ev}_{\lambda}(u_1), \text{ev}_{\lambda}(u_2))_{M^{\mathcal{A}}(\mu+\lambda)} = \text{ev}_{\lambda}((u_1, u_2)_{\mu \widetilde{M}^{\mathcal{R}}}) \quad \text{for } u_1, u_2 \in {}^{\mu} \widetilde{M}^{\mathcal{R}}. \quad (5.6)$$

The form $(\cdot, \cdot)_{\mu \widetilde{M}^{\mathcal{R}}}$ can be extended by linearity to an ω -contravariant form $(\cdot, \cdot)_{\mu \widetilde{M}}$ on ${}^{\mu} \widetilde{M}$.

5.3. The Shapovalov Determinant

Let $\widetilde{V}$ be a $(U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R})$ -module with a chosen ω -contravariant form. Let B_{η} be an $\mathcal{R}$ basis for the η -weight space $\widetilde{V}_{\eta}$ of $\widetilde{V}$. Let $\det \widetilde{V}_{B_{\eta}}$ be the determinant of the form evaluated on the basis B_{η} . Changing the basis B_{η} changes the determinant by a unit in $\mathcal{R}$, and we sometimes write $\det \widetilde{V}_{\eta}$ to mean the determinant calculated on an unspecified basis ($\det \widetilde{V}_{\eta}$ which is only defined up to multiplication by unit in $\mathcal{R}$). The *Shapovalov determinant* is

$$\det \widetilde{M}_{\eta}^{\mathcal{R}} := \det \left((b_i, b_j)_{\widetilde{M}^{\mathcal{R}}} \right)_{b_i, b_j \in B_{\eta}}. \quad (5.7)$$

Define the *partition function* $p : \mathfrak{h}^* \rightarrow \mathbb{Z}_{\geq 0}$ by

$$p(\gamma) := \dim M(0)_{\gamma}. \quad (5.8)$$

Then, $p(\gamma) = \dim M(\lambda)_{\gamma+\lambda}$ for any λ , and $\eta \notin Q^-$ implies that $p(\eta) = 0$ and $\det \widetilde{M}_{\eta}^{\mathcal{R}} = 1$.

Theorem 5.1 (see [5, Proposition 1.9A], [6, Theorem 3.4], [4]). *For any weight η ,*

$$\det \widetilde{M}_\eta^{\mathcal{R}} = c_\eta \prod_{\substack{1 \leq i < j \leq N \\ m > 0}} \left(z_i z_j^{-1} - q^{2m+2i-2j} z_i^{-1} z_j \right)^{p(\eta + m\epsilon_i - m\epsilon_j)}, \quad (5.9)$$

where c_η is a unit in $\mathcal{R} \otimes_{\mathcal{A}} \mathbb{Q}(q) = \mathbb{Q}(q)[z_1^{\pm 1}, \dots, z_N^{\pm 1}]$.

Proposition 5.2. *Fix $\mu, \eta \in \mathfrak{h}_{\mathbb{Z}}^*$ with $\eta - \mu \in Q^-$. Choose an $\mathcal{A}$ -basis $B_{\eta-\mu}$ for $U_q^{\mathcal{A}}(\mathfrak{gl}_N)_{\eta-\mu}$. Consider the $\mathcal{R}$ -bases $\widetilde{B}_{\eta-\mu} := \{b \cdot v_+ \mid b \in B_{\eta-\mu}\}$ for $\widetilde{M}_{\eta-\mu}^{\mathcal{R}}$ and ${}^\mu \widetilde{B}_\eta := \{b \cdot v_{\mu+} \mid b \in B_{\eta-\mu}\}$ for ${}^\mu \widetilde{M}_\eta^{\mathcal{R}}$. Then $\det {}^\mu \widetilde{M}_{({}^\mu \widetilde{B}_\eta)}^{\mathcal{R}} = \sigma_\mu(\det \widetilde{M}_{\widetilde{B}_{\eta-\mu}}^{\mathcal{R}})$.*

