Research Article

(L, M)-Fuzzy σ-Algebras

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The notion of (L, M)-fuzzy σ-algebras is introduced in the lattice value fuzzy set theory. It is a generalization of Klement’s fuzzy σ-algebras. In our definition of (L, M)-fuzzy σ-algebras, each L-fuzzy subset can be regarded as an L-measurable set to some degree.

1. Introduction and Preliminaries

In 1980, Klement established an axiomatic theory of fuzzy σ-algebras in [1] in order to prepare a measure theory for fuzzy sets. In the definition of Klement’s fuzzy σ-algebra (X, σ), σ was defined as a crisp family of fuzzy subsets of a set X satisfying certain set of axioms. In 1991, Biacino and Lettieri generalized Klement’s fuzzy σ-algebras to L-fuzzy setting [2].

In this paper, when both L and M are complete lattices, we define an (L, M)-fuzzy σ-algebra on a nonempty set X by means of a mapping σ : LX → M satisfying three axioms. Thus each L-fuzzy subset of X can be regarded as an L-measurable set to some degree.

When σ is an (L, M)-fuzzy σ-algebra on X, (X, σ) is called an (L, M)-fuzzy measurable space. An (L, 2)-fuzzy σ-algebra is also called an L-σ-algebra. A Klement σ-algebra can be viewed as a stratified [0, 1]-σ-algebra. A Biacino-Lettieri L-σ-algebra can be viewed as a stratified L-σ-algebra. A (2, M)-fuzzy σ-algebra is also called an M-fuzzifying σ-algebra. A crisp σ-algebra can be regarded as a (2, 2)-fuzzy σ-algebra.

Throughout this paper, both L and M denote complete lattices, and L has an order-reversing involution. X is a nonempty set. LX is the set of all L-fuzzy sets (or L-sets for short) on X. We often do not distinguish a crisp subset A of X and its character function χA. The smallest element and the largest element in M are denoted by ⊥M and ⊤M, respectively.

The binary relation < in M is defined as follows: for a, b ∈ M, a < b if and only if for every subset D ⊆ M, the relation b ≤ supD always implies the existence of d ∈ D with a ≤ d [3]. {a ∈ M : a < b} is called the greatest minimal family of b in the sense of [4], denoted by
\[ \beta(b). \text{ Moreover, for } b \in M, \text{ we define } \alpha(b) = \{ a \in M : a \dot{\prec} b \}. \text{ In a completely distributive lattice } M, \text{ there exist } \alpha(b) \text{ and } \beta(b) \text{ for each } b \in M, \text{ and } b = \bigvee \beta(b) = \bigwedge \alpha(b) \text{ (see [4]).} \]

In [4], Wang thought that \( \beta(0) = \{0\} \) and \( \alpha(1) = \{1\} \). In fact, it should be that \( \beta(0) = \emptyset \) and \( \alpha(1) = \emptyset \).

For a complete lattice \( L, A \in L^X \) and \( a \in L \), we use the following notation:

\[ A_{[a]} = \{ x \in X : A(x) \geq a \}. \tag{1.1} \]

If \( L \) is completely distributive, then we can define

\[ A_a^\alpha = \{ x \in X : a \notin \alpha(A(x)) \}. \tag{1.2} \]

Some properties of these cut sets can be found in [5–10].

**Theorem 1.1** (see [4]). Let \( M \) be a completely distributive lattice and \( \{a_i : i \in \Omega\} \subseteq M \). Then

1. \( \alpha(\bigwedge_{i \in \Omega} a_i) = \bigcup_{i \in \Omega} \alpha(a_i) \), that is, \( \alpha \) is an \( \bigwedge - \bigcup \) map;
2. \( \beta(\bigvee_{i \in \Omega} a_i) = \bigcup_{i \in \Omega} \beta(a_i) \), that is, \( \beta \) is a union-preserving map.

