Research Article

Strong Convergence Theorems for Equilibrium Problems and Fixed Point Problems in Hilbert Spaces

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We introduce an Ishikawa iterative scheme by the viscosity approximate method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in Hilbert space. Then, we prove some strong convergence theorems which extend and generalize S. Takahashi and W. Takahashi's results (2007).

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . Let F be a bifunction from $C \times C$ to R , where R is the set of real numbers. The equilibrium problem for $F : C \times C \rightarrow R$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \quad (1.1)$$

The set of solutions of (1.1) is denoted by $EP(F)$. Given a mapping $T : C \rightarrow H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1); for more details, see [1, 2].

Recall that a self-mapping S of a closed convex subset C of H is nonexpansive [3] if there holds that

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (1.2)$$

We denote the set of fixed points of S by $F(S)$. There are some methods for approximation of fixed points of a nonexpansive mapping. In 2000, Moudafi [4] introduced the viscosity approximation method for nonexpansive mappings (see [5] for further developments in both Hilbert and Banach spaces). Some methods have been proposed to solve the equilibrium problem; see, for instance, [1, 2, 6, 7]. Recently, Combettes and Hirstoaga [6] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. S. Takahashi and W. Takahashi [7] introduced a Mann iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space and proved a strong convergence theorem.

On the other hand, Ishikawa [8] introduced the following iterative process defined recursively by

$$\begin{aligned}x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) S y_n, \\ y_n &= \beta_n x_n + (1 - \beta_n) S x_n, \quad \forall n \in N,\end{aligned}\tag{1.3}$$

where the initial guess x_0 is taking in C arbitrarily, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$.

In this paper, motivated by the ideas in [4–8], we introduce an Ishikawa iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ by

$$\begin{aligned}F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) S y_n, \\ y_n &= \beta_n x_n + (1 - \beta_n) S u_n, \quad \forall n \in N,\end{aligned}\tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$.

We will prove in Section 3 that if the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ of parameters satisfy appropriate conditions, then the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ generated by (1.4) converge strongly to $z \in F(S) \cap EP(F)$. The results in this paper extend and generalize S. Takahashi and W. Takahashi's results [7].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H . $x_n \rightarrow x$ implies that $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x . In a real Hilbert space H , we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2\tag{2.1}$$

for all $x, y \in H$ and $\lambda \in R$; see [9].

For any $x \in H$, there exists a unique nearest point in C , denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection of H onto C . It is also known that $y = P_C(x)$ is equivalent to $\langle x - y, y - z \rangle \geq 0$ for all $z \in C$.

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;
- (A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y); \quad (2.2)$$

- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

We recall some lemmas needed later.

Lemma 2.1 (see [2]). *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \quad (2.3)$$

Lemma 2.2 (see [5]). *Let C be a nonempty closed convex subset of H , and let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\} \quad (2.4)$$

for all $x \in H$. Then, the following statements hold:

- (1) T_r is single-valued;
- (2) T_r is firmly nonexpansive, that is, for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y \rangle; \quad (2.5)$$

- (3) $F(T_r) = \text{EP}(F)$;
- (4) $\text{EP}(F)$ is closed and convex.

Lemma 2.3 (see [10]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - c_n)a_n + b_n, \quad \forall n \in \mathbb{N}, \quad (2.6)$$

where $\{b_n\}$ is a sequence of real numbers and $\{c_n\}$ is a sequence in $(0, 1)$ such that

$$(i) \sum_{n=1}^{\infty} c_n = \infty,$$

$$(ii) \limsup_{n \rightarrow \infty} (b_n / c_n) \leq 0 \text{ or } \sum_{n=1}^{\infty} |b_n| < \infty.$$

Then, $\lim_{n \rightarrow \infty} a_n = 0$.