Proof. For $b, b' \in B_{\eta-\mu}$,

$$(b \cdot v_{\mu+}, b' \cdot v_{\mu+})_{\mu \widetilde{M}^{\mathcal{R}}} = \sigma_\mu \circ \psi \circ \pi_0(\omega(b')b) = \sigma_\mu((b \cdot v_{0+}, b' \cdot v_{0+})_{\widetilde{M}^{\mathcal{R}}}). \quad (5.10)$$

The result follows by taking determinants. $\square$

5.4. Contravariant Forms on Tensor Products

If V and W are $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ -modules with ω -contravariant forms $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_W$, define an $\mathcal{A}$ -bilinear form $(\cdot, \cdot)_{W \otimes V}$ by $(w_1 \otimes v_1, w_2 \otimes v_2)_{W \otimes V} = (w_1, w_2)_W (v_1, v_2)_V$. Similarly, for a $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$ module $\widetilde{W}$ with $\mathcal{R}$ -bilinear ω -contravariant form $(\cdot, \cdot)_{\widetilde{W}}$, define a $\mathcal{R}$ -bilinear form $(\cdot, \cdot)_{\widetilde{W} \otimes_{\mathbb{Q}(q)} V}$ on $\widetilde{W} \otimes_{\mathbb{Q}(q)} V$ by

$$(u_1 \otimes v_1, u_2 \otimes v_2)_{\widetilde{W} \otimes_{\mathbb{Q}(q)} V} = (u_1, u_2)_{\widetilde{W}} (v_1, v_2)_V. \quad (5.11)$$

Since ω is a coalgebra involution (i.e., $\Delta(\omega(a)) = (\omega \otimes \omega)\Delta(a)$, for $a \in U_q(\mathfrak{gl}_N)$), the forms $(\cdot, \cdot)_{V \otimes W}$ and $(\cdot, \cdot)_{\mu \widetilde{W} \otimes_{\mathbb{Q}(q)} V}$ are ω -contravariant.

In the case when $\widetilde{W} = {}^\mu \widetilde{M}^{\mathcal{R}}$, evaluation of the ω -contravariant form $(\cdot, \cdot)_{\mu \widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V}$ at λ gives an ω -contravariant form $(\cdot, \cdot)_{M^{\mathcal{A}}(\mu+\lambda) \otimes_{\mathcal{A}} V}$:

$$\begin{aligned} (u_1 \otimes v_1, u_2 \otimes v_2)_{M^{\mathcal{A}}(\mu+\lambda) \otimes_{\mathcal{A}} V} &= \text{ev}_\lambda \left((u_1 \otimes v_1, u_2 \otimes v_2)_{\mu \widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V} \right) \\ &= (\text{ev}_\lambda(u_1) \otimes v_1, \text{ev}_\lambda(u_2) \otimes v_2)_{M(\mu+\lambda) \otimes_{\mathcal{A}} V}, \end{aligned} \quad (5.12)$$

for $u_1, u_2 \in {}^\mu \widetilde{M}$ and $v_1, v_2 \in V$. As in Section 4.3, this form can be extended to the $\mathcal{A}$ -submodule of the rational module where no denominators evaluate to zero.

6. The Misra-Miwa Formula for $F_{\vec{i}}$ from $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ Representation Theory

Let us prepare the setting for our main result (Theorem 6.1). Fix $\ell \geq 2$ and a partition λ . Let N be a positive integer greater than the number of parts of λ . All calculations below are in terms of representations of $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$.

- (1) Let $V = \Delta^{\mathcal{A}}(\varepsilon_1)$ be the standard N -dimensional module. Since $\Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{Q}(q) = \Delta(\lambda)$, (3.1) implies

$$\left(\Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} V \right) \otimes_{\mathcal{A}} \mathbb{Q}(q) \simeq \bigoplus \Delta^{\mathcal{A}}(\lambda + \varepsilon_{k_j}) \otimes_{\mathcal{A}} \mathbb{Q}(q), \quad (6.1)$$

where the sum is over those indices $1 = k_1 < k_2 < \cdots < k_{m_\lambda} \leq N$ for which $\lambda + \varepsilon_{k_j}$ is a partition. For ease of notation, let $\mu^{(j)} = \lambda + \varepsilon_{k_j}$.