For \( a \in L \) and \( D \subseteq X \), we define two \( L \)-fuzzy sets \( a \land D \) and \( a \lor D \) as follows:

\[ (a \land D)(x) = \begin{cases} a, & x \in D; \\ 0, & x \notin D. \end{cases} \]

\[ (a \lor D)(x) = \begin{cases} 1, & x \in D; \\ a, & x \notin D. \end{cases} \tag{1.3} \]

Then for each \( L \)-fuzzy set \( A \) in \( L^X \), it follows that

\[ A = \bigvee_{a \in L} (a \land A_{[a]}). \tag{1.4} \]

**Theorem 1.2** (see [5, 7, 10]). If \( L \) is completely distributive, then for each \( L \)-fuzzy set \( A \) in \( L^X \), we have

1. \( A = \bigvee_{a \in L} (a \land A_{[a]}) = \bigwedge_{a \in L} (a \lor A[a]) \);
2. for all \( a \in L \), \( A_{[a]} = \bigcap_{b \in \beta(a)} A_{[b]} \);
3. for all \( a \in L \), \( A[a] = \bigcap_{a \in \alpha(b)} A[b] \).

For a family of \( L \)-fuzzy sets \( \{A_i : i \in \Omega\} \) in \( L^X \), it is easy to see that

\[ \left( \bigwedge_{i \in \Omega} A_i \right)_{[a]} = \bigcap_{i \in \Omega} (A_i)_{[a]} \tag{1.5}. \]

If \( L \) is completely distributive, then it follows [7] that

\[ \left( \bigwedge_{i \in \Omega} A_i \right)^a = \bigcap_{i \in \Omega} (A_i)^a \tag{1.6} \].
Definition 1.3. Let $X$ be a nonempty set. A subset $\sigma$ of $[0,1]^X$ is called a Klement fuzzy $\sigma$-algebra if it satisfies the following three conditions:

1. for any constant fuzzy set $\alpha$, $\alpha \in \sigma$;
2. for any $A \in [0,1]^X$, $1 - A \in \sigma$;
3. for any \{ $A_n : n \in \mathbb{N}$ \} $\subseteq \sigma$, $\bigvee_{n \in \mathbb{N}} A_n \in \sigma$.

The fuzzy sets in $\sigma$ are called fuzzy measurable sets, and the pair $(X, \sigma)$ a fuzzy measurable space.

Definition 1.4. Let $L$ be a complete lattice with an order-reversing involution $'$ and $X$ a nonempty set. A subset $\sigma$ of $L^X$ is called an $L$-$\sigma$-algebra if it satisfies the following three conditions:

1. for any $a \in L$, constant $L$-fuzzy set $a \land \chi_X \in \sigma$;
2. for any $A \in L^X$, $A' \in \sigma$;
3. for any \{ $A_n : n \in \mathbb{N}$ \} $\subseteq \sigma$, $\bigvee_{n \in \mathbb{N}} A_n \in \sigma$.

The $L$-fuzzy sets in $\sigma$ are called $L$-measurable sets, and the pair $(X, \sigma)$ an $L$-measurable space.

2. $(L, M)$-Fuzzy $\sigma$-Algebras

L. Biacino and A. Lettieri defined that an $L$-$\sigma$-algebra $\sigma$ is a crisp subset of $L^X$. Now we consider an $M$-fuzzy subset $\sigma$ of $L^X$.

Definition 2.1. Let $X$ be a nonempty set. A mapping $\sigma : L^X \rightarrow M$ is called an $(L, M)$-fuzzy $\sigma$-algebra if it satisfies the following three conditions:

(LMS1) $\sigma(\chi_\emptyset) = T_M$;
(LMS2) for any $A \in L^X$, $\sigma(A) = \sigma(A')$;
(LMS3) for any \{ $A_n : n \in \mathbb{N}$ \} $\subseteq L^X$, $\sigma(\bigvee_{n \in \mathbb{N}} A_n) \geq \bigwedge_{n \in \mathbb{N}} \sigma(A_n)$.

An $(L, M)$-fuzzy $\sigma$-algebra $\sigma$ is said to be stratified if and only if it satisfies the following condition:

(LMS1)* $\forall a \in L$, $\sigma(a \land \chi_X) = T_M$.

If $\sigma$ is an $(L, M)$-fuzzy $\sigma$-algebra, then $(X, \sigma)$ is called an $(L, M)$-fuzzy measurable space.

