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Research Article

Bi-Lipschitz Mappings and Quasinearly Subharmonic Functions

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After considering a variant of the generalized mean value inequality of quasinearly subharmonic functions, we consider certain invariance properties of quasinearly subharmonic functions. Kojić has shown that in the plane case both the class of quasinearly subharmonic functions and the class of regularly oscillating functions are invariant under conformal mappings. We give partial generalizations to her results by showing that in $\mathbb{R}^n$, $n \geq 2$, these both classes are invariant under bi-Lipschitz mappings.

1. Introduction

Notation. Our notation is rather standard; see, for example, [1–3] and the references therein. We recall here only the following. The Lebesgue measure in $\mathbb{R}^n$, $n \geq 2$, is denoted by m_n. We write $B^n(x,r)$ for the ball in $\mathbb{R}^n$, with center x and radius r. Recall that $m_n(B^n(x,r)) = \nu_n r^n$, where $\nu_n := m_n(B^n(0,1))$. If D is an open set in $\mathbb{R}^n$, and $x \in D$, then we write $\delta_D(x)$ for the distance between the point x and the boundary ∂D of D. Our constants C are nonnegative, mostly ≥ 1, and may vary from line to line.

1.1. Subharmonic Functions and Generalizations

Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$. Let $u : \Omega \to [-\infty, +\infty)$ be a Lebesgue measurable function. We adopt the following definitions.

(i) u is *subharmonic* if u is upper semicontinuous and if

$$u(x) \leq \frac{1}{v_n r^n} \int_{B^n(x,r)} u(y)\, dm_n(y) \tag{1.1}$$

for all balls $\overline{B^n(x,r)} \subset \Omega$. A subharmonic function may be $\equiv -\infty$ on any component of Ω; see [3, page 9] and [4, page 60].

(ii) u is *nearly subharmonic* if $u^+ \in \mathcal{L}^1_{\text{loc}}(\Omega)$ and

$$u(x) \leq \frac{1}{v_n r^n} \int_{B^n(x,r)} u(y)\, dm_n(y) \tag{1.2}$$

for all balls $\overline{B^n(x,r)} \subset \Omega$. Observe that this definition, see [5, page 51], is slightly more general than the standard one [3, page 14].

(iii) Let $K \geq 1$. Then u is *K-quasinearly subharmonic* if $u^+ \in \mathcal{L}^1_{\text{loc}}(\Omega)$ and

$$u_L(x) \leq \frac{K}{v_n r^n} \int_{B^n(x,r)} u_L(y)\, dm_n(y) \tag{1.3}$$

for all $L \geq 0$ and for all balls $\overline{B^n(x,r)} \subset \Omega$. Here $u_L := \max\{u, -L\} + L$.

The function u is *quasinearly subharmonic* if u is K-quasinearly subharmonic for some $K \geq 1$. For the definition and properties of quasinearly subharmonic functions, see, for example, [1, 4–7] and the references therein.

Proposition 1.1 (cf. [5, Proposition 2.1, pages 54-55]). *The following holds.*

(i) *A subharmonic function is nearly subharmonic but not conversely.*

(ii) *A function is nearly subharmonic if and only if it is* 1*-quasinearly subharmonic.*

(iii) *A nearly subharmonic function is quasinearly subharmonic but not conversely.*

(iv) *If $u : \Omega \to [0, +\infty)$ is Lebesgue measurable, then u is K-quasinearly subharmonic if and only if $u \in \mathcal{L}^1_{\text{loc}}(\Omega)$ and*

$$u(x) \leq \frac{K}{v_n r^n} \int_{B^n(x,r)} u(y)\, dm_n(y) \tag{1.4}$$

for all balls $\overline{B^n(x,r)} \subset \Omega$.

1.2. Bi-Lipschitz Mappings

Let D be an open set in $\mathbb{R}^n$, $n \geq 2$. Let $M \geq 1$ be arbitrary. A function $f : D \to \mathbb{R}^n$ is *M-bi-Lipschitz* if

$$\frac{|y-x|}{M} \leq |f(y) - f(x)| \leq M|y-x| \tag{1.5}$$

for all $x, y \in D$. A function is *bi-Lipschitz* if it is M-bi-Lipschitz for some $M \geq 1$. It is easy to see that if $f : D \to \mathbb{R}^n$ is M-bi-Lipschitz, then also $f^{-1} : D' \to \mathbb{R}^n$ is M-bi-Lipschitz, where $D' := f(D)$.