3. Strong Convergence Theorem

In this section, we show a strong convergence theorem which solves the problem of finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of H . Let F be a bifunction from $C \times C$ to R satisfying (A1)–(A4) and let S be a nonexpansive mapping of C into H such that $F(S) \cap \text{EP}(F) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by $x_1 \in H$ and (1.4). If $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:*

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_n &= 0, & \sum_{n=1}^{\infty} \alpha_n &= \infty, & \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| &< \infty, \\ 0 < \liminf_{n \rightarrow \infty} \beta_n &\leq \limsup_{n \rightarrow \infty} \beta_n < 1, & \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| &< \infty, \\ \liminf_{n \rightarrow \infty} r_n &> 0, & \sum_{n=1}^{\infty} |r_{n+1} - r_n| &< \infty, \end{aligned} \quad (3.1)$$

then, $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to $z \in F(S) \cap \text{EP}(F)$, where $z = P_{F(S) \cap \text{EP}(F)} f(z)$.

Proof. Let $Q = P_{F(S) \cap \text{EP}(F)}$. Then Qf is a contraction of H into itself. In fact, there exists $a \in [0, 1)$ such that $\|f(x) - f(y)\| \leq a\|x - y\|$ for all $x, y \in H$. So, we have that

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq a\|x - y\| \quad (3.2)$$

for all $x, y \in H$. Since H is complete, there exists a unique element $z \in H$ such that $z = Qf(z)$. Such a $z \in H$ is an element of C .

Let $v \in F(S) \cap \text{EP}(F)$. Then from $u_n = T_{r_n} x_n$, we have

$$\|u_n - v\| = \|T_{r_n} x_n - T_{r_n} v\| \leq \|x_n - v\| \quad (3.3)$$

for all $n \in N$. Put $M = \max\{\|x_1 - v\|, (1/(1-a))\|f(v) - v\|\}$. It is obvious that $\|x_1 - v\| \leq M$.

Suppose $\|x_n - v\| \leq M$. Then, we have

$$\begin{aligned}
 \|x_{n+1} - v\| &\leq \alpha_n \|f(x_n) - v\| + (1 - \alpha_n) \|Sy_n - v\| \\
 &\leq \alpha_n \|f(x_n) - f(v)\| + \alpha_n \|f(v) - v\| + (1 - \alpha_n) \|Sy_n - v\| \\
 &\leq a\alpha_n \|x_n - v\| + \alpha_n \|f(v) - v\| + (1 - \alpha_n) \|y_n - v\|.
 \end{aligned} \tag{3.4}$$

On the other hand

$$\begin{aligned}
 \|y_n - v\| &\leq \beta_n \|x_n - v\| + (1 - \beta_n) \|Su_n - v\| \\
 &\leq \beta_n \|x_n - v\| + (1 - \beta_n) \|u_n - v\| \\
 &\leq \beta_n \|x_n - v\| + (1 - \beta_n) \|x_n - v\| \\
 &= \|x_n - v\|.
 \end{aligned} \tag{3.5}$$

Putting (3.5) into (3.4), we have

$$\begin{aligned}
 \|x_{n+1} - v\| &\leq a\alpha_n \|x_n - v\| + \alpha_n \|f(v) - v\| + (1 - \alpha_n) \|x_n - v\| \\
 &= [1 - \alpha_n(1 - a)] \|x_n - v\| + \alpha_n(1 - a) \frac{\|f(v) - v\|}{1 - a} \\
 &\leq [1 - \alpha_n(1 - a)]M + \alpha_n(1 - a)M = M.
 \end{aligned} \tag{3.6}$$