An $(L, 2)$-fuzzy $\sigma$-algebra is also called an $L$-$\sigma$-algebra, and an $(L, 2)$-fuzzy measurable space is also called an $L$-measurable space.

A $(2, M)$-fuzzy $\sigma$-algebra is also called an $M$-fuzzifying $\sigma$-algebra, and a $(2, M)$-fuzzy measurable space is also called an $M$-fuzzifying measurable space.

Obviously a crisp measurable space can be regarded as a $(2, 2)$-fuzzy measurable space.

If $\sigma$ is an $(L, M)$-fuzzy $\sigma$-algebra, then $\sigma(A)$ can be regarded as the degree to which $A$ is an $L$-measurable set.
Remark 2.2. If a subset $\sigma$ of $L^X$ is regarded as a mapping $\sigma : L^X \rightarrow 2$, then $\sigma$ is an $L$-$\sigma$-algebra if and only if it satisfies the following conditions:

1. $\chi_{\emptyset} \in \sigma$;
2. $A \in \sigma \Rightarrow A' \in \sigma$;
3. for any $\{A_n : n \in \mathbb{N}\} \subseteq \sigma$, $\bigvee_{n \in \mathbb{N}} A_n \in \sigma$.

Thus we easily see that a Klement $\sigma$-algebra is exactly a stratified $[0,1]$-$\sigma$-algebra, and a Biacino-Lettieri $L$-$\sigma$-algebra is exactly a stratified $L$-$\sigma$-algebra.

Moreover, when $L = 2$, a mapping $\sigma : 2^X \rightarrow M$ is an $M$-fuzzifying $\sigma$-algebra if and only if it satisfies the following conditions:

1. $\sigma(\emptyset) = T_M$;
2. for any $A \in 2^X$, $\sigma(A) = \sigma(A')$;
3. for any $\{A_n : n \in \mathbb{N}\} \subseteq 2^X$, $\sigma(\bigvee_{n \in \mathbb{N}} A_n) \geq \bigwedge_{n \in \mathbb{N}} \sigma(A_n)$.

Example 2.3. Let $(X, \sigma)$ be a crisp measurable space. Define $\chi_\sigma : 2^X \rightarrow [0,1]$ by

$$\chi_\sigma(A) = \begin{cases} 1, & A \in \sigma; \\ 0, & A \notin \sigma. \end{cases}$$

Then it is easy to prove that $(X, \chi_\sigma)$ is a $[0,1]$-fuzzifying measurable space.

Example 2.4. Let $X$ be a nonempty set and $\sigma : 2^X \rightarrow [0,1]$ a mapping defined by

$$\sigma(A) = \begin{cases} 1, & A \in \{\emptyset, X\}; \\ 0.5, & A \notin \{\emptyset, X\}. \end{cases}$$

Then it is easy to prove that $(X, \sigma)$ is a $[0,1]$-fuzzifying measurable space. If $A \in 2^X$ with $A \notin \{\emptyset, X\}$, then 0.5 is the degree to which $A$ is measurable.

Example 2.5. Let $X$ be a nonempty set and $\sigma : [0,1]^X \rightarrow [0,1]$ a mapping defined by

$$\sigma(A) = \begin{cases} 1, & A \in \{\chi_{\emptyset}, \chi_X\}; \\ 0.5, & A \notin \{\chi_{\emptyset}, \chi_X\}. \end{cases}$$
Then it is easy to prove that \( (X, \sigma) \) is a \( ([0,1], [0,1]) \)-fuzzy measurable space. If \( A \in [0,1]^X \) with \( A \notin \{ \chi \theta, \chi X \} \), then 0.5 is the degree to which \( A \) is \( [0,1] \)-measurable.

**Corollary 2.10.** A mapping \( \sigma : L^X \to M \) is an \( (L, M) \)-fuzzy \( \sigma \)-algebra if and only if for each \( a \in M \setminus \{ \bot M \}, \sigma[a] \) is an \( L \)-\( \sigma \)-algebra.