- (2) Fix an $\mathcal{A}$ -basis $\{v_1, \dots, v_N\}$ of V where v_k has weight ε_k and $Y_i(v_k) = \delta_{i,k} v_{k+1}$. Recursively, define singular weight vectors $v_{\mu^{(j)}}$ in $(\Delta^{\mathcal{A}}(\lambda) \otimes V) \otimes_{\mathcal{A}} \mathbb{Q}(q)$ by

- (i) $v_{\mu^{(1)}} = v_\lambda \otimes v_1$
- (ii) for each k , the submodule W_k of $(\Delta(\lambda) \otimes_{\mathcal{A}} V) \otimes_{\mathcal{A}} \mathbb{Q}(q)$ generated by $\{v_\lambda \otimes v_i \mid 1 \leq i \leq k\}$ contains all weight vectors of $(\Delta(\lambda) \otimes_{\mathcal{A}} V) \otimes_{\mathcal{A}} \mathbb{Q}(q)$ of weight greater than or equal to $\lambda + \varepsilon_k$. Thus, using (6.1), for each $1 \leq j \leq m_\lambda$ there is a one-dimensional space of singular vectors of weight $\mu^{(j)}$ in W_{k_j} , and this is not contained in $W_{k_{j-1}}$ (since $k_j > k_{j-1}$). This implies that there unique singular vector $v_{\mu^{(j)}}$ of weight $\mu^{(j)}$ in

$$v_\lambda \otimes v_{k_j} + \bigoplus_{1 \leq i < j} U_q(\mathfrak{gl}_N) v_{\mu^{(i)}} \subseteq \left(\Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} V \right) \otimes_{\mathcal{A}} \mathbb{Q}(q), \quad (6.2)$$

where we recall that $U_q(\mathfrak{gl}_N) = U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathbb{Q}(q)$.

- (3) There is a unique ω -contravariant form on $\Delta^{\mathcal{A}}(\lambda)$ normalized so that $(v_\lambda, v_\lambda) = 1$ and a unique ω -contravariant form on V normalized so that $(v_1, v_1) = 1$. As in Section 5.4, define a ω -contravariant form on $(\Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} V) \otimes_{\mathcal{A}} \mathbb{Q}(q)$ by $(u_1 \otimes w_1, u_2 \otimes w_2) = (u_1, u_2)(w_1, w_2)$. For each $1 \leq j \leq m_\lambda$, define an element $r_j(\lambda) \in \mathbb{Q}(q)$ by

$$r_j(\lambda) := (v_{\mu^{(j)}}, v_{\mu^{(j)}}). \quad (6.3)$$

Theorem 6.1. *The Misra-Miwa operators $F_{\vec{i}}$ from Section 3.3 satisfy*

$$F_{\vec{i}}|\lambda\rangle = \sum_{\vec{\tau}(b^{(j)})=\vec{i}} v^{\text{val}_{\phi_{2\ell}} r_j(\lambda)} \left| \mu^{(j)} \right\rangle, \quad (6.4)$$

where $b^{(j)}$ is the box $\mu^{(j)}/\lambda$, $\bar{c}(b^{(j)})$ is the color of box $b^{(j)}$ as in Figure 1, $\phi_{2\ell}$ is the 2ℓ th cyclotomic polynomial in q , and $\text{val}_{\phi_{2\ell}} r$ is the number of factors of $\phi_{2\ell}$ in the numerator of r minus the number of factors of $\phi_{2\ell}$ in the denominator of r .

The proof of Theorem 6.1 will occupy the rest of this section. We will first prove a similar statement, Proposition 6.6, where the role of the Weyl modules is played by the universal Verma modules from Section 4. For ease of notation, let $\widetilde{M}^{\mathcal{R}}$ denote the module ${}^0\widetilde{M}^{\mathcal{R}}$ from Section 4.2.