So, we have that $\|x_{n+1} - v\| \leq M$ for any $n \in N$. And hence $\{x_n\}$ is bounded. We also obtain that $\{u_n\}$, $\{y_n\}$, $\{Su_n\}$, $\{Sy_n\}$, and $\{f(x_n)\}$ are bounded. Next, we show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. In fact,

$$\begin{aligned}
 \|y_n - y_{n-1}\| &= \|\beta_n x_n + (1 - \beta_n) Su_n - [\beta_{n-1} x_{n-1} + (1 - \beta_{n-1}) Su_{n-1}]\| \\
 &= \|\beta_n (x_n - x_{n-1}) + (\beta_n - \beta_{n-1}) x_{n-1} + (1 - \beta_n) (Su_n - Su_{n-1}) + (\beta_{n-1} - \beta_n) Su_{n-1}\| \\
 &\leq |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \beta_n \|x_n - x_{n-1}\| + (1 - \beta_n) \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}| \|Su_{n-1}\|,
 \end{aligned} \tag{3.7}$$

and hence

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1 - \alpha_n)Sy_n - \alpha_{n-1}f(x_{n-1}) - (1 - \alpha_{n-1})Sy_{n-1}\| \\
&= \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1}f(x_{n-1}) \\
&\quad + (1 - \alpha_n)Sy_n - (1 - \alpha_n)Sy_{n-1} + (1 - \alpha_n)Sy_{n-1} - (1 - \alpha_{n-1})Sy_{n-1}\| \\
&\leq \alpha_n a \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + (1 - \alpha_n) \|y_n - y_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \|Sy_{n-1}\| \\
&\leq \alpha_n a \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + (1 - \alpha_n) \\
&\quad \times [\|\beta_n - \beta_{n-1}\| \|x_{n-1}\| + \beta_n \|x_n - x_{n-1}\| + (1 - \beta_n) \|u_n - u_{n-1}\| + \|\beta_n - \beta_{n-1}\| \|Su_{n-1}\|] \\
&\quad + |\alpha_n - \alpha_{n-1}| \|Sy_{n-1}\| \\
&= [\beta_n - \alpha_n(\beta_n - a)] \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| [\|f(x_{n-1})\| + \|Sy_{n-1}\|] \\
&\quad + (1 - \alpha_n) [\|\beta_n - \beta_{n-1}\| \|x_{n-1}\| + \|Su_{n-1}\|] + (1 - \alpha_n)(1 - \beta_n) \|u_n - u_{n-1}\| \\
&\leq [\beta_n - \alpha_n(\beta_n - a)] \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K_1 + (1 - \alpha_n) [\|\beta_n - \beta_{n-1}\| K_2 \\
&\quad + (1 - \alpha_n)(1 - \beta_n) \|u_n - u_{n-1}\|,
\end{aligned} \tag{3.8}$$

where $K_1 = \sup\{\|f(x_n)\| + \|Sy_n\| : n \in N\}$ and $K_2 = \sup\{\|x_n\| + \|Su_n\| : n \in N\}$.

On the other hand, from $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \tag{3.9}$$

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C. \tag{3.10}$$

Putting $y = u_{n+1}$ in (3.9) and $y = u_n$ in (3.10), we have

$$\begin{aligned}
F(u_n, u_{n+1}) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle &\geq 0, \\
F(u_{n+1}, u_n) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle &\geq 0.
\end{aligned} \tag{3.11}$$

So, from the monotonicity of F , we get

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \tag{3.12}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \quad (3.13)$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in N$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}, \end{aligned} \quad (3.14)$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{b} |r_{n+1} - r_n| L, \end{aligned} \quad (3.15)$$

where $L = \sup\{\|u_n - x_n\| : n \in N\}$. So from (3.8), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq [\beta_n - \alpha_n(\beta_n - a)] \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K_1 \\ &\quad + (1 - \alpha_n) |\beta_n - \beta_{n-1}| K_2 + (1 - \alpha_n)(1 - \beta_n) \left[\|x_n - x_{n-1}\| + \frac{1}{b} |r_n - r_{n-1}| L \right] \\ &= (1 - \alpha_n(1 - a)) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K_1 \\ &\quad + (1 - \alpha_n) |\beta_n - \beta_{n-1}| K_2 + (1 - \alpha_n)(1 - \beta_n) \frac{1}{b} |r_n - r_{n-1}| L. \end{aligned} \quad (3.16)$$

Using Lemma 2.1 in [10], we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.17)$$