**Theorem 2.7.** A mapping \( \sigma : L^X \to M \) is an \( (L, M) \)-fuzzy \( \sigma \)-algebra if and only if for each \( a \in M \setminus \{ \bot M \}, \sigma[a] \) is an \( L \)-\( \sigma \)-algebra.

**Proof.** The proof is obvious and is omitted. \( \square \)

**Corollary 2.8.** A mapping \( \sigma : 2^X \to M \) is an \( M \)-fuzzifying \( \sigma \)-algebra if and only if for each \( a \in M \setminus \{ \bot M \}, \sigma[a] \) is a \( \sigma \)-algebra.

**Theorem 2.9.** If \( M \) is completely distributive, then a mapping \( \sigma : L^X \to M \) is an \( (L, M) \)-fuzzy \( \sigma \)-algebra if and only if for each \( a \in \alpha(\bot M), \sigma[a] \) is an \( L \)-\( \sigma \)-algebra.

**Proof.**

**Necessity.** Suppose that \( \sigma : L^X \to M \) is an \( (L, M) \)-fuzzy \( \sigma \)-algebra and \( a \in \alpha(\bot M) \). Now we prove that \( \sigma[a] \) is an \( L \)-\( \sigma \)-algebra.

\[(LS1) \text{ By } \sigma(\chi \theta) = \top_M \text{ and } \alpha(\top_M) = \emptyset \text{, we know that } a \notin \alpha(\sigma(\chi \theta)); \text{ this implies that } \chi \theta \in \sigma[a].\]

\[(LS2) \text{ If } A \in \sigma[a], \text{ then } a \notin \alpha(\sigma(A)) = \alpha(\sigma(A')); \text{ this shows that } A' \in \sigma[a].\]

\[(LS3) \text{ If } \{A_i : i \in \Omega \} \subseteq \sigma[a], \text{ then for all } i \in \Omega, a \notin \alpha(\sigma(A_i)). \text{ Hence } a \notin \bigcup_{i \in \Omega} \alpha(\sigma(A_i)). \text{ By } \sigma(\bigcup_{i \in \Omega} A_i) \supseteq \bigwedge_{i \in \Omega} \sigma(A_i), \text{ we know that}\]

\[
\alpha\left(\sigma\left(\bigcup_{i \in \Omega} A_i\right)\right) \subseteq \alpha\left(\bigwedge_{i \in \Omega} \sigma(A_i)\right) = \bigcup_{i \in \Omega} \alpha(\sigma(A_i)). \tag{2.5}\]

This shows that \( a \notin \alpha(\sigma(\bigcup_{i \in \Omega} A_i)) \). Therefore, \( \bigcup_{i \in \Omega} A_i \in \sigma[a] \). The proof is completed. \( \square \)

**Corollary 2.10.** If \( M \) is completely distributive, then a mapping \( \sigma : 2^X \to M \) is an \( M \)-fuzzifying \( \sigma \)-algebra if and only if for each \( a \in \alpha(\bot M), \sigma[a] \) is a \( \sigma \)-algebra.
Corollary 2.11. If $M$ is completely distributive, and $\sigma$ is an $(L, M)$-fuzzy $\sigma$-algebra, then

1. $\sigma[b] \subseteq \sigma[a]$ for any $a, b \in M \setminus \{\bot_M\}$ with $a \in \beta(b)$;

2. $\sigma[b] \subseteq \sigma[a]$ for any $a, b \in \alpha(\bot_M)$ with $b \in \alpha(a)$.

Theorem 2.12. Let $M$ be completely distributive, and let $\{\sigma^a : a \in \alpha(\bot_M)\}$ be a family of $L$-$\sigma$-algebras. If $\sigma^a = \bigcap \{\sigma^b : a \in \alpha(b)\}$ for all $a \in \alpha(\bot_M)$, then there exists an $(L, M)$-fuzzy $\sigma$-algebra $\sigma$ such that $\sigma[a] = \sigma^a$.

Proof. Suppose that $\sigma^a = \bigcap \{\sigma^b : a \in \alpha(b)\}$ for all $a \in \alpha(\bot_M)$. Define $\sigma : L^X \to M$ by

$$\sigma(A) = \bigwedge_{a \in M} (a \lor \sigma^a(A)) = \bigwedge \{a \in M : A \in \sigma^a\}. \quad (2.6)$$

By Theorem 1.2, we can obtain that $\sigma[a] = \sigma^a$.