Definition 6.2. Recursively define singular weight vectors $v_{\varepsilon_k+} \in (\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V) \otimes_{\mathcal{R}} \mathbb{K}$ and elements $s_k \in \mathbb{K}$ for $1 \leq k \leq N$ by

- (i) $v_{\varepsilon_1+} = v_+ \otimes v_1$,
- (ii) since $\{v_+ \otimes v_j \mid 1 \leq j \leq N\}$ generates $\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V$ as a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{\leq 0}$ module, Proposition 4.2 implies that, for each $1 \leq k \leq N$, there is a unique singular vector v_{ε_k+} in $v_+ \otimes v_{k+\oplus_{1 \leq j < k} \mathbb{K}_q(\mathfrak{gl}_N)} v_{\varepsilon_j+} \subseteq (\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V) \otimes_{\mathcal{R}} \mathbb{K}$, where $U_q^{\mathbb{K}}(\mathfrak{gl}_N) := U_q(\mathfrak{gl}_N) \otimes_{(q)} \mathbb{K}$ and the factor of $\mathbb{K}$ acts by multiplication on $\widetilde{M}^{\mathcal{R}}$.

Let $s_k = (v_{\varepsilon_k+}, v_{\varepsilon_k+})$.

The s_k are quantized versions of the Jantzen numbers first calculated in [11, Section 5] and quantized in [12]. It follows immediately from the definition that $s_1 = 1$.

Lemma 6.3. For any weight η , up to multiplication by a power of q ,

$$\prod_{1 \leq k \leq N} s_k^{p(\eta - \varepsilon_k)} = \prod_{1 \leq k \leq N} \frac{\det \widetilde{M}_{\eta - \varepsilon_k}^{\mathcal{R}}}{\sigma_{\varepsilon_k} \det \widetilde{M}_{\eta - \varepsilon_k}^{\mathcal{R}}}, \quad (6.5)$$

where, as in Section 5.3, $\det \widetilde{M}_{\eta - \varepsilon_k}^{\mathcal{R}}$ is the determinant of the Shapovalov form evaluated on an $\mathcal{R}$ -basis for the $\eta - \varepsilon_k$ weight space of $\widetilde{M}^{\mathcal{R}}$.

Comment 2. In order for Lemma 6.3 to hold as stated, for each $1 \leq k \leq N$, one must calculate the $\det \widetilde{M}_{\eta - \varepsilon_k}^{\mathcal{R}}$ in the numerator and denominator with respect to the same $\mathcal{R}$ -basis. The power of q which appears depends on this choice of $\mathcal{R}$ -bases.

Proof of Lemma 6.3. For each $\gamma \in \text{span}_{\mathbb{Z}_{\leq 0}}(R^+)$ fix an $\mathcal{R}$ -basis B_γ for $U_q^{\mathcal{R}}(\mathfrak{gl}_N)_\gamma^{\leq 0}$. Consider the following three $\mathbb{K}$ -bases for $((\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_\eta) \otimes_{\mathcal{R}} \mathbb{K}$:

$$\begin{aligned} A_\eta &:= \{(b \cdot v_+) \otimes v_k \mid b \in B_{\eta - \varepsilon_k}, 1 \leq k \leq N\}, \\ C_\eta &:= \{b \cdot (v_+ \otimes v_k) \mid b \in B_{\eta - \varepsilon_k}, 1 \leq k \leq N\}, \\ D_\eta &:= \{b \cdot v_{\varepsilon_k+} \mid b \in B_{\eta - \varepsilon_k}, 1 \leq k \leq N\}. \end{aligned} \quad (6.6)$$

Let $\det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_B$ denote the determinant of $(\cdot, \cdot)_{(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_\eta}$ calculated on B , where B is one of A_η, C_η , or D_η . Let $\det {}^\nu \widetilde{M}_{B_{\eta-\nu}}^{\mathcal{R}}$ denote $\det {}^\nu \widetilde{M}_\eta^{\mathcal{R}}$ calculated with respect to the basis $B_{\eta-\nu} \cdot v_{\nu+}$.

By the definition of the ω -contravariant form on $\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V$ (see Section 4.5),

$$\det \left(\widetilde{M}^{\mathcal{R}} \otimes V \right)_{A_{\eta}} = \prod_{k=1}^N \left(\det \widetilde{M}_{B_{\eta-\varepsilon_k}}^{\mathcal{R}} \right)^{\dim V_{\varepsilon_k}} (\det V_{\varepsilon_k})^{\dim \widetilde{M}_{\eta-\varepsilon_k}^{\mathcal{R}}}. \quad (6.7)$$