From (3.15) and $|r_{n+1} - r_n| \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0. \quad (3.18)$$

It follows from (3.7) that

$$\lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0. \quad (3.19)$$

Since $x_n = \alpha_{n-1}f(x_{n-1}) + (1 - \alpha_{n-1})Sy_{n-1}$, we have

$$\begin{aligned}\|x_n - Sy_n\| &\leq \|x_n - Sy_{n-1}\| + \|Sy_{n-1} - Sy_n\| \\ &\leq \alpha_{n-1}\|f(x_{n-1}) - Sy_{n-1}\| + \|y_{n-1} - y_n\|.\end{aligned}\quad (3.20)$$

From $\alpha_n \rightarrow 0$, we have $\|x_n - Sy_n\| \rightarrow 0$. For $v \in F(S) \cap \text{EP}(F)$, we have

$$\begin{aligned}\|u_n - v\|^2 &= \|T_{r_n}x_n - T_{r_n}v\|^2 \\ &\leq \langle T_{r_n}x_n - T_{r_n}v, x_n - v \rangle \\ &= \langle u_n - v, x_n - v \rangle \\ &= \frac{1}{2}(\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2),\end{aligned}\quad (3.21)$$

and hence

$$\|u_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - u_n\|^2. \quad (3.22)$$

Therefore, from the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|y_n - v\|^2 &\leq \beta_n\|x_n - v\|^2 + (1 - \beta_n)\|Su_n - v\|^2 \\ &\leq \beta_n\|x_n - v\|^2 + (1 - \beta_n)\|u_n - v\|^2 \\ &\leq \beta_n\|x_n - v\|^2 + (1 - \beta_n)[\|x_n - v\|^2 - \|x_n - u_n\|^2] \\ &= \|x_n - v\|^2 - (1 - \beta_n)\|x_n - u_n\|^2,\end{aligned}\quad (3.23)$$

and hence

$$\begin{aligned}\|x_{n+1} - v\|^2 &= \|\alpha_n f(x_n) + (1 - \alpha_n)Sy_n - v\|^2 \\ &\leq \alpha_n\|f(x_n) - v\|^2 + (1 - \alpha_n)\|Sy_n - v\|^2 \\ &\leq \alpha_n\|f(x_n) - v\|^2 + (1 - \alpha_n)\|y_n - v\|^2 \\ &\leq \alpha_n\|f(x_n) - v\|^2 + (1 - \alpha_n)[\|x_n - v\|^2 - (1 - \beta_n)\|x_n - u_n\|^2] \\ &\leq \alpha_n\|f(x_n) - v\|^2 + \|x_n - v\|^2 - (1 - \alpha_n)(1 - \beta_n)\|x_n - u_n\|^2.\end{aligned}\quad (3.24)$$

So, we have

$$\begin{aligned}(1 - \alpha_n)(1 - \beta_n)\|x_n - u_n\|^2 &\leq \alpha_n\|f(x_n) - v\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\ &\leq \alpha_n\|f(x_n) - v\|^2 + \|x_n - x_{n+1}\|(\|x_n - v\| + \|x_{n+1} - v\|).\end{aligned}\quad (3.25)$$

Without loss of generality, let us assume that there exists two real numbers β^* and $\bar{\beta}$ such that $1 > \bar{\beta} \geq \beta_n \geq \beta^* > 0$ for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned} (1 - \alpha_n)(1 - \bar{\beta})\|x_n - u_n\|^2 &\leq (1 - \alpha_n)(1 - \beta_n)\|x_n - u_n\|^2 \\ &\leq \alpha_n\|f(x_n) - v\|^2 + \|x_n - x_{n+1}\|(\|x_n - v\| + \|x_{n+1} - v\|). \end{aligned} \quad (3.26)$$