Let Ω be an open subset of $\mathbb{R}^n$. Let $p_D \in D$ and $x_\Omega \in \Omega$. We write

$$M\text{-BiLip}(p_D, x_\Omega, D, \Omega) := \{h : D \to \mathbb{R}^n : h \text{ is } M\text{-bi-Lipschitz}, h(p_D) = x_\Omega, h(D) \subset \Omega\}. \tag{1.6}$$

2. On the Generalized Mean Value Inequality

Lemma 2.1. *Let D be a bounded open set in $\mathbb{R}^n$, $n \geq 2$. Fix a point $p_D \in D$. Let Ω be a domain in $\mathbb{R}^n$. Let $u : \Omega \to [0, +\infty)$ be a K-quasinearly subharmonic function. Then there is $C = C(K, n, D, M, p_D) \geq 1$ such that*

$$u(x_\Omega) \leq \frac{C}{m_n(h(D))} \int_{h(D)} u(y)\, dm_n(y) \tag{2.1}$$

for every point $x_\Omega \in \Omega$ and all $h \in M$-BiLip $(p_D, x_\Omega, D, \Omega)$, $M \geq 1$.

Proof. Take $x_\Omega \in \Omega$ and $h \in M$-BiLip $(p_D, x_\Omega, D, \Omega)$, $M \geq 1$, arbitrarily. (Observe that the set of bi-Lipschitz mappings is (in general) nonempty.) Write

$$R_D := \sup_{y \in D} |p_D - y|, \qquad r_D := \delta_D(p_D). \tag{2.2}$$

Using the fact that $h \mid B^n(p_D, r_D) : B^n(p_D, r_D) \to h(B^n(p_D, r_D))$ is a homeomorphism, one sees easily that $B^n(x_\Omega, r_D/M) \subset h(D)$. Since h is M-bi-Lipschitz, it follows from a result of Radó-Reichelderfer, see, for example, [8, Theorem 2.2, page 99], that $m_n(h(D)) \leq n! M^n m_n(D)$. (Observe that bi-Lipschitz mappings satisfy the property N and are differentiable almost everywhere, see, for example, [9, Theorem 33.2, page 112, Theorem 32.1, page 109].) Therefore,

$$\begin{aligned}
u(x_\Omega) &\leq \frac{K}{v_n(r_D/M)^n} \int_{B^n(x_\Omega, r_D/M)} u(y)\, dm_n(y) \\
&\leq \frac{KM^n(R_D/r_D)^n}{m_n(B^n(p_D, R_D))} \int_{h(D)} u(y)\, dm_n(y) \\
&\leq \frac{KM^n(R_D/r_D)^n}{m_n(D)} \int_{h(D)} u(y)\, dm_n(y) \\
&\leq \frac{KM^n(R_D/r_D)^n}{m_n(h(D))/n!M^n} \int_{h(D)} u(y)\, dm_n(y) \\
&\leq \frac{n!KM^{2n}(R_D/r_D)^n}{m_n(h(D))} \int_{h(D)} u(y)\, dm_n(y).
\end{aligned} \tag{2.3}$$

Thus (2.1) holds with $C = C(K, n, M, D, p_D)$. □

Theorem 2.2. *Let D be an open set in $\mathbb{R}^n$, $n \geq 2$, with $m_n(D) < +\infty$. Fix a point $p_D \in D$. Let Ω be an open set in $\mathbb{R}^n$. Let $u : \Omega \to [0, +\infty)$ be a K-quasinearly subharmonic function. Then there is a constant $C = C(K, n, D, M, p_D) \geq 1$ such that* (2.1) *holds for every point $x_\Omega \in \Omega$ and all $h \in M$-BiLip $(p_D, x_\Omega, D, \Omega)$, $M \geq 1$.*