Corollary 2.13. Let $M$ be completely distributive, and let $\{\sigma^a : a \in \alpha(\bot_M)\}$ be a family of $\sigma$-algebras. If $\sigma^a = \bigcap \{\sigma^b : a \in \alpha(b)\}$ for all $a \in \alpha(\bot_M)$, then there exists an $M$-fuzzifying $\sigma$-algebra $\sigma$ such that $\sigma[a] = \sigma^a$.

Theorem 2.14. Let $M$ be completely distributive, and let $\{\sigma_a : a \in M \setminus \{\bot_M\}\}$ be a family of $L$-$\sigma$-algebra. If $\sigma_a = \bigcap \{\sigma_b : a \in \beta(a)\}$ for all $a \in M \setminus \{\bot_M\}$, then there exists an $(L, M)$-fuzzy $\sigma$-algebra $\sigma$ such that $\sigma[a] = \sigma_a$.

Proof. Suppose that $\sigma_a = \bigcap \{\sigma_b : b \in \beta(a)\}$ for all $a \in M \setminus \{\bot_M\}$. Define $\sigma : L^X \to M$ by

$$\sigma(A) = \bigvee_{a \in M} (a \land \sigma_a(A)) = \bigvee \{a \in M : A \in \sigma_a\}. \quad (2.7)$$

By Theorem 1.2, we can obtain $\sigma[a] = \sigma_a$.

Corollary 2.15. Let $M$ be completely distributive, and let $\{\sigma_a : a \in M \setminus \{\bot_M\}\}$ be a family of $\sigma$-algebra. If $\sigma_a = \bigcap \{\sigma_b : b \in \beta(a)\}$ for all $a \in M \setminus \{\bot_M\}$, then there exists an $M$-fuzzifying $\sigma$-algebra $\sigma$ such that $\sigma[a] = \sigma_a$.

Theorem 2.16. Let $\{\sigma_i : i \in \Omega\}$ be a family of $(L, M)$-fuzzy $\sigma$-algebra on $X$. Then $\bigwedge_{i \in \Omega} \sigma_i$ is an $(L, M)$-fuzzy $\sigma$-algebra on $X$, where $\bigwedge_{i \in \Omega} \sigma_i : L^X \to M$ is defined by $(\bigwedge_{i \in \Omega} \sigma_i)(A) = \bigwedge_{i \in \Omega} \sigma_i(A)$.

Proof. This is straightforward.
3. \((L, M)\)-Fuzzy Measurable Functions

In this section, we will generalize the notion of measurable functions to fuzzy setting.

**Theorem 3.1.** Let \((Y, \tau)\) be an \((L, M)\)-fuzzy measurable space and \(f : X \rightarrow Y\) a mapping. Define a mapping \(f_L^- (\tau) : L^X \rightarrow M\) by for all \(A \in L^X\),

\[
    f_L^- (\tau)(A) = \bigvee \{ \tau(B) : f_L^- (B) = A \}, \quad \text{where } \forall x \in X, \ f_L^- (B)(x) = B(f(x)). \quad (3.1)
\]

Then \((X, f_L^- (\tau))\) is an \((L, M)\)-fuzzy measurable space.

**Proof.** \((\text{LMS1})\) holds from the following equality:

\[
    f_L^- (\tau)(\chi_\emptyset) = \bigvee \{ \tau(B) : f_L^- (B) = \chi_\emptyset \} = \tau(\chi_\emptyset) = \top_M. \quad (3.2)
\]

\((\text{LMS2})\) can be shown from the following fact: for all \(A \in L^X\),

\[
    f_L^- (\tau)(A) = \bigvee \{ \tau(B) : f_L^- (B) = A \}
    = \bigvee \{ \tau(B') : f_L^- (B') = f_L^- (B)' = A' \} \quad (3.3)
    = f_L^- (\tau)(A').
\]

