Research Article

A Class of Weak Hopf Algebras

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We introduce a class of noncommutative and noncocommutative weak Hopf algebras with infinite Ext quivers and study their structure. We decompose them into a direct sum of two algebras. The coalgebra structures of these weak Hopf algebras are described by their Ext quiver. The weak Hopf extension of Hopf algebra $H_n$ has a quotient Hopf algebra and a sub-Hopf algebra which are isomorphic to $H_n$.

1. Introduction

Weak Hopf algebra was introduced by Li in 1998 as a generalization of Hopf algebras [1]. It had been proved in [1, 2]; for some sorts of finite dimensional weak Hopf algebras $H$, the quantum quasiadouble $D(H)$ of $H$ is quasi-braided equipped with some quasi-R-matrix $R$. Hence $R$ is a solution of the Quantum Yang-Baxter Equation.

First two examples of noncommutative and noncocommutative weak Hopf algebras were given in [3]. Up to now, many examples of weak Hopf algebras have been found [2, 4–7]. So far, all examples of weak Hopf algebras were based on some Hopf algebras and were constructed by weak extension.

In this paper, we first give a Hopf algebra, denoted by $H_n$. By weak extension, we construct a weak Hopf algebra $W(n_1, n_2, n_3)$ corresponding to $H_n$ and study their structure. $W(n_1, n_2, n_3)$ has a quotient Hopf algebra and a sub-Hopf algebra which are isomorphic to $H_n$. And as an algebra, $W(n_1, n_2, n_3)$ can be decomposed into a direct sum of two algebras, one of which is $H_n$. The coalgebra structures of these weak Hopf algebras are described by their Ext quiver [8, 9].

We organize our paper as follows. In Section 2, we introduce the Hopf algebra $H_n$. In Section 3, we define a class of weak Hopf algebras $W(n_1, n_2, n_3)$. In Section 4, we study
the structure of \( W(n_1, n_2, n_3) \) and decompose \( W(n_1, n_2, n_3) \) into a direct sum of \( H_n \) and some algebra of polynomials as an algebra. We give the Ext-quiver of coalgebra of \( W(n_1, n_2, n_3) \) and prove that \( W(n_1, n_2, n_3) \) has a quotient Hopf algebra and a sub-Hopf algebra which are isomorphic to \( H_n \).

### 2. A Quiver Hopf Algebra

The Hopf Algebra \( F(q) \) is defined in [10]. Let \( q \in k \setminus 0 \). As a \( k \)-algebra \( F(q) \) is generated by \( a, b, x \) subject to the relations

\[
ab = 1, \quad ba = 1, \quad xa = qax, \quad xb = q^{-1}bx.
\]

The coalgebra structure of \( F(q) \) is determined by

\[
\Delta(a) = a \otimes a, \quad \Delta(b) = b \otimes b, \quad \Delta(x) = x \otimes a + 1 \otimes x.
\]

\[
\epsilon(1) = \epsilon(a) = \epsilon(b) = 1, \quad \epsilon(x) = 0.
\]

We generalize \( F(q) \) to \( H_n \), which is defined as follows. Let \( k \) be a field, \( q \in k \setminus 0 \), \( i = 1, 2, \ldots, n \). As a \( k \)-algebra \( H_n \) is generated by \( K, K^{-1}, \) and \( X_i, i = 1, 2, \ldots, n \) subject to the relations

\[
KK^{-1} = 1, \quad K^{-1}K = 1, \quad X_iK = qKX_i, \quad X_iK^{-1} = q^{-1}K^{-1}X_i.
\]

The coalgebra structure of \( H_n \) is determined by

\[
\Delta(K) = K \otimes K, \quad \Delta\left(K^{-1}\right) = K^{-1} \otimes K^{-1},
\]

\[
\Delta(X_i) = X_i \otimes K + 1 \otimes X_i
\]

\[
\epsilon(K) = \epsilon\left(K^{-1}\right) = 1, \quad \epsilon(X_i) = 0.
\]

The antipode \( S \) is induced by

\[
S(K) = K^{-1}, \quad S\left(K^{-1}\right) = K, \quad S(X_i) = -K^{-1}X_i.
\]

### 3. A Class of Weak Hopf Algebras

In this section, we construct a class of weak Hopf algebra corresponding to \( H_n \).

First recall the definition of weak Hopf algebra [1].

