Research Article

A Study on Degree of Approximation by $(E, 1)$ Summability Means of the Fourier-Laguerre Expansion

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A very new theorem on the degree of approximation of the generating function by $(E, 1)$ means of its Fourier-Laguerre series at the frontier point $x = 0$ is obtained.

1. Introduction

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of its n th partial sums $\{s_n\}$.
If

$$E_n^1 = \frac{1}{2^n} \sum_{k=0}^n {}^nC_k s_{n-k} \longrightarrow s \quad \text{as } n \longrightarrow \infty, \quad (1.1)$$

then we say that $\{s_n\}$ is summable by $(E, 1)$ means (see the study by Hardy [1]), and it is written as $s_n \rightarrow s (E, 1)$, where $\{s_n\}$ is the sequence of n th partial sums of the series $\sum_{n=0}^{\infty} u_n$.

The Fourier-Laguerre expansion of a function $f(x) \in L(0, \infty)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \quad (1.2)$$

where

$$a_n = \left\{ \Gamma(\alpha + 1) \binom{n + \alpha}{n} \right\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) L_n^{(\alpha)}(y) dy \quad (1.3)$$

and $L_n^{(\alpha)}(x)$ denotes the n th Laguerre polynomial of order $\alpha > -1$, defined by generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \omega^n = (1 - \omega)^{-\alpha-1} \exp\left(\frac{-x\omega}{1 - \omega}\right), \quad (1.4)$$

and existence of integral (1.3) is presumed.

We write

$$\phi(y) = \{\Gamma(\alpha + 1)\}^{-1} e^{-y} y^\alpha \{f(y) - f(0)\}. \quad (1.5)$$

Gupta [2] estimated the order of the function by Cesàro means of series (1.2) at the point $x = 0$ after replacing the continuity condition in Szegő's theorem [3] by a much lighter condition. He established the following theorem.

Theorem 1.1. *If*

$$F(t) = \int_0^t \frac{|f(y)|}{y} dy = o\left\{\log\left(\frac{1}{t}\right)\right\}^{1+p}, \quad t \rightarrow 0, \quad -1 < p < \infty, \quad (1.6)$$

$$\int_1^\infty e^{-y/2} y^{(3\alpha-3k-1)/3} |f(y)| dy < \infty,$$

then

$$\sigma_n^k(0) = o(\log n)^{p+1} \quad (1.7)$$

provided that $k > \alpha + 1/2$, $\alpha > -1$, with $\sigma_n^k(0)$ being the n th Cesàro mean of order k .

Denoting the harmonic means by $\{t_n\}$, Singh [4] estimated the order of function by harmonic means of series (1.2) at point $x = 0$ by weaker conditions than those of Theorem 1.1. He proved the following theorem.

Theorem 1.2. *For $-5/6 < \alpha < -1/2$,*

$$t_n(0) - f(0) = o(\log n)^{p+1}, \quad (1.8)$$

provided that

$$\int_t^\delta \frac{|\phi(y)|}{y^{\alpha+1}} dy = o\left\{\log\left(\frac{1}{t}\right)\right\}^{1+p}, \quad t \rightarrow 0, \quad -1 < p < \infty, \quad (1.9)$$

δ is a fixed positive constant,

$$\begin{aligned} \int_{\delta}^n e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| dy &= o\left\{ n^{-(2\alpha+1)/4} (\log n)^{1+p} \right\}, \\ \int_n^{\infty} e^{y/2} y^{-1/3} |\phi(y)| dy &= o\left\{ (\log n)^{p+1} \right\}, \quad n \rightarrow \infty. \end{aligned} \quad (1.10)$$

2. Main Theorem

The objects of present paper are as follows:

- (1) We prove our theorem for $(E, 1)$ means which is entirely different from (C, k) and harmonic means.
- (2) We employ a condition which is weaker than condition (1.9) of Theorem 1.2.
- (3) In our theorem the range of α is increased to $-1 < \alpha < -1/2$, which is more useful for application.

In fact, we establish the following theorem.