\((\text{LMS3})\) for any \(\{ A_n : n \in \mathbb{N} \} \subseteq L^X\), by

\[
    f_L^- (\tau) \left( \bigvee_{n \in \mathbb{N}} A_n \right) = \bigvee \left\{ \tau(B) : f_L^- (B) = \bigvee_{n \in \mathbb{N}} A_n \right\}
    \geq \bigvee \left\{ \tau \left( \bigvee_{n \in \mathbb{N}} B_n \right) : f_L^- (B_n) = A_n \right\} \quad (3.4)
    \geq \bigwedge_{n \in \mathbb{N}} f_L^- (\tau)(A_n)
\]

we can prove \((\text{LMS3})\). \qed

**Definition 3.2.** Let \((X, \sigma)\) and \((Y, \tau)\) be \((L, M)\)-fuzzy measurable spaces. A mapping \(f : X \rightarrow Y\) is called \((L, M)\)-fuzzy measurable if \(\sigma(f_L^- (B)) \geq \tau(B)\) for all \(B \in L^Y\).

An \((L, 2)\)-fuzzy measurable mapping is called an \(L\)-measurable mapping, and a \((2, M)\)-fuzzy measurable mapping is called an \(M\)-fuzzifying measurable mapping.

Obviously a Klement fuzzy measurable mapping can be viewed as an \([0,1]\)-measurable mapping.

The following theorem gives a characterization of \((L, M)\)-fuzzy measurable mappings.
Theorem 3.3. Let \((X, \sigma)\) and \((Y, \tau)\) be two \((L, M)\)-fuzzy measurable spaces. A mapping \(f : X \to Y\) is \((L, M)\)-fuzzy measurable if and only if \(f^{-1}_L(\tau)(A) \leq \sigma(A)\) for all \(A \in L^X\).

Proof.

Necessity. If \(f : X \to Y\) is \((L, M)\)-fuzzy measurable, then \(\sigma(f^{-1}_L(B)) \geq \tau(B)\) for all \(B \in L^Y\). Hence for all \(B \in L^Y\), we have

\[
f^{-1}_L(\tau)(A) = \bigvee \{ \tau(B) : f^{-1}_L(B) = A \} \leq \bigvee \{ \sigma(f^{-1}_L(B)) : f^{-1}_L(B) = A \} \leq \sigma(A).
\]

Sufficiency. If \(f^{-1}_L(\tau)(A) \leq \sigma(A)\) for all \(A \in L^X\), then \(\tau(B) \leq f^{-1}_L(\tau)(f^{-1}_L(B)) \leq \sigma(f^{-1}_L(B))\) for all \(B \in L^Y\); this shows that \(f : X \to Y\) is \((L, M)\)-fuzzy measurable.

The next three theorems are trivial.

Theorem 3.4. If \(f : (X, \sigma) \to (Y, \tau)\) and \(g : (Y, \tau) \to (Z, \rho)\) are \((L, M)\)-fuzzy measurable, then \(g \circ f : (X, \sigma) \to (Z, \rho)\) is \((L, M)\)-fuzzy measurable.

Theorem 3.5. Let \((X, \sigma)\) and \((Y, \tau)\) be \((L, M)\)-fuzzy measurable spaces. Then a mapping \(f : (X, \sigma) \to (Y, \tau)\) is \((L, M)\)-fuzzy measurable if and only if \(f : (X, \sigma[a]) \to (Y, \tau[a])\) is \(L\)-measurable for any \(a \in M \setminus \{\bot_M\}\).

Theorem 3.6. Let \(M\) be completely distributive, and let \((X, \sigma)\) and \((Y, \tau)\) be \((L, M)\)-fuzzy measurable spaces. Then a mapping \(f : (X, \sigma) \to (Y, \tau)\) is \((L, M)\)-fuzzy measurable if and only if \(f : (X, \sigma[a]) \to (Y, \tau[a])\) is \(L\)-measurable for any \(a \in \sigma(\bot_M)\).