Proof. Let $t > 1$ be arbitrary. It is easy to see that $tm_n(D \cap B^n(p_D, r_t)) \geq m_n(D)$ for some $r_t > 0$. Write $D_t := D \cap B^n(p_D, r_t)$ and $p_{D_t} = p_D$. One sees easily that D_t satisfies the assumptions of Lemma 2.1; that is, D_t is a bounded domain, $h(D_t) \subset h(D) \subset \Omega$ and $h(p_{D_t}) = h(p_D) = x_\Omega$. Hence there is a constant $C_1 = C_1(K, n, D, M, p_D) \geq 1$ such that

$$u(x_\Omega) \leq \frac{C_1}{m_n(h(D_t))} \int_{h(D_t)} u(y)\, dm_n(y) \tag{2.4}$$

for every point $x_\Omega \in \Omega$ and all $h \in M$-BiLip $(p_{D_t}, x_\Omega, D_t, \Omega)$. Since h and h^{-1} are M-bi-Lipschitz, it follows that $m_n(h(D)) \leq n! M^n m_n(D)$ and $m_n(D_t) \leq n! M^n m_n(h(D_t))$; see again [8, Theorem 2.2, page 99]. Thus for $C_2 = C_2(n, M) = (n!)^2 M^{2n}$,

$$\frac{m_n(D_t)}{m_n(D)} \leq C_2 \cdot \frac{m_n(h(D_t))}{m_n(h(D))}. \tag{2.5}$$

Proceed then as follows:

$$\begin{aligned} \frac{1}{m_n(h(D_t))} \int_{h(D_t)} u(y)\, dm_n(y) &\leq C_2 \cdot \frac{m_n(D)}{m_n(D_t)} \cdot \frac{1}{m_n(h(D))} \int_{h(D_t)} u(y)\, dm_n(y) \\ &\leq C_2 \cdot \frac{t}{m_n(h(D))} \int_{h(D_t)} u(y)\, dm_n(y) \\ &\leq C_2 \cdot \frac{t}{m_n(h(D))} \int_{h(D)} u(y)\, dm_n(y). \end{aligned} \tag{2.6}$$

Therefore

$$u(x_\Omega) \leq \frac{C_1 C_2 t}{m_n(h(D))} \int_{h(D)} u(y)\, dm_n(y), \tag{2.7}$$

concluding the proof. □

3. An Invariance of the Class of Quasinearly Subharmonic Functions

Suppose that G and U are open sets in the complex plane $\mathbb{C}$. If $f : U \to G$ is analytic and $u : G \to [-\infty, +\infty)$ is subharmonic, then $u \circ f$ is subharmonic; see, for example, [3, page 37] and [4, Corollary 3.3.4, page 70]. Using Koebe's one-quarter and distortion theorems, Kojić proved the following partial generalization.

Theorem 3.1 (see [6, Theorem 1, page 245]). *Let Ω and G be open sets in $\mathbb{C}$. Let $u : \Omega \to [0, +\infty)$ be a K-quasinearly subharmonic function. If $\varphi : G \to \Omega$ is conformal, then the composition mapping $u \circ \varphi : G \to [0, +\infty)$ is C-quasinearly subharmonic for some $C = C(K)$.*

For the definition and properties of conformal mappings, see, for example, [9, pages 13–15] and [8, pages 171-172].

Below we give a partial generalization to Kojić's result. Our result gives also a partial generalization to the standard result according to which in $\mathbb{R}^n$, $n \geq 2$, the class of subharmonic functions is invariant under orthogonal transformations; see [10, page 55].

Theorem 3.2. *Let Ω and U be open sets in $\mathbb{R}^n$, $n \geq 2$. Let $u : \Omega \to [0, +\infty)$ be a K-quasinearly subharmonic function. If $f : U \to \Omega$ is M-bi-Lipschitz, then the composition mapping $u \circ f : U \to [0, +\infty)$ is C-quasinearly subharmonic for some $C = C(K, n, M)$.*

Proof. It is sufficient to show that there exists a constant $C = C(K, n, f) > 0$ such that

$$(u \circ f)(x_0) \leq \frac{C}{m_n(B^n(x_0, r_0))} \int_{B^n(x_0, r_0)} (u \circ f)(x) dm_n(x) \tag{3.1}$$

for all $\overline{B^n(x_0, r_0)} \subset U$. To see this, observe first that

$$B^n\left(x_0', \frac{r_0}{M}\right) \subset f(B^n(x_0, r_0)) \subset B^n(x_0', Mr_0), \tag{3.2}$$

where $x_0' = f(x_0)$.

