Research Article

Unicity of Meromorphic Function Sharing One Small Function with Its Derivative

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We deal with the problem of uniqueness of a meromorphic function sharing one small function with its k’s derivative and obtain some results.

1. Introduction and Main Results

In this article, a meromorphic function means meromorphic in the open complex plane. We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and the standard notations such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\bar{N}(r,f)$, and so on.

Let $f$ and $g$ be two nonconstant meromorphic functions; a meromorphic function $a(z)(\neq \infty)$ is called a small functions with respect to $f$ provided that $T(r,a) = S(r,f)$. Note that the set of all small function of $f$ is a field. Let $b(z)$ be a small function with respect to $f$ and $g$. We say that $f$ and $g$ share $b(z)$ CM(IM) provided that $f-b$ and $g-b$ have same zeros counting multiplicities (ignoring multiplicities).

Moreover, we use the following notations.

Let $k$ be a positive integer. We denote by $N_k(r,1/(f-a))$ the counting function for the zeros of $f-a$ with multiplicity $\leq k$ and by $\bar{N}_k(r,1/(f-a))$ the corresponding one for which the multiplicity is not counted. Let $N_k(r,1/(f-a))$ be the counting function for the zeros of $f-a$ with multiplicity $\geq k$, and let $\bar{N}_k(r,1/(f-a))$ be the corresponding one for which the multiplicity is not counted. Set $N_k(r,1/(f-a)) = \bar{N}(r,1/(f-a)) + \bar{N}(r,1/(f-a)) + \cdots + \bar{N}(r,1/(f-a))$. And we define

$$\delta_p(a,f) = 1 - \limsup_{r \to \infty} \frac{N_p(r,1/(f-a))}{T(r,f)}. \quad (1.1)$$
Obviously, $1 \geq \Theta(a, f) \geq \delta_r(a, f) \geq \delta(a, f) \geq 0$. For more details, reader can see [1, 2].

Brück (see [3]) considered the uniqueness problems of an entire function sharing one value with its derivative and proved the following result.

**Theorem A.** Let $f$ be nonconstant entire function. If $f$ and $f'$ share the value 1 CM and if $N(r, 1/f') = S(r, f)$, then $(f' - 1)/(f - 1) \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.


**Theorem B (see[5]).** Let $f$ be a nonconstant meromorphic function and, let $k$ be a positive integer. Suppose that $f$ and $f^{(k)}$ share 1 CM and

$$2N(r, f) + N\left(r, \frac{1}{f^{'}}\right) + N\left(r, \frac{1}{f^{(k)}}\right) < (\lambda + o(1))T\left(r, f^{(k)}\right),$$

for $r \in I$, where $I$ is a set of infinite linear measure and $\lambda$ satisfies $0 < \lambda < 1$, then $(f^{(k)} - 1)/(f - 1) \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

**Theorem C (see[6]).** Let $f$ be a nonconstant, nonentire meromorphic function and $a(z)(\neq 0, \infty)$ be a small function with respect to $f$. If

1. $f$ and $a(z)$ have no common poles,
2. $f - a$ and $f^{(k)} - a$ share the value 0 CM,
3. $4\delta(0, f) + 2(k + 8)\Theta(\infty, f) > 2k + 19$, then $f \equiv f^{(k)}$, where $k$ is a positive integer.

In the same paper, Yu [6] posed four open questions. Lahiri and Sarkar [7] and Zhang [8] studied the problem of a meromorphic or an entire function sharing one small function with its derivative with weighted shared method and obtained the following result, which answered the open questions posed by Yu [6].

**Theorem D (see[8]).** Let $f$ be a non-constant meromorphic function and, let $k$ be a positive integer. Also let $a(z)(\neq 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose that $f - a$ and $f^{(k)} - a$ share 0 IM and

$$4N(r, f) + 3N\left(r, \frac{1}{f^{(k)}}\right) + 2N\left(r, \frac{1}{f'/a}\right) < (\lambda + o(1))T\left(r, f^{(k)}\right),$$

for $0 < \lambda < 1$, $r \in I$, and $I$ is a set of infinite linear measure. Then $(f^{(k)} - a) \setminus (f - a) \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