Corollary 3.7. Let \((X, \sigma)\) and \((Y, \tau)\) be \(M\)-fuzzifying measurable spaces. Then a mapping \(f : (X, \sigma) \to (Y, \tau)\) is \(M\)-fuzzifying measurable if and only if \(f : (X, \sigma[a]) \to (Y, \tau[a])\) is measurable for any \(a \in M \setminus \{\bot_M\}\).

Corollary 3.8. Let \(M\) be completely distributive, and let \((X, \sigma)\) and \((Y, \tau)\) be \(M\)-fuzzifying measurable spaces. Then a mapping \(f : (X, \sigma) \to (Y, \tau)\) is \(M\)-fuzzifying measurable if and only if \(f : (X, \sigma[a]) \to (Y, \tau[a])\) is measurable for any \(a \in \sigma(\bot_M)\).

4. \((I, I)\)-Fuzzy \(\sigma\)-Algebras Generated by \(I\)-Fuzzifying \(\sigma\)-Algebras

In this section, \(B\) will be used to denote the \(\sigma\)-algebra of Borel subsets of \(I = [0, 1]\).

Theorem 4.1. Let \((X, \sigma)\) be an \(I\)-fuzzifying measurable space. Define a mapping \(\xi(\sigma) : I^X \to I\) by

\[
\xi(\sigma)(A) = \bigwedge_{B \in B} \sigma\left( A^{-1}(B) \right) .
\]

(4.1)
Then $\zeta(\sigma)$ is a stratified $(I, I)$-fuzzy $\sigma$-algebra, which is said to be the $(I, I)$-fuzzy $\sigma$-algebra generated by $\sigma$.

**Proof.** (LMS1) For any $B \in \mathcal{B}$ and for any $a \in I$, if $a \in B$, then $(a \wedge \chi_X)^{-1}(B) = X$; if $a \notin B$, then $(a \wedge \chi_X)^{-1}(B) = \emptyset$. However, we have that $\sigma((a \wedge \chi_X)^{-1}(B)) = 1$. This shows that $\zeta(\sigma)(a \wedge \chi_X) = 1$.

(LMS2) for all $A \in I^X$ and for all $B \in \mathcal{B}$, we have

$$
\zeta(\sigma)(A') = \bigwedge_{B \in \mathcal{B}} \sigma\left( (1 - A)^{-1}(B) \right)
= \bigwedge_{B \in \mathcal{B}} \sigma\left( \{ x \in X : 1 - A(x) \in B \} \right)
= \bigwedge_{B \in \mathcal{B}} \sigma\left( \{ x \in X : \exists b \in B, \text{ s.t. } A(x) = 1 - b \} \right)
= \bigwedge_{B \in \mathcal{B}} \sigma\left( A^{-1}(B) \right)
= \zeta(\sigma)(A).
$$

(LMS3) for any $\{ A_n : n \in \mathbb{N} \} \subseteq I^X$ and for all $B \in \mathcal{B}$, by

$$
\zeta(\sigma)\left( \bigvee_{n \in \mathbb{N}} A_n \right) = \bigwedge_{B \in \mathcal{B}} \sigma\left( \left( \bigvee_{n \in \mathbb{N}} A_n \right)^{-1}(B) \right)
= \bigwedge_{B \in \mathcal{B}} \sigma\left( \bigcup_{n \in \mathbb{N}} A_n^{-1}(B) \right)
\geq \bigwedge_{B \in \mathcal{B}} \bigwedge_{n \in \mathbb{N}} \sigma\left( A_n^{-1}(B) \right)
= \bigwedge_{n \in \mathbb{N}} \bigwedge_{B \in \mathcal{B}} \sigma\left( A_n^{-1}(B) \right) = \bigwedge_{n \in \mathbb{N}} \zeta(\sigma)(A_n),
$$
we obtain $\zeta(\sigma)(\bigvee_{n \in \mathbb{N}} A_n) \geq \bigwedge_{n \in \mathbb{N}} \zeta(\sigma)(A_n)$. \qed

**Corollary 4.2.** Let $(X, \sigma)$ be a measurable space. Define a subset $\zeta(\sigma) \subseteq I^X$ (can be viewed as a mapping $\zeta(\sigma) : I^X \to 2$) by

$$
\zeta(\sigma) = \left\{ A \in I^X : \forall B \in \mathcal{B}, A^{-1}(B) \in \sigma \right\}.
$$

Then $\zeta(\sigma)$ is a stratified $I$-$\sigma$-algebra.