It follows that $\|x_n - u_n\| \rightarrow 0$. We also have

$$\begin{aligned} \|Su_n - x_n\| &\leq \|Sy_n - x_n\| + \|Su_n - Sy_n\| \\ &\leq \|Sy_n - x_n\| + \|u_n - y_n\| \\ &\leq \|Sy_n - x_n\| + \|u_n - x_n\| + \|x_n - y_n\| \\ &= \|Sy_n - x_n\| + \|u_n - x_n\| + (1 - \beta_n)\|x_n - Su_n\|. \end{aligned} \quad (3.27)$$

It follows that

$$\beta^*\|Su_n - x_n\| \leq \beta_n\|Su_n - x_n\| \leq \|Sy_n - x_n\| + \|u_n - x_n\|. \quad (3.28)$$

Hence, $\|Su_n - x_n\| \rightarrow 0$. Since

$$\|Su_n - u_n\| \leq \|Su_n - x_n\| + \|x_n - u_n\|, \quad (3.29)$$

we also have $\lim_{n \rightarrow \infty} \|Su_n - u_n\| = 0$. Next, we show that

$$\limsup_{n \rightarrow \infty} \langle f(z) - z, x_n - z \rangle \leq 0, \quad (3.30)$$

where $z = P_{F(S) \cap \text{EP}(F)} f(z)$. To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} \langle f(z) - z, x_{n_i} - z \rangle = \limsup_{n \rightarrow \infty} \langle f(z) - z, x_n - z \rangle. \quad (3.31)$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{ij}}\}$ of $\{u_{n_i}\}$ which converges weakly to w . Without loss of generality, we can assume that $\{u_{n_i}\} \rightharpoonup w$. From $\|Su_n - u_n\| \rightarrow 0$, we obtain $Su_{n_i} \rightharpoonup w$. Let us show $w \in \text{EP}(F)$. By $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \quad (3.32)$$

From (A2), we also have

$$\frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq F(y, u_n), \quad (3.33)$$

and hence,

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, u_{n_i}). \quad (3.34)$$

Since $(u_{n_i} - x_{n_i})/r_{n_i} \rightarrow 0$ and $u_{n_i} \rightarrow w$, from (A4), we have

$$f(y, w) \leq 0, \quad \forall y \in C. \quad (3.35)$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$ and hence $F(y_t, w) \leq 0$. So we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, w) \leq tF(y_t, y). \quad (3.36)$$

Dividing by t , we get

$$F(y_t, y) \geq 0. \quad (3.37)$$

Letting $t \rightarrow 0$ and from (A3), we get

$$F(w, y) \geq 0 \quad (3.38)$$

for all $y \in C$ and hence $w \in \text{EP}(F)$. We shall show that $w \in F(S)$. Assume $w \notin F(S)$. Since $u_{n_i} \rightarrow w$ and $w \neq Sw$, from the Opial theorem [11] we have

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|u_{n_i} - w\| &< \liminf_{i \rightarrow \infty} \|u_{n_i} - Sw\| \\ &\leq \liminf_{i \rightarrow \infty} \{\|u_{n_i} - Su_{n_i}\| + \|Su_{n_i} - Sw\|\} \\ &\leq \liminf_{i \rightarrow \infty} \|u_{n_i} - w\|. \end{aligned} \quad (3.39)$$

This is a contradiction. So, we get $w \in F(S)$. Therefore, $w \in F(S) \cap \text{EP}(F)$. Since $z = P_{F(S) \cap \text{EP}(F)} f(z)$, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle f(z) - z, x_n - z \rangle &= \lim_{i \rightarrow \infty} \langle f(z) - z, x_{n_i} - z \rangle \\ &= \lim_{i \rightarrow \infty} \langle f(z) - z, u_{n_i} - z \rangle \\ &= \langle f(z) - z, w - z \rangle \leq 0. \end{aligned} \quad (3.40)$$