In this article, we will pay our attention to the value sharing of $f$ and $[f^n]^{(k)}$ that share a small function and obtain the following results, which are the improvements and complements of the above theorems.
**Theorem 1.1.** Let \( k(\geq 1), \ n(\geq 1) \) be integers and let \( f \) be a non-constant meromorphic function. Also let \( a(z)(\neq 0, \ \infty) \) be a small function with respect to \( f \). If \( f \) and \( [f^n]^{(k)} \) share \( a(z) \) IM and

\[
4\overline{N}(r,f) + 2\overline{N}\left(r, \left( f/a \right)^2 \right) + 2N_2\left(r, \frac{1}{(f^n)^{(k)}} \right) + \overline{N}\left(r, \frac{1}{(f^n)^{(k)}} \right) \\
\leq (\lambda + o(1))T(r, (f^n)^{(k)}),
\]

or \( f \) and \([f^n]^{(k)}\) share \( a(z) \) CM and

\[
2\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{(f/a)} \right) + N_2\left(r, \frac{1}{(f^n)^{(k)}} \right) \leq (\lambda + o(1))T(r, (f^n)^{(k)}),
\]

for \( 0 < \lambda < 1, \ r \in I, \) and \( I \) is a set of infinite linear measure, then \( (f - a) \setminus ([f^n]^{(k)} - a) \equiv c, \) for some constant \( c \in \mathbb{C} \setminus \{0\} \).

**Theorem 1.2.** Let \( k(\geq 1), \ n(\geq 1) \) be integers and \( f \) be a non-constant meromorphic function. Also let \( a(z)(\neq 0, \ \infty) \) be a small function with respect to \( f \). If \( f \) and \([f^n]^{(k)}\) share \( a(z) \) IM and

\[
(2k + 6)\Theta(\infty,f) + 3\Theta(0,f) + 2\delta_{k+2}(0,f) > 2k + 10,
\]

or \( f \) and \([f^n]^{(k)}\) share \( a(z) \) CM and

\[
(k + 3)\Theta(\infty,f) + \delta_{2}(0,f) + \delta_{k+2}(0,f) > k + 4,
\]

then \( f \equiv (f^n)^{(k)} \).

Clearly, Theorem 1.1 improves and extends Theorems B and D, while 1.2 improves and extends Theorem C.

### 2. Some Lemmas

In this section, first of all, we give some definitions which will be used in the whole paper.

**Definition 2.1.** Let \( F \) and \( G \) be two meromorphic functions defined in \( \mathbb{C} \); assume, that \( F \) and \( G \) share \( 1 \) IM; let \( z_0 \) be a zero of \( F - 1 \) with multiplicity \( p \) and a zero of \( G - 1 \) with multiplicity \( q \). We denote by \( N_L^{(1)}(r, 1/F - 1) \) the counting function of the zeros of \( F - 1 \) where \( p = q = 1 \) and by \( N_L^{(2)}(r, 1/F - 1) \) the counting function of zeros of \( F - 1 \) where \( p = q \geq 2 \). We denotes by \( N_L(r, 1/F - 1) \) the counting function of the zeros of \( F - 1 \) where \( p > q \geq 1 \); each point is counted according to its multiplicity, and \( \overline{N}_L(r, 1/F - 1) \) denote its reduced form. In the same way, we can define \( N_L^{(1)}(r, 1/G - 1), \ N_L^{(2)}(r, 1/G - 1), \ N_L(r, 1/G - 1), \) and so on.
Definition 2.2. In this paper $N_0(r, 1/F')$ denotes the counting function of the zeros of $F'$ which are not the zeros of $F$ and $F - 1$, and $\overline{N}_0(r, 1/F')$ denotes its reduced form. In the same way, we can define $N_0(r, 1/G')$ and $\overline{N}_0(r, 1/G')$.

Next we present some lemmas which will be needed in the sequel. Let $F, G$ be two non-constant meromorphic functions defined in $\mathbb{C}$. We shall denote by $H$ the following function:

$$H = \left( \frac{F''}{F'} - 2 \frac{F'}{F - 1} \right) - \left( \frac{G''}{G'} - 2 \frac{G'}{G - 1} \right).$$

(2.1)

Lemma 2.3 (see[2]). Let $F, G$ be two nonconstant meromorphic functions defined in $\mathbb{C}$. If $F$ and $G$ are sharing 1 IM, then

$$N(r, H) \leq \overline{N}(r, F) + \overline{N}_2\left( r, \frac{1}{F} \right) + \overline{N}_2\left( r, \frac{1}{G} \right) + \overline{N}_L\left( r, \frac{1}{F - 1} \right) + \overline{N}_L\left( r, \frac{1}{G - 1} \right) + \overline{N}_0\left( r, \frac{1}{F} \right) + \overline{N}_0\left( r, \frac{1}{G} \right) + S(r, f).$$