From Corollary 4.2, we see that the functor $\zeta$ in Theorem 4.1 is a generalization of Klement functor $\zeta$. 


Theorem 4.3. Let \((X, \sigma)\) and \((Y, \tau)\) be two I-fuzzifying measurable spaces, and \(f : X \to Y\) is a map. Then \(f : (X, \sigma) \to (Y, \tau)\) is I-fuzzifying measurable if and only if \(f : (X, \zeta(\sigma)) \to (Y, \zeta(\tau))\) is \((I, I)\)-fuzzy measurable.

Proof.

Necessity. Suppose that \(f : (X, \sigma) \to (Y, \tau)\) is I-fuzzifying measurable. Then \(\sigma(f^{-1}(A)) \geq \tau(A)\) for any \(A \in 2^X\). In order to prove that \(f : (X, \zeta(\sigma)) \to (Y, \zeta(\tau))\) is \((I, I)\)-fuzzy measurable, we need to prove that \(\zeta(\sigma)(f^{-1}(A)) \geq \zeta(\tau)(A)\) for any \(A \in I^X\).

In fact, for any \(A \in I^X\), by

\[
\zeta(\sigma)(f^{-1}(A)) = \bigwedge_{B \in B} \sigma\left(\left(f^{-1}(A)\right)^{-1}(B)\right) = \bigwedge_{B \in B} \sigma\left((A \circ f)^{-1}(B)\right) \\
= \bigwedge_{B \in B} \sigma(B \circ A \circ f) = \bigwedge_{B \in B} \sigma\left(f^{-1}\left(A^{-1}(B)\right)\right) \\
\geq \bigwedge_{B \in B} \tau\left(A^{-1}(B)\right) = \zeta(\tau)(A),
\]

we can prove the necessity.

Sufficiency. Suppose that \(f : (X, \zeta(\sigma)) \to (Y, \zeta(\tau))\) is \((I, I)\)-fuzzy measurable. Then \(\zeta(\sigma)(f^{-1}(A)) \geq \zeta(\tau)(A)\) for any \(A \in I^X\). In particular, it follows that \(\zeta(\sigma)(f^{-1}(A)) \geq \zeta(\tau)(A)\) for any \(A \in 2^X\). In order to prove that \(f : (X, \sigma) \to (Y, \tau)\) is I-fuzzifying measurable, we need to prove that \(\sigma(f^{-1}(A)) \geq \tau(A)\) for any \(A \in 2^X\). In fact, for any \(A \in 2^X\) and for any \(B \in B\), if \(0, 1 \in B\), then \(A^{-1}(B) = X\); if \(0, 1 \notin B\), then \(A^{-1}(B) = \emptyset\); if only one of 0 and 1 is in \(B\), then \(A^{-1}(B) = A\) or \(A^{-1}(B) = A'\). However, we have

\[
\sigma(f^{-1}(A)) = \sigma(f^{-1}(A)) \\
= \sigma(f^{-1}(A)) \land \sigma(f^{-1}(A))' \\
= \bigwedge_{B \in B} \sigma\left(\left(f^{-1}(A)\right)^{-1}(B)\right) \\
= \zeta(\sigma)(f^{-1}(A)) \\
\geq \zeta(\tau)(A) \\
= \zeta(\tau)(A) \land \zeta(\tau)(A') \\
= \bigwedge_{B \in B} \tau\left(A^{-1}(B)\right) = \tau(A).
\]

This shows that \(f : (X, \sigma) \to (Y, \tau)\) is I-fuzzifying measurable. \(\square\)

Corollary 4.4. Let \((X, \sigma)\) and \((Y, \tau)\) be two measurable spaces, and \(f : X \to Y\) is a mapping. Then \(f : (X, \sigma) \to (Y, \tau)\) is measurable if and only if \(f : (X, \zeta(\sigma)) \to (Y, \zeta(\tau))\) is \(I\)-measurable.
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References

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