For $1 \leq k \leq N$, V_{ε_k} is one dimensional and $\det V_{\varepsilon_k}$ is a power of q . Hence, up to multiplication by a power of q , (6.7) simplifies to

$$\det \left(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V \right)_{A_{\eta}} = \prod_{k=1}^N \det \widetilde{M}_{B_{\eta-\varepsilon_k}}^{\mathcal{R}}. \quad (6.8)$$

Notice that $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \cdot v_{\varepsilon_k+}$ is isomorphic to ${}^{\varepsilon_k} \widetilde{M}$, and D_{η} is the union of $\mathcal{R}$ -bases for each of these submodules. For each $1 \leq k \leq N$, and each $\eta \in \mathfrak{h}_{\mathbb{Z}}^*$ define an $\mathcal{R}$ basis of ${}^{\varepsilon_k} \widetilde{M}_{\eta}$ by

$${}^{\varepsilon_k} \widetilde{B}_{\eta} := \{ b \cdot v_{\varepsilon_k+} \mid b \in B_{\eta-\varepsilon_k} \}. \quad (6.9)$$

Using $(v_{\varepsilon_k+}, v_{\varepsilon_k+}) = s_k$,

$$\det \left(\widetilde{M}^{\mathcal{R}} \otimes V \right)_{D_{\eta}} = \prod_{k=1}^N s_k^{\dim ({}^{\varepsilon_k} \widetilde{M}_{\eta}^{\mathcal{R}})} \det {}^{\varepsilon_k} \widetilde{M}_{({}^{\varepsilon_k} \widetilde{B}_{\eta})}^{\mathcal{R}} = \prod_{k=1}^N s_k^{p(\eta-\varepsilon_k)} \sigma_{\varepsilon_k} \left(\det \widetilde{M}_{\widetilde{B}_{\eta-\varepsilon_k}}^{\mathcal{R}} \right), \quad (6.10)$$

where the last equality uses Proposition 5.2. Here, as in Section 5.3, $\det {}^{\varepsilon_k} \widetilde{M}_{({}^{\varepsilon_k} \widetilde{B}_{\eta})}^{\mathcal{R}}$ is the Shapovalov determinant calculated with respect to the basis ${}^{\varepsilon_k} \widetilde{B}_{\eta}$.

The change of basis from A_{η} to C_{η} is unitriangular and the change of basis from C_{η} to D_{η} is unitriangular. Thus, $\det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_{A_{\eta}} = \det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_{D_{\eta}}$, and so the right sides of (6.8) and (6.10) are equal. The lemma follows from this equality by rearranging. $\square$

Lemma 6.4. *Up to multiplication by a power of q ,*

$$s_k = \prod_{1 \leq j < k} \left(\frac{z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k}{\sigma_{\varepsilon_j} (z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k)} \right). \quad (6.11)$$

Proof. Fix $1 \leq k \leq N$. Setting $\eta = \varepsilon_k$ in Lemma 6.3 and applying Theorem 5.1 we see that, up to multiplication by a power of q ,

$$\begin{aligned} \prod_{1 \leq x \leq N} s_x^{p(\varepsilon_k - \varepsilon_x)} &= \prod_{1 \leq x \leq N} \frac{\det \widetilde{M}_{\varepsilon_k - \varepsilon_x}^{\mathcal{R}}}{\sigma_{\varepsilon_x} \det \widetilde{M}_{\varepsilon_k - \varepsilon_x}^{\mathcal{R}}} \\ &= \prod_{1 \leq x \leq N} \prod_{\substack{1 \leq i < j \leq N \\ m > 0}} \left(\frac{c_{\varepsilon_k - \varepsilon_x} (z_i z_j^{-1} - q^{2m+2i-2j} z_i^{-1} z_j)}{\sigma_{\varepsilon_x} (c_{\varepsilon_k - \varepsilon_x}) \sigma_{\varepsilon_x} (z_i z_j^{-1} - q^{2m+2i-2j} z_i^{-1} z_j)} \right)^{p(\varepsilon_k - \varepsilon_x + m\varepsilon_i - m\varepsilon_j)}, \end{aligned} \quad (6.12)$$