(2.2)

If $F$ and $G$ are sharing 1 CM, then

$$N(r, H) \leq \overline{N}(r, F) + \overline{N}_2\left( r, \frac{1}{F} \right) + \overline{N}_2\left( r, \frac{1}{G} \right) + \overline{N}_0\left( r, \frac{1}{F} \right) + \overline{N}_0\left( r, \frac{1}{G} \right) + S(r, f).$$

(2.3)

Lemma 2.4 (see[1]). Let $f$ be a meromorphic function and $a$ is a finite complex number. Then

(i) $T(r, 1/(f - a)) = T(r, f) + O(1)$,

(ii) $m(r, f^{(k)}/f^{(0)}) = S(r, f)$ for $k > 1 \geq 0$,

(iii) $T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, 1/(f - a_1(z))) + \overline{N}(r, 1/(f - a_2(z))) + S(r, f),$

where $a_1(z)$ and $a_2(z)$ are two meromorphic functions such that $T(r, a_i) = S(r, f), (i = 1, 2)$.

Lemma 2.5 (see[7]). Let $f$ be a non-constant meromorphic function, and $k, p$ are two positive integers. Then

$$N_p\left( r, \frac{1}{f^{(k)}} \right) \leq N_{p+k}\left( r, \frac{1}{f} \right) + k\overline{N}(r, f) + S(r, f).$$

(2.4)

Lemma 2.6 (see[9]). Let $f$ be a non-constant meromorphic function and let $n$ be a positive integer. $P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f$ where $a_i$ are meromorphic functions such that $T(r, a_i) = S(r, f) (i = 1, 2, \ldots, n)$, and $a_n \neq 0$. Then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

(2.5)
3. Proof of Theorem 1.1

Let \( F = f(z)/a(z), \) \( G = (f^n(z))^{(k)}/a(z), \) then

\[
F - 1 = \frac{f(z) - a(z)}{a(z)}, \quad G - 1 = \frac{(f^n(z))^{(k)} - a(z)}{a(z)}. \tag{3.1}
\]

From the definitions of \( F, G \) and recalling that \( F \) and \( G \) share value 1 IM(CM), we get

\[
N_E^1\left(r, \frac{1}{F - 1}\right) = N_E^1\left(r, \frac{1}{G - 1}\right) + S(r, f), \tag{3.2}
\]

\[
\overline{N}_E^2\left(r, \frac{1}{F - 1}\right) = \overline{N}_E^2\left(r, \frac{1}{G - 1}\right) + S(r, f),
\]

\[
\overline{N}_L\left(\frac{1}{G - 1}\right) \leq \overline{N}\left(\frac{1}{F}\right) + \overline{N}(r, F) + S(r, F), \tag{3.3}
\]

\[
\overline{N}\left(\frac{1}{F - 1}\right) = \overline{N}\left(\frac{1}{G - 1}\right) + S(r, F) = N_E^1\left(r, \frac{1}{F - 1}\right) + \overline{N}_E^2\left(r, \frac{1}{F - 1}\right)
\]

\[
+ \overline{N}_L\left(r, \frac{1}{F - 1}\right) + \overline{N}_L\left(r, \frac{1}{G - 1}\right) + S(r, f). \tag{3.4}
\]

We will distinguish two cases below.

**Case 1** \((H \neq 0).\) From (2.1) it is easy to see that \( m(r, H) = S(r, f). \)

**Subcase 1.1.** Suppose that \( f \) and \((f^n)^{(k)}\) share \( a(z) \) IM. According to (3.1), \( F \) and \( G \) share 1 IM except the zeros and poles of \( a(z). \) By (3.1), we have

\[
\overline{N}(r, F) = \overline{N}(r, f) + S(r, f), \quad \overline{N}(r, G) = \overline{N}(r, f) + S(r, f). \tag{3.5}
\]

Let \( z_0 \) be a simple zero of \( F - 1 \) and \( G - 1, \) but \( a(z_0) \neq 0, \infty. \) Through a simple calculation we know that \( z_0 \) is a zero of \( H, \) so

\[
N_E^1\left(r, \frac{1}{F - 1}\right) \leq N\left(r, \frac{1}{H}\right) + S(r, f) \leq T(r, H) + S(r, f) \leq N(r, H) + S(r, f). \tag{3.6}
\]
From (3.4)–(3.6) and Lemma 2.3, we have

\[
\mathcal{N}\left( r, \frac{1}{G-1} \right) \leq \mathcal{N}(r, F) + 2\mathcal{N}_L\left( r, \frac{1}{F-1} \right) + 2\mathcal{N}_L\left( r, \frac{1}{G-1} \right) + \mathcal{N}_2\left( r, \frac{1}{F} \right) \\
+ \mathcal{N}_2\left( r, \frac{1}{G} \right) + \mathcal{N}_2^0\left( r, \frac{1}{F-1} \right) + \mathcal{N}_0\left( r, \frac{1}{F} \right) + \mathcal{N}_0\left( r, \frac{1}{G} \right) + S(r, f)
\]  
(3.7)