**Definition 3.1.** A \( k \)-bialgebra \( H = (H, \mu, \eta, \Delta, \epsilon) \) is called a weak Hopf algebra if there exists \( T \in \text{Hom}_k(H, H) \) such that \( id \ast T \ast id = id \) and \( T \ast id \ast T = T \) where \( T \) is called a weak antipode of \( H \).
A weak Hopf algebra is called pointed if it is pointed as a coalgebra. If a weak Hopf algebra \( H \) is pointed, then the set of all group-like elements \( G(H) \) is a regular monoid [6].

Now we construct weak Hopf algebra \( W \) corresponding to \( H_n \). The set \( G(W) \) of group-like elements of weak Hopf algebra \( W \) is a regular monoid which has generators \( g, \overline{g}, 1 \), subject to \( g g = g \overline{g}, \overline{g} g = g, \overline{g}^2 = \overline{g} \).

To construct all possible weak extension we need the following discussion.

Recall, for any coalgebra \( C \), that the group-like elements in \( C \) are the set \( G(C) = \{ a \in C \mid a \neq 0 \text{ and } \Delta(a) = a \otimes a \} \); necessarily \( \varepsilon(a) = 1 \) for \( a \in G(C) \). Note that a simple subcoalgebra \( D \) of \( C \) is one-dimensional \( \iff D = ka \) for some \( a \in G(C) \). A coalgebra is pointed if all of its simple subcoalgebras are one-dimensional. For \( a, b \in G(C) \), the \( a, b \)-primitive elements in \( C \) are the set \( P_{a,b}(C) = \{ c \in C \mid \Delta(c) = c \otimes a + b \otimes c \} \); necessarily \( \varepsilon(c) = 0 \) for \( c \in P_{a,b}(C) \). Note that \( (a - b) = \{ l(a - b) \mid l \in k \} \subset P_{a,b}(C) \); an \( a, b \)-primitive element \( c \) is nontrivial if \( c \notin k(a - b) = \{ l(a - b) \mid l \in k \} \). If \( a = b = 1 \), the 1,1-primitives are simply called primitive; otherwise they are called skew primitive.

The following result is a generalization of [11].

**Lemma 3.2.** Let \( W \) be the weak Hopf algebra defined above. One has

\[
Pg_{a,b}(W) \subseteq Pg_{a,gb}(W), \quad \overline{g}Pg_{a,b}(W) \subseteq Pg_{a,\overline{g}b}(W).
\]

**Proof.** Let \( u \in Pg_{a,b}(W) \), then \( \Delta(u) = u \otimes a + b \otimes u \). Hence,

\[
\Delta(gu) = \Delta(g)\Delta(u) = (g \otimes g) (u \otimes a + b \otimes u) = gu \otimes ga + gb \otimes gu \in Pg_{a,gb}(W).
\]

The second inclusion is proved similarly. \( \square \)

**Corollary 3.3.** For \( W \), one has

\[
\dim P_{g^{i+1},g}(W) = \dim P_{g^{i+1},g^{-1}}(W), \quad i \geq 2,
\]

\[
\dim P_{g^{-i},g}(W) = \dim P_{g^{-i},g}(W), \quad i \geq 2,
\]

\[
\dim P_{g,g}(W) = \dim P_{g,g}(W) = \dim P_{g,g}(W) = \dim P_{g^{-1},g}(W).
\]

**Proof.** We only prove the first equation. In fact, the map \( \psi : P_{g^{i+1},g^{-1}}(W) \to P_{g^{i+1},g^{-1}}(W), u \mapsto gu \) is a linear map with inverse \( \psi^{-1} : P_{g^{-i},g}(W) \to P_{g^{-i},g}(W), v \mapsto \overline{gv} \). Hence, \( P_{g^{i+1},g^{-1}}(W) \) and \( P_{g^{-i},g}(W) \) are isomorphic as vector spaces. \( \square \)

Since all the dimensions in Corollary 3.3 are same, we have the following corollary.