where, for each $1 \leq x \leq N$, $c_{\varepsilon_k - \varepsilon_x}$ is a unit in $\mathbb{Q}(q)[z_1^{\pm 1}, \dots, z_N^{\pm 1}]$. The value $p(\varepsilon_k - \varepsilon_x + m\varepsilon_i - m\varepsilon_j)$ is 0 unless $m = 1$ and $x \leq i < j \leq k$. If $i > x$, then σ_{ε_x} acts as the identity on $z_i z_j^{-1} - q^{2+2i-2j} z_i^{-1} z_j$, so the corresponding factors in the numerator and denominator cancel. Hence, we need only consider factors on the right hand side where $m = 1$, $i = x$, and $x < j \leq k$. If $x > k$, then $\varepsilon_k - \varepsilon_x \notin Q^-$, and hence $p(\varepsilon_k - \varepsilon_x) = 0$, so on the left hand side we only need to consider those factors where $1 \leq x \leq k$. Up to multiplication by a power of q , the expression reduces to

$$\begin{aligned} \prod_{1 \leq x \leq k} s_x^{p(\varepsilon_k - \varepsilon_x)} &= \prod_{1 \leq x < k} \left(\frac{c_{\varepsilon_k - \varepsilon_x}}{\sigma_{\varepsilon_x} (c_{\varepsilon_k - \varepsilon_x})} \right)^{p(\varepsilon_k - \varepsilon_j)} \prod_{x < j \leq k} \left(\frac{z_x z_j^{-1} - q^{2+2x-2j} z_x^{-1} z_j}{\sigma_{\varepsilon_x} (z_x z_j^{-1} - q^{2+2x-2j} z_x^{-1} z_j)} \right)^{p(\varepsilon_k - \varepsilon_j)} \\ &= \prod_{1 < j \leq k} \left(\prod_{1 \leq x < j} \frac{z_x z_j^{-1} - q^{2+2x-2j} z_x^{-1} z_j}{\sigma_{\varepsilon_x} (z_x z_j^{-1} - q^{2+2x-2j} z_x^{-1} z_j)} \right)^{p(\varepsilon_k - \varepsilon_j)}. \end{aligned} \quad (6.13)$$

The last two expressions are equal because they are each a product over pairs (x, j) with $1 \leq x < j \leq k$, and the factors of $c_{\varepsilon_k - \varepsilon_x} / (\sigma_{\varepsilon_x} (c_{\varepsilon_k - \varepsilon_x}))$ have been dropped because they are powers of q . Using the fact that $s_1 = 1$ and making the change of variables $j \rightarrow x$ and $x \rightarrow j$ on the right side, (6.13) becomes

$$\prod_{1 < x \leq k} s_x^{p(\varepsilon_k - \varepsilon_x)} = \prod_{1 < x \leq k} \left(\prod_{1 \leq j < x} \frac{z_j z_x^{-1} - q^{2+2j-2x} z_j^{-1} z_x}{\sigma_{\varepsilon_j} (z_j z_x^{-1} - q^{2+2j-2x} z_j^{-1} z_x)} \right)^{p(\varepsilon_k - \varepsilon_x)}. \quad (6.14)$$

For $k \geq 2$, the lemma now follows by induction. For $k = 1$, the result simply says that $s_1 = 1$, which we already know. $\square$

Figure 2: The partition enclosed by the thick lines is $\lambda = (10, 10, 8, 8, 8, 6, 6, 6, 1, 1)$. If $k = 6$ then $A(\lambda, < 6) = \{a_1, a_3\}$, $R(\lambda, < 6) = \{g_2, g_5\}$, and $\text{ev}_\lambda(s_6) = ([2]/[3])([3]/[4])([4]/[5])([7]/[8])([8]/[9]) = ([2]/[5])([7]/[9]) = ([c(g_5) - c(b)][c(g_2) - c(b)])/([c(a_3) - c(b)][c(a_1) - c(b)])$. The factors in the numerator of the first expression are displayed. These are the q -integers corresponding to the hook lengths of the boxes in the same column as the addable box b in row 6.