It follows by the second fundamental theorem, (3.5), and (3.7) that

\[
T(r, G) \leq \mathcal{N}(r, G) + \mathcal{N}\left( r, \frac{1}{G} \right) + \mathcal{N}\left( r, \frac{1}{G-1} \right) - \mathcal{N}_0\left( r, \frac{1}{G} \right) + S(r, G) \\
\leq 2\mathcal{N}(r, f) + 2\mathcal{N}\left( r, \frac{1}{F-1} \right) + 2\mathcal{N}\left( r, \frac{1}{G-1} \right) + \mathcal{N}\left( r, \frac{1}{G} \right) + S(r, f).
\]  
(3.8)

By Lemma 2.5, we have

\[
T\left( r, (f^n)^{(k)} \right) \leq 4\mathcal{N}(r, f) + 2\mathcal{N}\left( r, \frac{1}{(f/a)^{n}} \right) + 2\mathcal{N}_2\left( r, \frac{1}{(f^n)^{(k)}} \right) + \mathcal{N}\left( r, \frac{1}{(f^n)^{(k)}} \right) + S(r, f),
\]  
(3.9)

which contradicts (1.4).

Subcase 1.2. Suppose that \( f \) and \( (f^n)^{(k)} \) share \( a(z) \) CM.

Let \( z_0 \) be a simple zero of \( F-1 \) and \( G-1 \), but \( a(z_0) \neq 0, \infty \). By a simple calculation, we can still get \( H(z_0) = 0 \). Therefore

\[
\mathcal{N}_{1}\left( r, \frac{1}{F-1} \right) \leq \mathcal{N}\left( r, \frac{1}{H} \right) + S(r, f) \leq \mathcal{N}(r, H) + S(r, f).
\]  
(3.10)

Noting that \( \mathcal{N}_{1}(r, 1/(F-1)) = \mathcal{N}_{1}(r, 1/(G-1)) + S(r, f) \), by (3.4) and Lemma 2.3, we can deduce

\[
\mathcal{N}\left( r, \frac{1}{G-1} \right) \leq \mathcal{N}(r, F) + \mathcal{N}_{2}\left( r, \frac{1}{F} \right) + \mathcal{N}_{2}\left( r, \frac{1}{G} \right) + \mathcal{N}_0\left( r, \frac{1}{F} \right) + \mathcal{N}_0\left( r, \frac{1}{G} \right) \\
+ \mathcal{N}_2\left( r, \frac{1}{F-1} \right) + S(r, f).
\]  
(3.11)
By the second fundamental theorem, (3.5), and (3.11), we have

\[
T(r, G) \leq \overline{N}(r, G) + \overline{N} \left( r, \frac{1}{G} \right) + \overline{N} \left( r, \frac{1}{G-1} \right) - N_0 \left( r, \frac{1}{G} \right) + S(r, G)
\]

\[
\leq 2\overline{N}(r, f) + N_2 \left( r, \frac{1}{G} \right) + \overline{N} \left( r, \frac{1}{F} \right) + S(r, f).
\]

(3.12)

Taking into account (3.1), we have

\[
T \left( r, (f^n)^{(k)} \right) \leq 2\overline{N}(r, f) + \overline{N} \left( r, \frac{1}{(f/a)^{(k)}} \right) + N_2 \left( r, \frac{1}{(f/a)^{(k)}} \right) + S(r, f).
\]

(3.13)

This contradicts (1.5).

Case 2 \((H \equiv 0)\). Integration yields

\[
\frac{1}{F-1} = \frac{A}{G-1} + B,
\]

(3.14)

where \(A, B\) are constants and \(A \neq 0\). It is easy to see that \(F\) and \(G\) share 1 CM. Now we claim that \(B = 0\).