Then

$$\begin{aligned}
(u \circ f)(x_0) = u(x_0') &\leq \frac{K}{m_n(B^n(x_0', r_0/M))} \int_{B^n(x_0', r_0/M)} u(y) dm_n(y) \\
&\leq \frac{K}{v_n(r_0/M)^n} \int_{B^n(x_0', r_0/M)} (u \circ f)\left(f^{-1}(y)\right) dm_n(y) \\
&\leq \frac{KM^{2n}}{v_n(Mr_0)^n} \int_{B^n(x_0', r_0/M)} (u \circ f)\left(f^{-1}(y)\right) dm_n(y) \\
&\leq \frac{KM^{2n}}{m_n(f(B^n(x_0, r_0)))} \int_{f(B^n(x_0, r_0))} (u \circ f)\left(f^{-1}(y)\right) dm_n(y) \\
&\leq \frac{KM^{2n}}{m_n(f(B^n(x_0, r_0)))} \int_{f(B^n(x_0, r_0))} (u \circ f)\left(f^{-1}(y)\right) |J_{f^{-1}}(y)| \cdot \frac{1}{|J_{f^{-1}}(y)|} dm_n(y) \\
&\leq \frac{KM^{2n}}{m_n(f(B^n(x_0, r_0)))} \int_{f(B^n(x_0, r_0))} (u \circ f)\left(f^{-1}(y)\right) |J_{f^{-1}}(y)| \cdot \left|J_f\left(f^{-1}(y)\right)\right| dm_n(y) \\
&\leq \frac{KM^{2n}}{m_n(f(B^n(x_0, r_0)))} \int_{f(B^n(x_0, r_0))} (u \circ f)\left(f^{-1}(y)\right) |J_{f^{-1}}(y)| \cdot n! M^n dm_n(y) \\
&\leq \frac{n! KM^{3n}}{m_n(f(B^n(x_0, r_0)))} \int_{f(B^n(x_0, r_0))} (u \circ f)\left(f^{-1}(y)\right) |J_{f^{-1}}(y)| dm_n(y)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{n!KM^{3n}}{m_n\big(f(B^n(x_0,r_0))\big)}\int_{B^n(x_0,r_0)}(u\circ f)(x)dm_n(x)\\
&\leq \frac{n!KM^{3n}}{m_n\big(B^n(x_0',r_0/M)\big)}\int_{B^n(x_0,r_0)}(u\circ f)(x)dm_n(x)\\
&\leq \frac{n!KM^{4n}}{m_n(B^n(x_0,r_0))}\int_{B^n(x_0,r_0)}(u\circ f)(x)dm_n(x).
\end{aligned} \tag{3.3}$$

Above we have used the routineous fact that for M-bi-Lipschitz mappings,

$$\left|J_f\left(f^{-1}(y)\right)\right| \leq n!M^n, \tag{3.4}$$

and the already cited change of variable result of Radó-Reichelderfer; see [8, Theorem 2.2, page 99]. (Recall again that bi-Lipschitz mappings satisfy the property N and are differentiable almost everywhere.) □

4. An Invariance of Regularly Oscillating Functions

Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$. Let $f : \Omega \to \mathbb{R}^m$ be continuous. Write

$$L(x,f) := \limsup_{y\to x}\frac{|f(y)-f(x)|}{|y-x|}. \tag{4.1}$$

The function $x \mapsto L(x,f)$ is a Borel function in Ω. If f is differentiable at x, then $L(x,f) = |f'(x)|$; see [9, page 11], [11, page 19], and [12, page 93].

A function $f : \Omega \to \mathbb{R}$ is *regularly oscillating*, if there is $K \geq 1$ such that

$$L(x,f) \leq Kr^{-1}\sup_{y\in B^n(x,r)}|f(y)-f(x)|, \quad \overline{B^n(x,r)} \subset \Omega. \tag{4.2}$$

The class of such functions is denoted by $\mathrm{OC}^1_K(\Omega)$. The class of all regularly oscillating functions is denoted by $\mathrm{RO}(\Omega)$; see [11, page 19], [13, page 17], [14], [6, page 245], and [12, page 96].

Using again Koebe's results, Kojić proved also the following result.

Theorem 4.1 (see [6, Theorem 2, page 245]). *Let Ω and G be open sets in $\mathbb{C}$. Let $u \in \mathrm{OC}^1_K(\Omega)$. If $f : G \to \Omega$ is conformal, then $u\circ f \in \mathrm{OC}^1_C(G)$, where C depends only on K.*

Below we give a partial generalization to Kojić's above result.