Proposition 6.5. Let λ be a partition. Let $A(\lambda, < k)$ (resp. $R(\lambda, < k)$) be the set of boxes which can be added to (resp. removed from) λ on rows λ_j with $j < k$ such that the result is still a partition. Let $b = (\lambda + \varepsilon_k)/\lambda$ and let $c(\cdot)$ be as in Figure 1. Then, up to multiplication by a power of q ,

$$\text{ev}_\lambda(s_k) = \begin{cases} \frac{\prod_{r \in R(\lambda, < k)} [c(r) - c(b)]}{\prod_{a \in A(\lambda, < k)} [c(a) - c(b)]}, & \text{if } \lambda + \varepsilon_k \text{ is a partition,} \\ 0, & \text{if } \lambda + \varepsilon_k \text{ is not a partition.} \end{cases} \quad (6.15)$$

Proof. For $1 \leq j \leq N$, let g_j be the last box in row j of λ . By Lemma 6.4, up to multiplication by a power of q ,

$$\text{ev}_\lambda(s_k) = \text{ev}_\lambda \left(\prod_{1 \leq j < k} \frac{z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k}{\sigma_{\varepsilon_j} (z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k)} \right) = \prod_{1 \leq j < k} \frac{[c(g_j) - c(b)]}{[c(g_j) - c(b) + 1]}, \quad (6.16)$$

where the last equality is a simple calculation from definitions. The denominator on the right side is never zero, and the numerator is zero exactly when $\lambda_k = \lambda_{k-1}$, so that $\lambda + \varepsilon_k$ is no longer a partition. If $\lambda_j = \lambda_{j+1}$ for any $j < k$, then there is cancellation, giving (6.15). See Figure 2. $\square$

Proposition 6.6. Let $N_j^l(\mu/\lambda)$ be as in Section 3.3. For any partition λ ,

$$\begin{aligned} \text{val}_{\phi_{2\ell}} \text{ev}_\lambda(s_k) &= N_j^l(\mu/\lambda), \quad \text{if } \mu = \lambda + \varepsilon_k \text{ is a partition, and } \mu/\lambda \text{ is an } \bar{i} \text{ colored box,} \\ \text{ev}_\lambda(s_k) &= 0, \quad \text{otherwise.} \end{aligned} \quad (6.17)$$

Proof. By Proposition 6.5, $\text{ev}_\lambda(s_k) = 0$ if $\lambda + \varepsilon_k$ is not a partition. If $\lambda + \varepsilon_k$ is a partition, then

$$\begin{aligned} \{b \in A(\lambda, < k) : \bar{c}(b) = \bar{c}(\mu/\lambda)\} &= \{b \in A_{\bar{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\}, \\ \{b \in R(\lambda, < k) : \bar{c}(b) = \bar{c}(\mu/\lambda)\} &= \{b \in R_{\bar{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\}, \end{aligned} \quad (6.18)$$

where the notation is as in Section 3.3. Since

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = q^{-x} (q - q^{-1})^{-1} \prod_{d|2x} \phi_d, \quad (6.19)$$

$[x]$ is divisible by $\phi_{2\ell}$ if and only if x is divisible by ℓ , and $[x]$ is never divisible by $\phi_{2\ell}^2$. The result now follows from Proposition 6.5. $\square$

Proof of Theorem 6.1. Fix λ and $1 \leq k \leq m_\lambda$. From definitions, $(\text{ev}_\lambda \otimes 1)v_{\varepsilon_{k,+}} = v_{\mu^{(j)}}$. Thus, using (5.12),

$$r_j(\lambda) = (v_{\mu^{(j)}}, v_{\mu^{(j)}}) = ((\text{ev}_\lambda \otimes 1)v_{\varepsilon_{k,+}}, (\text{ev}_\lambda \otimes 1)v_{\varepsilon_{k,+}}) = \text{ev}_\lambda(v_{\varepsilon_{k,+}}, v_{\varepsilon_{k,+}}) = \text{ev}_\lambda(s_{k_j}). \quad (6.20)$$

The result now follows from Proposition 6.6. $\square$

Acknowledgments

The authors thank M. Kashiwara, A. Kleshchev, T. Tanisaki, R. Virk, and B. Webster for helpful discussions. A. Ram was partly supported by the NSF Grant no. DMS-0353038 and the Australian Research Council Grants nos. DP0986774 and DP0879951. P. Tingley was partly supported by the Australia Research Council Grant no. DP0879951 and the NSF Grant no. DMS-0902649.

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