**Corollary 3.4.** One has

\[
\dim P_{g,g}(W) \leq \dim P_{g,g}(W), \quad \dim P_{g,1}(W) \leq \dim P_{g,1}(W).
\]
Proof. The map \( \varphi : P_{g,1}(W) \rightarrow P_{g,\overline{g}}(W), \ u \mapsto g\overline{g}u \) is a linear map. If \( \varphi(u) = g\overline{g}u = l(g - g\overline{g}) \),
for some \( l \in k \), then \( u \in kG(W) \), the vector space spanned by all group-like elements, because
\( W \) is graded. Hence, \( u = l(g - 1) \). Therefore, the linear map \( \varphi \) is an injection. Consequently,
\[
\dim P_{\overline{g},1}(W) \leq \dim P_{g,\overline{g}}(W).
\] (3.5)
The proof of the second inequality is similar. \( \square \)

By the above discussion we know that weak Hopf algebra \( W \) is determined by \( P_{1,\overline{g}}(W) \),
\( P_{g,1}(W) \), and \( P_{g,\overline{g}}(W) \). Take \( x_1, \ldots, x_{n_1} \) to be linearly independent nontrivial elements in
\( P_{1,\overline{g}}(W) \), and \( y_1, \ldots, y_{n_2} \) linearly independent nontrivial elements in \( P_{g,1}(W) \). Let
\[
P_{g,\overline{g}}(W) = (g\overline{g}P_{1,\overline{g}}(W) + gP_{g,1}(W)) \oplus V,
\] (3.6)
and \( z_1, \ldots, z_{n_3} \) a basis of \( V \). Then \( W \) is determined by \( x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}, z_1, \ldots, z_{n_3} \).

To summarize, we define weak Hopf algebra \( W(n_1, n_2, n_3) \) corresponding to \( H_n \) as follows.

Definition 3.5. Let \( k \) be a field. For any positive integers \( n_1, n_2, n_3 \), and nonzero element \( q \in k \),
we define \( W(n_1, n_2, n_3) \) to be associative algebra over field \( k \) generated by \( 1, g, \overline{g}, x_i, y_j, z_k, i = 1, 2, \ldots, n_1, j = 1, 2, \ldots, n_2, k = 1, 2, \ldots, n_3 \), subject to
\[
\begin{align*}
g\overline{g} &= g\overline{g}, & g\overline{g}^2 &= g, & \overline{g}^2 g &= \overline{g}, \\
gx_i &= qx_ig, & \overline{g}x_i &= q^{-1}x_i\overline{g}, & i = 1, 2, \ldots, n_1, \\
gy_j &= qy_ig, & \overline{g}y_j &= q^{-1}y_i\overline{g}, & j = 1, 2, \ldots, n_2, \\
gz_k\overline{g} &= qz_k, & k = 1, 2, \ldots, n_3.
\end{align*}
\] (3.7)

\( W(n_1, n_2, n_3) \) can be endowed with coalgebra structure by
\[
\Delta(g) = g \otimes g, \quad \Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \\
\Delta(y_j) = y_j \otimes 1 + \overline{g} \otimes y_j, \\
\Delta(z_k) = z_k \otimes g + g\overline{g} \otimes z_k,
\] (3.11-3.14)
\[
\varepsilon(1) = \varepsilon(g) = \varepsilon(\overline{g}) = 1, \quad \varepsilon(x_i) = 0, \quad \varepsilon(y_j) = 0, \quad \varepsilon(z_k) = 0,
\] (3.15)
while the weak antipode \( T \) is induced by
\[
T(1) = 1, \quad T(g) = \overline{g}, \quad T(\overline{g}) = g, \quad T(x_i) = -x_i\overline{g}, \quad T(y_j) = -gy_j, \quad T(z_k) = -z_k\overline{g},
\] (3.16-3.17)
Theorem 3.6. For any positive integers \( n_1, n_2, n_3 \), \( W(n_1, n_2, n_3) \) is a weak Hopf algebra.

Proof. First we must check that the coproduct \( \Delta \) is an algebra map. It suffices to prove that \( \Delta \) preserves the relations (3.7)–(3.10). It is easy to see that \( \Delta \) preserves the relations (3.7). And

\[
\Delta(gx_i) = (g \otimes g)(x_i \otimes g + 1 \otimes x_i)
= gx_i \otimes g^2 + g \otimes gx_i
= (qx_i g) \otimes g^2 + g \otimes (qx_i g)
= q(x_i \otimes g + 1 \otimes x_i)(g \otimes g)
= \Delta(qx_i g),
\]

\[
\Delta(gy_j) = (g \otimes g)(y_j \otimes 1 + \overline{g} \otimes y_j)
= gy_j \otimes g + \overline{g} \otimes gy_j
= (qy_j g) \otimes g + \overline{g} \otimes (qy_j g)
= q(y_j \otimes 1 + \overline{g} \otimes y_j)(g \otimes g)
= \Delta(qy_j g),
\]

\[
\Delta(gz_k \overline{g}) = (g \otimes g)(z_k \otimes g + \overline{g} \otimes z_k)(\overline{g} \otimes \overline{g})
= g z_k \overline{g} \otimes gg \overline{g} + gg \overline{g} \otimes g z_k \overline{g}
= (qz_k) \otimes g + \overline{g} \otimes (qz_k)
= \Delta(qz_k).
\]

Next we prove that \( T \) is the weak antipode. It suffices to prove that for each generator \( g, \overline{g}, x_i, y_j, z_k \), the action of \( T \ast id \ast T \) is the same as that of \( T \), and the action of \( id \ast T \ast id \) is the same as that of \( id \).