Theorem 2.1. *If*

$$E_n^1 = \frac{1}{2^n} \sum_{k=0}^n {}^nC_k s_k \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad (2.1)$$

then the degree of approximation of Fourier-Laguerre expansion (1.2) at the point $x = 0$ by $(E, 1)$ means E_n^1 is given by

$$E_n^1(0) - f(0) = o\{\xi(n)\} \quad (2.2)$$

provided that

$$\Phi(t) = \int_0^t |\phi(y)| dy = o\left\{ t^{\alpha+1} \xi\left(\frac{1}{t}\right) \right\}, \quad t \rightarrow 0, \quad (2.3)$$

δ is a fixed positive constant and $-1 < \alpha < -1/2$,

$$\int_{\delta}^n e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| dy = o\left\{ n^{-(2\alpha+1)/4} \xi(n) \right\}, \quad (2.4)$$

$$\int_n^{\infty} e^{y/2} y^{-1/3} |\phi(y)| dy = o\{\xi(n)\}, \quad n \rightarrow \infty, \quad (2.5)$$

where $\xi(t)$ is a positive monotonic increasing function of t such that $\xi(n) \rightarrow \infty$ as $n \rightarrow \infty$.

3. Lemmas

Lemma 3.1 (see the study by Szegő, 1959, [3, page 175]). *Let α be arbitrary and real, let c and $\epsilon \in$ be fixed positive constants, and let $n \rightarrow \infty$. Then*

$$L_n^{(\alpha)}(x) = O\left(x^{-(2\alpha+1)/4} n^{(2\alpha-1)/4}\right) \quad \text{if } \frac{c}{n} \leq x \leq \epsilon, \quad (3.1)$$

$$L_n^{(\alpha)}(x) = O(n^\alpha) \quad \text{if } 0 \leq x \leq \frac{\epsilon}{n}. \quad (3.2)$$

4. Proof of the Main Theorem

Since

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{\alpha}, \quad (4.1)$$

therefore,

$$\begin{aligned} s_n(0) &= \sum_{k=0}^n a_k L_k^{(\alpha)}(0) = \{\Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) \sum_{k=0}^n L_k^{(\alpha)}(y) dy \\ &= \{\Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) L_n^{(\alpha+1)}(y) dy. \end{aligned} \quad (4.2)$$

Now,

$$\begin{aligned} E_n^1(0) &= \frac{1}{2^n} \sum_{k=0}^n {}^nC_k s_k(0) \\ &= \frac{1}{2^n} \sum_{k=0}^n {}^nC_k \{\Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) L_k^{(\alpha+1)}(y) dy. \end{aligned} \quad (4.3)$$

Using orthogonal property of Laguerre's polynomial and (1.5), we have

$$\begin{aligned} E_n^1(0) - f(0) &= \frac{1}{2^n} \sum_{k=0}^n {}^nC_k \int_0^\infty \phi(y) L_k^{(\alpha+1)}(y) dy \\ &= \left(\int_0^{1/n} + \int_{1/n}^\delta + \int_\delta^n + \int_n^\infty \right) \frac{1}{2^n} \sum_{k=0}^n {}^nC_k \phi(y) L_k^{(\alpha+1)}(y) dy \\ &= I_1 + I_2 + I_3 + I_4 \quad (\text{say}). \end{aligned} \quad (4.4)$$

Using orthogonal property and condition (3.2) (taking $\alpha + 1$ for α and δ for ϵ) of Lemma 3.1, we get

$$\begin{aligned}
 I_1 &= \frac{1}{2^n} \sum_{k=0}^n {}^nC_k O\{n^{\alpha+1}\} \int_0^{1/n} |\phi(y)| dy \\
 &= \frac{1}{2^n} \sum_{k=0}^n {}^nC_k O\{n^{\alpha+1}\} o\left\{\frac{1}{n^{\alpha+1}} \xi(n)\right\}, \\
 I_1 &= o\left(\frac{1}{2^n} \sum_{k=0}^n {}^nC_k \xi(n)\right) \\
 &= o\{\xi(n)\} \quad \text{since } \sum_{k=0}^n {}^nC_k = 2^n.
 \end{aligned} \tag{4.5}$$