From $x_{n+1} - z = \alpha_n(f(x_n) - z) + (1 - \alpha_n)(Sy_n - z)$, we have

$$(1 - \alpha_n)^2 \|Sy_n - z\|^2 \geq \|x_{n+1} - z\|^2 - 2\alpha_n \langle f(x_n) - z, x_{n+1} - z \rangle, \quad (3.41)$$

$$\begin{aligned} \|y_n - z\|^2 &= \|\beta_n x_n + (1 - \beta_n)Su_n - z\|^2 \\ &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \|Su_n - z\|^2 \\ &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \|u_n - z\|^2 \\ &\leq \|x_n - z\|^2. \end{aligned} \quad (3.42)$$

It follows that

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1 - \alpha_n)^2 \|Sy_n - z\|^2 + 2\alpha_n \langle f(x_n) - z, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n)^2 \|y_n - z\|^2 + 2\alpha_n \langle f(x_n) - f(z), x_{n+1} - z \rangle \\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n a \|x_n - z\| \|x_{n+1} - z\| \\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - z\|^2 + \alpha_n a \{ \|x_n - z\|^2 + \|x_{n+1} - z\|^2 \} \\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle. \end{aligned} \quad (3.43)$$

Hence

$$\|x_{n+1} - z\|^2 \leq \frac{(1 - \alpha_n)^2 + \alpha_n a}{1 - \alpha_n a} \|x_n - z\|^2 + \frac{2\alpha_n}{1 - \alpha_n a} \langle f(z) - z, x_{n+1} - z \rangle. \quad (3.44)$$

From $\alpha_n \rightarrow 0$, we know that there exists a positive integer n_0 , such that $1 > 1 - \alpha_n a > 1/2$ for all $n \geq n_0$. Then

$$\begin{aligned} \frac{(1 - \alpha_n)^2 + \alpha_n a}{1 - \alpha_n a} &= \frac{1 - 2\alpha_n + \alpha_n a}{1 - \alpha_n a} + \frac{\alpha_n^2}{1 - \alpha_n a} \\ &= 1 - \frac{2(1 - a)\alpha_n}{1 - \alpha_n a} + \frac{\alpha_n^2}{1 - \alpha_n a} \\ &\leq 1 - 2(1 - a)\alpha_n + 2\alpha_n^2, \quad \forall n \geq n_0. \end{aligned} \quad (3.45)$$

Putting above inequality into (3.44), we get

$$\|x_{n+1} - z\|^2 \leq (1 - 2(1 - a)\alpha_n)\|x_n - z\|^2 + 2\overline{M}\alpha_n^2 + \frac{2\alpha_n}{1 - \alpha_n a}\sigma_n, \quad \forall n \geq n_0, \quad (3.46)$$

where $\overline{M} = \sup\{\|x_n - z\|^2 : n \in N\}$, and $\sigma_n = \langle f(z) - z, x_{n+1} - z \rangle$.

It follows from Lemma 2.3 that

$$x_n \longrightarrow z \in F(S) \cap \text{EP}(F). \quad (3.47)$$

It follows from $\|x_n - u_n\| \rightarrow 0$ and (3.42) that $u_n \rightarrow z$ and $y_n \rightarrow z$. □

By Theorem 3.1, we can obtain the following new result.

Corollary 3.2. *Let C be a nonempty closed convex subset of H . Let S be a nonexpansive mapping of C into H such that $F(S) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) S y_n, \\ y_n &= \beta_n x_n + (1 - \beta_n) S P_C x_n, \quad \forall n \in N. \end{aligned} \quad (3.48)$$

If $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfy the following conditions:

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_n &= 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\ 0 < \liminf_{n \rightarrow \infty} \beta_n &\leq \limsup_{n \rightarrow \infty} \beta_n < 1, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \end{aligned} \quad (3.49)$$

then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$.

Proof. Put $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in N$ in Theorem 3.1. Then, we get $u_n = P_C x_n$. So from Theorem 3.1, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$. □

Remark 3.3. Theorem 3.1 and Corollary 3.2, respectively, extend and generalize Theorem 3.2 and Corollary 3.3 in [7] from the Mann iterative form to the Ishikawa iterative form.

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