Since

\[
(\Delta \otimes id)\Delta(x_i) = (\Delta \otimes id)(x_i \otimes g + 1 \otimes x_i)
= (x_i \otimes g + 1 \otimes x_i) \otimes g + 1 \otimes 1 \otimes x_i
= x_i \otimes g \otimes g + 1 \otimes x_i \otimes g + 1 \otimes 1 \otimes x_i,
\]

we get

\[
(id \ast T \ast id)(x_i) = x_i \overline{g} g + (-x_i \overline{g}) g + x_i = x_i = id(x_i),
\]

\[
(T \ast id \ast T)(x_i) = (-x_i \overline{g}) \overline{g} g + x_i \overline{g} + (-x_i \overline{g})
= -x_i (\overline{g} g \overline{g}) = -x_i \overline{g} = T(x_i).
\]
Since
\[
(\Delta \otimes \text{id}) \Delta(y_j) = (\Delta \otimes \text{id})(y_j \otimes 1 + \overline{g} \otimes y_j)
\]
\[
= (y_j \otimes 1 + \overline{g} \otimes y_j) \otimes 1 + \overline{g} \otimes \overline{g} \otimes y_j
\]
\[
= y_j \otimes 1 \otimes 1 + \overline{g} \otimes y_j \otimes 1 + \overline{g} \otimes \overline{g} \otimes y_j,
\]}

it follows that
\[
(id \ast T \ast id)(y_j) = y_j + \overline{g}(-gy_j) + \overline{g}gy_j = y_j = id(y_j),
\]
\[
(T \ast id \ast T)(y_j) = (-gy_j) + gy_j + g\overline{g}(-gy_j) = -gy_j = T(y_j).
\]}

Since
\[
(\Delta \otimes \text{id})\Delta(z_k) = (\Delta \otimes \text{id})(z_k \otimes g + g\overline{g} \otimes z_k)
\]
\[
= (z_k \otimes g + g\overline{g} \otimes z_k) \otimes g + g\overline{g} \otimes g\overline{g} \otimes z_k
\]
\[
= z_k \otimes g \otimes g + g\overline{g} \otimes z_k \otimes g + g\overline{g} \otimes g\overline{g} \otimes z_k,
\]}

we get
\[
(id \ast T \ast id)(z_k) = z_k g\overline{g} + g\overline{g}(-z_k \overline{g})g + g\overline{g}g\overline{g}z_k
\]
\[
= z_k - q\overline{g}z_k \overline{g} + g\overline{g}z_k
\]
\[
= z_k - z_k + z_k = z_k = id(z_k),
\]
\[
(T \ast id \ast T)(z_k) = (-z_k \overline{g})g\overline{g} + g\overline{g}z_k \overline{g} + g\overline{g}g\overline{g}(-z_k \overline{g})
\]
\[
= (-z_k \overline{g}) + z_k \overline{g} + (-z_k \overline{g}) = -z_k \overline{g} = T(z_k).
\]

4. The Structure of $W(n_1, n_2, n_3)$

In this section we study the algebra and coalgebra structure of $W(n_1, n_2, n_3)$.

It is easy to prove that the elements $g\overline{g}$ and $1 - g\overline{g}$ are a pair of orthogonal central idempotents. Set $W_1 = W(n_1, n_2, n_3)g\overline{g}$, $W_2 = W(n_1, n_2, n_3)(1 - g\overline{g})$. We have the following.

Theorem 4.1. $W(n_1, n_2, n_3)$ can be written as a direct sum of two-sided ideals $W(n_1, n_2, n_3) = W_1 \oplus W_2$. And one has the following.