Further, using orthogonal property and condition (3.1) (taking $\alpha + 1$ for α , 1 for c , and δ for ϵ) of Lemma 3.1, we get

$$I_2 = \frac{1}{2^n} \sum_{k=0}^n {}^nC_k O\{n^{(2\alpha+1)/4}\} \int_{1/n}^{\delta} |\phi(y)| y^{-(2\alpha+3)/4} dy. \tag{4.6}$$

Now,

$$\begin{aligned}
 \sum_{k=0}^n {}^nC_k n^{(2\alpha+1)/4} &= \left\{ \sum_{k=0}^{[n/2]} + \sum_{k=[n/2]+1}^n \right\} {}^nC_k n^{(2\alpha+1)/4} \\
 &= (n)^{(2\alpha+1)/4} \sum_{k=0}^{[n/2]} {}^nC_k + {}^nC_{[n/2]} (n)^{(2\alpha+5)/4} \\
 &\leq (n)^{(2\alpha+1)/4} \sum_{k=0}^n {}^nC_k + {}^nC_{[n/2]} (n)^{(2\alpha+5)/4}, \\
 \sum_{k=0}^n {}^nC_k n^{(2\alpha+1)/4} &= (n)^{(2\alpha+1)/4} 2^n + {}^nC_{[n/2]} (n)^{(2\alpha+5)/4}
 \end{aligned} \tag{4.7}$$

since

$$\begin{aligned}
 2^n &= \sum_{k=0}^n {}^nC_k \\
 &= {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_{[n/2]} + {}^nC_{[n/2]+1} + \cdots \\
 &\geq {}^nC_{[n/2]} + {}^nC_{[n/2]} + \cdots + {}^nC_n \\
 &\geq {}^nC_{[n/2]} + {}^nC_{[n/2]} + \cdots + {}^nC_{[n/2]} = \left\{ \left\lfloor \frac{n}{2} \right\rfloor + 1 \right\} {}^nC_{[n/2]} \geq \frac{n}{2} {}^nC_{[n/2]}.
 \end{aligned} \tag{4.8}$$

Therefore,

$$\frac{n}{2} {}^nC_{[n/2]} \leq 2^n. \quad (4.9)$$

By (4.7) and (4.9), we have,

$$\begin{aligned} \sum_{k=0}^n {}^nC_k \left\{ n^{(2\alpha+1)/4} \right\} &\leq (n)^{(2\alpha+1)/4} 2^n + 2(2^n) (n)^{(2\alpha+1)/4} \\ &= O\left\{ (n)^{(2\alpha+1)/4} 2^n \right\}. \end{aligned} \quad (4.10)$$

Thus,

$$\begin{aligned} I_2 &= O\left\{ (n)^{(2\alpha+1)/4} \right\} \int_{1/n}^{\delta} y^{-(2\alpha+3)/4} |\phi(y)| dy \\ &= O\left\{ (n)^{(2\alpha+1)/4} \right\} \left[\left\{ y^{-(2\alpha+3)/4} \Phi(y) \right\}_{1/n}^{\delta} + \int_{1/n}^{\delta} \frac{(2\alpha+3)}{4} y^{-(2\alpha+7)/4} \Phi(y) dy \right] \\ &= O\left\{ (n)^{(2\alpha+1)/4} \right\} \left[y^{-(2\alpha+3)/4} o\left\{ y^{\alpha+1} \xi\left(\frac{1}{y}\right) \right\} \right] + \left\{ \int_{1/n}^{\delta} y^{-(2\alpha+7)/4} o\left\{ y^{\alpha+1} \xi\left(\frac{1}{y}\right) \right\} dy \right\} \\ &= O\left\{ (n)^{(2\alpha+1)/4} \right\} \left[o\left\{ y^{(2\alpha+1)/4} \xi\left(\frac{1}{y}\right) \right\} \right]_{1/n}^{\delta} + o\left\{ \int_{1/n}^{\delta} y^{(2\alpha-3)/4} \xi\left(\frac{1}{y}\right) dy \right\} \\ &= O\left\{ (n)^{(2\alpha+1)/4} \right\} \left[O(1) + o\left\{ n^{-(2\alpha+1)/4} \xi(n) \right\} \right] + \xi(n) o\left\{ \int_{1/n}^{\delta} y^{(2\alpha-3)/4} dy \right\} \\ &= o\{\xi(n)\} + o\left\{ (n)^{(2\alpha+1)/4} \xi(n) \right\} \left\{ \int_{1/n}^{\delta} y^{(2\alpha-3)/4} dy \right\} \\ &= o\{\xi(n)\} + o\left\{ n^{(2\alpha+1)/4} \xi(n) \right\} \left\{ \frac{y^{(2\alpha-3)/4+1}}{\{(2\alpha-3)/4\} + 1} \right\}_{1/n}^{\delta} \\ &= o\{\xi(n)\} + o\left\{ n^{(2\alpha+1)/4} \xi(n) \right\} \left\{ \frac{y^{(2\alpha+1)/4}}{(2\alpha+1)/4} \right\}_{1/n}^{\delta} \\ &= o\{\xi(n)\} + o\left\{ n^{(2\alpha+1)/4} \xi(n) \right\} \left\{ n^{-(2\alpha+1)/4} \right\} \\ &= o\{\xi(n)\} + o\{\xi(n)\}, \\ I_2 &= o\{\xi(n)\}. \end{aligned} \quad (4.11)$$