If \(\overline{N}(r, f) \neq S(r, f)\), then by (3.14) we get \(B = 0\). So our claim holds. Hence we can assume that

\[
\overline{N}(r, f) = S(r, f).
\]

(3.15)

If \(B \neq 0\), then we can rewrite (3.14) as

\[
\frac{1}{F-1} = \frac{B(G-1 + A/B)}{G-1}.
\]

(3.16)

So

\[
\overline{N} \left( r, \frac{1}{G-1 + A/B} \right) = \overline{N}(r, F) = S(r, f).
\]

(3.17)

If \(A \neq B\), then by Lemma 2.4 and (3.17) we have

\[
T(r, G) \leq \overline{N}(r, G) + \overline{N} \left( r, \frac{1}{G} \right) + \overline{N} \left( r, \frac{1}{G-1 + A/B} \right) + S(r, f)
\]

\[
\leq \overline{N} \left( r, \frac{1}{G} \right) + S(r, f) \leq T(r, G) + S(r, f).
\]

(3.18)
Hence

\[ T(r, G) = \overline{N}\left(r, \frac{1}{G}\right) + S(r, f), \]  

(3.19)

that is,

\[ T\left(r, (f^n)^{(k)}\right) = \overline{N}\left(r, \frac{1}{(f^n)^{(k)}}\right) + S(r, f). \]  

(3.20)

This is a contradiction with (1.4) and (1.5). If \( A = B \), then from (3.14) we get \( 1/(F - 1) = AG/(G - 1) \). We rewrite it as

\[ -\frac{a^2}{f^n(Af - a - aA)} = \frac{(f^n)^{(k)}}{f^n}. \]  

(3.21)

So by Lemmas 2.4 and 2.6 and (3.15), we have

\[ (n + 1)T(r, f) = T\left(r, \frac{(f^n)^{(k)}}{f^n}\right) + S(r, f) \]

\[ \leq n\overline{N}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f) \leq nT(r, f) + S(r, f). \]  

(3.22)

This implies that \( T(r, f) = S(r, f) \), since \( n \geq 1 \). This is impossible. Hence our claim is right. So \( (G - 1)/(F - 1) = A \). Theorem 1.1 is, thus, completely proved.

4. Proof of Theorem 1.2

The proof is similar to the proof of Theorem 1.1. Let \( F \) and \( G \) be defined as in Theorem 1.1; hence, we have (3.1)–(3.5). We still distinguish two cases.

Case 1. \( H \neq 0 \)

Subcase 1.1. Suppose that \( f \) and \( (f^n)^{(k)} \) share \( a(z) \) IM, then we can still get (3.6) and (3.7). Then by the second fundamental theorem, Lemma 2.3, and (3.5) we have

\[ T(r, F) = \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - 1}\right) - N_0\left(r, \frac{1}{F}\right) + S(r, F) \]

\[ \leq 2\overline{N}(r, f) + 2\overline{N}\left(r, \frac{1}{G}\right) + 2\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, f). \]  

(4.1)
Applying Lemma 2.5 to the above inequality and noticing the definition of $F, G$, we get

\[ T(r, f) \leq (2k + 6)N(r, f) + 3N \left( r, \frac{1}{f} \right) + 2N_{k+2}N \left( r, \frac{1}{f} \right) + S(r, f) \]

\[ \leq [(2k + 6)(1 - \Theta(\infty, f)) + 3 - 3\Theta(0, f) + 2 - 2\delta_{k+2}(0, f)]T(r, f) + S(r, f). \]

This implies that

\[ (2k + 6)\Theta(\infty, f) + 3\Theta(0, f) + 2\delta_{k+2}(0, f) \leq 2k + 10. \]

(4.3)

This contradicts (1.6).

**Subcase 1.2.** Suppose that $f$ and $(f^n)_{(k)}$ share $a(z)$ CM. Similarly as above, we can easily obtain $N_{1}(r, 1/(F - 1)) = N_{1}(r, 1/(G - 1)) + S(r, f)$; by Lemma 2.3, we can deduce

\[ N \left( r, \frac{1}{F - 1} \right) \leq N(r, F) + N_{2} \left( r, \frac{1}{F} \right) + N_{0} \left( r, \frac{1}{F} \right) \]

\[ + N_{0} \left( r, \frac{1}{G} \right) + N_{2} \left( r, \frac{1}{G - 1} \right) + S(r, f). \]

(4.4)

So by the second fundamental theorem, (4.4), and using Lemma 2.5 again, we have

\[ T(r, f) \leq N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F - 1} \right) - N_{0} \left( r, \frac{1}{F} \right) + S(r, f) \]

\[ \leq 2N(r, f) + N_{2} \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{G} \right) + S(r, f) \]

\[ \leq [(k + 5) - (k + 3)\Theta(\infty, f) - \delta_{2}(0, f) - \delta_{k+2}(0, f)]T(r, f) + S(r, f). \]

This implies that

\[ (k + 3)\Theta(\infty, f) + \delta_{2}(0, f) + \delta_{k+2}(0, f) \leq k + 4. \]

(4.6)

This contradicts (1.7).