1. As an algebra, $W_1$ is isomorphic to $H_n$, where $n = n_1 + n_2 + n_3$.
2. As an algebra, $W_2$ is isomorphic to the free associative algebra $k(Y_1, \ldots, Y_t)$ of $t$ generators, where $t = n_1 + n_2$. 
Theorem 4.2. The weak Hopf algebras discussed in Remark 4.5. The isomorphisms described in Theorem 4.1 are not isomorphisms of bialgebras.

Proof. (1) Since $g\overline{g}$ and $1 - g\overline{g}$ are a pair of orthogonal central idempotents,

$$W(n_1, n_2, n_3) = W(n_1, n_2, n_3)g\overline{g} \oplus W(n_1, n_2, n_3)(1 - g\overline{g}) = W_1 \oplus W_2. \quad (4.1)$$

The isomorphism $W_1 \rightarrow H_n$ is induced by $x_ig\overline{g} \mapsto X_i$, $y_ig\overline{g} \mapsto X_{n_1+j_i}$, $z_kg\overline{g} \mapsto X_{n_1+n_2+k}$, $g\overline{g} \mapsto 1$, $g^2\overline{g} \mapsto K$.

(2) Note that $z_k(1 - g\overline{g}) = 0$ and $x_i(1 - g\overline{g})y_j(1 - g\overline{g}) = y_j(1 - g\overline{g})x_i(1 - g\overline{g})$. Since $x_i(1 - g\overline{g})$, $y_j(1 - g\overline{g})$ are generators of $W_2$, the isomorphism $W_2 \rightarrow k(Y_1, \ldots, Y_1)$ is defined by $(1 - g^2) \mapsto 1$, $x_i(1 - g^2) \mapsto Y_i$, $y_j(1 - g^2) \mapsto Y_{n_1+j}$. \qed

A weak Hopf ideal $J$ of a weak Hopf algebra $H$ is a bi-ideal such that $T(J) \subseteq J$, where $T$ is the weak antipode of $H$. It is easy to see that $H/J$ has a natural structure of a weak Hopf algebra.

Theorem 4.2. The ideal $J$ in $W(n_1, n_2, n_3)$ generated by $1 - g\overline{g}$ is a weak Hopf ideal. And the quotient weak Hopf algebra $W(n_1, n_2, n_3)/J$ is a Hopf algebra, which is isomorphic to $H_n$, where $n = n_1 + n_2 + n_3$.

Proof. Since

$$\Delta(1 - g\overline{g}) = 1 \otimes 1 - g\overline{g} \otimes g\overline{g}$$

$$= 1 \otimes 1 - g\overline{g} \otimes 1 + g\overline{g} \otimes 1 - g\overline{g} \otimes g\overline{g}$$

$$= (1 - g\overline{g}) \otimes 1 + g\overline{g} \otimes (1 - g\overline{g}),$$

$$T(1 - g\overline{g}) = T(1) - T(\overline{g})T(g) = 1 - g\overline{g},$$

$J$ is a weak Hopf ideal in $W(n_1, n_2, n_3)$.

The isomorphism $W(n_1, n_2, n_3)/J \rightarrow H_n$ is defined by $g + J \mapsto K$, $\overline{g} + J \mapsto K^{-1}$, $x_i + J \mapsto X_i$, $gy_j + J \mapsto X_{n_1+j}$, $z_k + J \mapsto X_{n_1+n_2+k}$. \qed

Now we give the Ext quiver of $W(n_1, n_2, n_3)$. For the definition and calculation of Ext quiver, we refer to [5, 8, 9, 12].

The Ext quiver of $W(n_1, n_2, n_3)$ is shown in Figure 1. The multiplicity of arrow $g \cdot \rightarrow \cdot 1$ is $n_1$. The multiplicity of arrow $1 \cdot \rightarrow \overline{g}$ is $n_2$. The multiplicity of other arrows is all $n$.

Theorem 4.3. The sub-coalgebra $H$ related to the subquiver in Figure 2 is isomorphic to $H_n$ as coalgebra.

Proof. The isomorphism $H \rightarrow H_n$ is induced by $g\overline{g} \mapsto 1$, $g \mapsto K$, $\overline{g} \mapsto K^{-1}$, $x_i \mapsto X_i$, $gy_j \mapsto X_{n_1+j}$, $z_k \mapsto X_{n_1+n_2+k}$. \qed

Remark 4.4. The isomorphisms described in Theorem 4.1 are not isomorphisms of bialgebras.

Remark 4.5. The weak Hopf algebras discussed in [4, 5] also have quotient Hopf algebras and sub-Hopf algebras which are isomorphic to the related Hopf algebras.
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