Theorem 4.2. *Let Ω and U be open sets in $\mathbb{R}^n$, $n \geq 2$. Let $u \in \mathrm{OC}^1_K(\Omega)$. If $\varphi : U \to \Omega$ is M-bi-Lipschitz, $M \geq 1$, then $u\circ\varphi \in \mathrm{OC}^1_{KM^2}(U)$.*

Proof. Let $\varphi : U \to \Omega$ be M-bi-Lipschitz. Take $x_0 \in U$ and $r_0 > 0$ arbitrarily such that $\overline{B^n(x_0, r_0)} \subset U$. Write $x_0' = \varphi(x_0)$ and $x' = \varphi(x)$ for $x \in U$. Then

$$\begin{aligned} L(x_0, u \circ \varphi) &= \limsup_{x \to x_0} \frac{|u(\varphi(x)) - u(\varphi(x_0))|}{|x - x_0|} \\ &= \limsup_{x \to x_0} \frac{|u(\varphi(x)) - u(\varphi(x_0))|}{|\varphi(x) - \varphi(x_0)|} \cdot \frac{|\varphi(x) - \varphi(x_0)|}{|x - x_0|} \\ &\le \limsup_{x' \to x_0'} \frac{|u(x') - u(x_0')|}{|x' - x_0'|} \cdot \limsup_{x \to x_0} \frac{|\varphi(x) - \varphi(x_0)|}{|x - x_0|} \\ &= L(x_0', u) \cdot \limsup_{x \to x_0} \frac{|\varphi(x) - \varphi(x_0)|}{|x - x_0|}. \end{aligned} \tag{4.3}$$

Using (3.2) (for $f = \varphi$), we get

$$\begin{aligned} L(x_0', u) &\le \frac{K}{r_0/M} \sup_{x' \in B^n(x_0', r_0/M)} |u(x') - u(x_0')| \\ &\le \frac{KM}{r_0} \sup_{x' \in B^n(x_0', r_0/M)} |u(x') - u(x_0')| \\ &\le \frac{KM}{r_0} \sup_{x' \in \varphi(B^n(x_0, r_0))} |u(x') - u(x_0')| \\ &\le \frac{KM}{r_0} \sup_{x \in B^n(x_0, r_0)} |u(\varphi(x)) - u(\varphi(x_0))| \\ &\le \frac{KM}{r_0} \sup_{x \in B^n(x_0, r_0)} |(u \circ \varphi)(x) - (u \circ \varphi)(x_0)|. \end{aligned} \tag{4.4}$$

On the other hand, since φ is M-bi-Lipschitz,

$$\limsup_{x \to x_0} \frac{|\varphi(x) - \varphi(x_0)|}{|x - x_0|} \le \limsup_{x \to x_0} \frac{M|x - x_0|}{|x - x_0|} = M < +\infty. \tag{4.5}$$

Therefore,

$$\begin{aligned} L(x_0, u \circ \varphi) &\le \frac{KM}{r_0} \sup_{x \in B^n(x_0, r_0)} |(u \circ \varphi)(x) - (u \circ \varphi)(x_0)| \cdot M \\ &\le \frac{KM^2}{r_0} \sup_{x \in B^n(x_0, r_0)} |(u \circ \varphi)(x) - (u \circ \varphi)(x_0)|. \end{aligned} \tag{4.6}$$

Thus $u \circ \varphi \in \mathrm{OC}^1_{KM^2}(U)$. □

In addition of regularly oscillating functions, one sometimes considers so-called HC^1 functions, too; see [11, page 19], [13, page 16], and [12, page 93]. Their definition reads as follows. Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$. Let $K \geq 1$. A function $f : \Omega \to \mathbb{R}$ is in $\mathrm{HC}^1_K(\Omega)$ if

$$L(x,f) \leq K r^{-1} \sup_{y \in B^n(x,r)} |f(y)|, \quad \overline{B^n(x,r)} \subset \Omega. \tag{4.7}$$

The class $\mathrm{HC}^1(\Omega)$ is the union of all $\mathrm{HC}^1_K(\Omega)$, $K \geq 1$. Clearly, $\mathrm{HC}^1_{2K}(\Omega) \subset \mathrm{OC}^1_K(\Omega)$.

Proceeding as above in the proof of Theorem 4.2 one gets the following result.

Theorem 4.3. *Let Ω and U be open sets in $\mathbb{R}^n$, $n \geq 2$. Let $u \in \mathrm{HC}^1_K(\Omega)$. If $\varphi : U \to \mathbb{R}^n$ is M-bi-Lipschitz, $M \geq 1$, then $u \circ \varphi \in \mathrm{HC}^1_{KM^2}(U)$.*

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