**Case 2** ($H \equiv 0$). Similarly, we can also get (3.14). Next we claim that $B = 0$. If $N(r, f) \neq S(r, f)$, then it follows that $B = 0$ from (3.14). Hence, we may assume that (3.15) holds. If $B \neq 0$ and $B \neq -1$, then

\[ \frac{A}{G - 1} = -\frac{BF - (B + 1)}{F - 1}, \]

(4.7)

and so

\[ N(r, G) = N \left( r, \frac{1}{F - (B + 1)/B} \right). \]

(4.8)
Again by second fundamental theorem and (4.4) we have

$$T(r, F) = \overline{N}\left(r, \frac{1}{F}\right) + S(r, f),$$

(4.9)

that is,

$$T(r, f) \leq \overline{N}\left(r, \frac{1}{F}\right) + S(r, f) \leq T(r, f) + S(r, f).$$

(4.10)

Then we have $T(r, f) = \overline{N}(r, 1/f)$, and it follows that $\Theta(0, f) = 0$ and from (3.15) we have $\Theta(\infty, f) = 1$; then with (1.6) and (1.7) we may deduce $\delta_{k+2}(0, f) > 1$. It is impossible, and we can assume that $B = -1$; thus, we can get

$$\frac{(f^n)^{(k)}}{a} - (A + 1) \equiv -A \cdot a \cdot \frac{1}{f}.$$  

(4.11)

It shows that $T(r, f) = T(r, (f^n)^{(k)})$.

If $A = -1$, by (4.11), then we have $f \cdot (f^n)^{(k)} \equiv a^2$, which with the above equality may lead to $T(r, f) = S(r, f)$, which is impossible. If $A \neq -1$, then by second fundamental theorem, Lemma 2.5, (3.15), and (4.11) we have

$$T\left(r, (f^n)^{(k)}\right) \leq \overline{N}\left(r, \frac{1}{(f^n)^{(k)} - a(A + 1)}\right) + \overline{N}\left(r, \frac{1}{(f^n)^{(k)}}\right) + S(r, f),$$

(4.12)

$$\leq k\overline{N}(r, f) + N_{k+2}\left(r, \frac{1}{f}\right) + S(r, f) \leq T(r, f) + S(r, f),$$

which with (3.15) may deduce $N_{k+2}(r, 1/f) = T(r, f) + S(r, f)$; so $\delta_{k+2}(0, f) = 0$, which with $\Theta(\infty, f) = 1$ and (1.6) may deduce $\Theta(0, f) > 1$, which is impossible. Hence our claim holds.

Next we will prove that $A = 1$. From (3.17) we have $G - 1 \equiv A(F - 1)$. Then

$$\overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F + 1/A - 1}\right).$$

(4.13)

If $A \neq 1$, then we have

$$T(r, F) \leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r, f).$$

(4.14)

By Lemma 2.5, we get

$$T(r, f) \leq (k + 1)\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + N_{k+2}\left(r, \frac{1}{f}\right) + S(r, f).$$

(4.15)
It implies that
\[(k + 1)\Theta(\infty, f) + \Theta(0, f) + \delta_{k+2}(0, f) \leq k + 2. \tag{4.16}\]

Combining (4.16) with (1.6) yields
\[2(k + 2) + \Theta(0, f) \geq 2(k + 3)\Theta(\infty, f) + 3\Theta(0, f) + 2\delta_{2+k}(0, f) - 4\Theta(\infty, f) > 2k + 6, \tag{4.17}\]
that is, \(\Theta(0, f) > 2\). This is a contradiction.

Combining (4.16) with (1.7) yields
\[k + 2 + 2\Theta(\infty, f) \geq (k + 3)\Theta(\infty, f) + \Theta(0, f) + \delta_{k+2}(0, f) > k + 4, \tag{4.18}\]
that is, \(\Theta(\infty, f) > 1\), which is also a contradiction. Hence \(A = 1\) and \(f \equiv (f^n)^{(k)}\). Now Theorem 1.2 has been completely proved.

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**References**