Now, we consider

$$\begin{aligned} I_3 &= \left[\left(\frac{1}{2^n} \right) \left\{ \sum_{k=0}^n {}^nC_k \int_{\delta}^n e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| e^{-y/2} y^{(2\alpha+3)/4} \left| L_n^{(\alpha+1)}(y) \right| dy \right\} \right] \\ &= \left(\frac{1}{2^n} \right) \left\{ \sum_{k=0}^n {}^nC_k O\{n^{(2\alpha+1)/4}\} \int_{\delta}^n e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| dy \right\}, \end{aligned} \quad (4.12)$$

$$I_3 = O\left\{ (n)^{(2\alpha+1)/4} \right\} o\left\{ (n)^{-(2\alpha+1)/4} \xi(n) \right\}, \quad \text{using (2.4),}$$

$$I_3 = o\{\xi(n)\}.$$

Finally,

$$\begin{aligned} I_4 &= \left[\left(\frac{1}{2^n} \right) \left\{ \sum_{k=0}^n {}^nC_k \int_n^{\infty} e^{y/2} y^{-(3\alpha+5)/6} |\phi(y)| e^{-y/2} y^{(3\alpha+5)/6} \left| L_n^{(\alpha+1)}(y) \right| dy \right\} \right] \\ &= \left(\frac{1}{2^n} \right) \left\{ \sum_{k=0}^n {}^nC_k O\{k^{(\alpha+1)/2}\} \int_n^{\infty} \frac{e^{y/2} y^{-1/3} |\phi(y)|}{y^{(\alpha+1)/2}} dy \right\}, \end{aligned} \quad (4.13)$$

$$I_4 = O\left[\left(\frac{1}{2^n} \right) (2^n) \left\{ n^{(\alpha+1)/2} n^{-(\alpha+1)/2} \right\} o\{\xi(n)\} \right], \quad \text{by (2.5)}$$

$$I_4 = o\{\xi(n)\}.$$

Combining (4.4), (4.5), (4.11), (4.12), and (4.13), we get

$$E_n^1(0) - f(0) = o\{\xi(n)\}. \quad (4.14)$$

This completes the proof of the theorem.

References

- [1] G. H. Hardy, *Divergent Series*, University Press, Oxford, UK, 1st edition, 1949.
- [2] D. P. Gupta, "Degree of approximation by Cesàro means of Fourier-Laguerre expansions," *Acta Scientiarum Mathematicarum*, vol. 32, pp. 255–259, 1971.
- [3] G. Szegő, *Orthogonal Polynomials*, Colloquium Publications American Mathematical Society, New York, NY, USA, 1959.
- [4] T. Singh, "Degree of approximation by harmonic means of Fourier-Laguerre expansions," *Publicationes Mathematicae Debrecen*, vol. 24, no. 1-2, pp. 53–57, 1977.