Research Article

On q -Operators and Summation of Some q -Series

Sana Guesmi and Ahmed Fitouhi

Faculté des Sciences de Tunis, Tunis 1060, Tunisia

Correspondence should be addressed to Ahmed Fitouhi, ahmed.fitouhi@fst.rnu.tn

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Using Jackson's q -derivative and the q -Stirling numbers, we establish some transformation theorems leading to the values of some convergent q -series.

1. Introduction

The operator $(x(d/dx))^n$ has many assets and plays a central role in arithmetic fields and in computation of some finite or infinite sums. For example, when we try to compute the sum $\sum_{k=0}^{+\infty} k^n x^k$, we use the operators $(x(d/dx))^n$, which give

$$\sum_{k=0}^{+\infty} k^n x^k = \left(x \frac{d}{dx} \right)^n \left(\frac{1}{1-x} \right), \quad |x| < 1, \quad n = 0, 1, 2, \dots \quad (1.1)$$

These operators are intimately related to the Stirling numbers of second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ by the formula (see [1])

$$\left(x \frac{d}{dx} \right)^n f(x) = \sum_{k=1}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} x^k \frac{d^k f}{dx^k}, \quad (1.2)$$

where f is a suitable function. We note that the q -analogue of formula (1.2) has been studied by many authors (see [2, 3] and references therein) and has found applications in many fields such as arithmetic partitions and asymptotic expansions.

This paper deals with the analogues of the operators $(x(d/dx))^n$ in *Quantum Calculus* and some q -transformation theorems that will be used to establish the sums of some q -series.

This paper is organized as follows. In Section 2, we present some preliminary notions and notations useful in the sequel. Section 3 gives three applications of a result proved in [2], states a transformation theorem using the q -Stirling numbers, and presents some related applications. Section 4 attempts to give a new q -analogue of formula (1.2) by studying the transformation theorem related to a q -derivative operator.

2. Notations and Preliminaries

To make this paper self-containing and easily decipherable, we recall some useful preliminaries about the *Quantum Calculus* and we select Gasper-Rahman's book [4], for the notations and for a deep study in this way. Throughout this paper, we fix $q \in]0, 1[$.

2.1. q -Shifted Factorials

For $a \in \mathbb{C}$, the q -shifted factorials are defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{+\infty} (1 - aq^k). \quad (2.1)$$

We also write

$$(a_1, \dots, a_k; q)_n = \prod_{j=1}^k (a_j; q)_n, \quad n = 0, 1, \dots, \infty. \quad (2.2)$$

We put

$$\begin{aligned} [x]_q &= \frac{1 - q^x}{1 - q}, \quad x \in \mathbb{C}, \\ [n]_q! &= \frac{(q; q)_n}{(1 - q)^n}, \quad n \in \mathbb{N}. \end{aligned} \quad (2.3)$$

For $a, x \in \mathbb{C}$ and $n \in \mathbb{N}$, we adopt the following notation [5]:

$$(x - a)_q^n = \begin{cases} 1 & \text{if } n = 0, \\ (x - a)(x - aq) \cdots (x - aq^{n-1}) & \text{if } n \geq 1. \end{cases} \quad (2.4)$$

The q -analogue of the Jordan factorial is given by

$$\begin{aligned} [x]_{k,q} &= [x]_q [x - 1]_q \cdots [x - k + 1]_q \\ &= \frac{(1 - q^x)(1 - q^{x-1}) \cdots (1 - q^{x-k+1})}{(1 - q)^k}, \end{aligned} \quad (2.5)$$

and the q -binomial coefficient is defined by

$$\begin{bmatrix} x \\ k \end{bmatrix}_q = \frac{[x]_{k,q}}{[k]_q!}. \quad (2.6)$$

2.2. The Jackson's q -Derivative

The q -derivative $D_q f$ of a function f is defined by (see [4])

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \quad \text{if } x \neq 0, \quad (2.7)$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. Note that when f is differentiable, at x , then $(D_q f)(x)$ tends to $f'(x)$ as q tends to 1^- .

It is easy to see that for suitable functions f and g , we have

$$D_q(fg)(x) = f(qx)D_q g(x) + g(x)D_q f(x), \quad (2.8)$$

$$D_q\left(\frac{f}{g}\right)(x) = \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(x)g(qx)}. \quad (2.9)$$

2.3. Elementary q -Special Functions

Two q -analogues of the exponential function are given by (see [4])

$$e_q(z) = \sum_{n=0}^{+\infty} \frac{z^n}{[n]_q!} = \frac{1}{((1-q)z; q)_\infty}, \quad |z| < (1-q)^{-1}, \quad (2.10)$$

$$E_q(z) = \sum_{n=0}^{+\infty} q^{n(n-1)/2} \frac{z^n}{[n]_q!} = (- (1-q)z; q)_\infty, \quad z \in \mathbb{C}.$$

They satisfy the relations

$$\begin{aligned} D_q e_q(z) &= e_q(z), & D_q E_q(z) &= E_q(qz), \\ e_q(z)E_q(-z) &= E_q(z)e_q(-z) = 1, & E_q(z) &= e_{1/q}(z). \end{aligned} \quad (2.11)$$

In 1910, F. H. Jackson defined a q -analogue of the Gamma function by (see [4, 6])

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}, \quad x \neq 0, -1, -2, \dots \quad (2.12)$$

It satisfies the following functional equations:

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1, \quad \Gamma_q(n+1) = [n]_q!, \quad n \in \mathbb{N}. \quad (2.13)$$

2.4. q -Stirling Numbers of Noncentral Type

In [7], Charalambides introduced the so-called noncentral q -Stirling numbers, which are q -analogues of the Stirling numbers and classified into two kinds.

The noncentral q -Stirling numbers of the first kind $s_q(n, k; r)$ are defined by the following generating relation:

$$[t-r]_{n,q} = q^{-\binom{n}{2}-rn} \sum_{k=0}^n s_q(n, k; r) [t]_q^k, \quad n = 0, 1, \dots, \quad (2.14)$$

and they are given by

$$s_q(n, k; r) = \frac{1}{(1-q)^{n-k}} \sum_{j=k}^n (-1)^{j-k} q^{\binom{n-j}{2}+r(n-j)} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{pmatrix} j \\ k \end{pmatrix}. \quad (2.15)$$

The noncentral q -Stirling numbers of the second kind $S_q(n, k; r)$ are defined by the following generating relation:

$$[t]_q^n = \sum_{k=0}^n q^{\binom{k}{2}-rk} S_q(n, k; r) [t-r]_{k,q}, \quad n = 0, 1, \dots, \quad (2.16)$$

and they are given by

$$\begin{aligned} S_q(n, k; r) &= \frac{1}{[k]_q!} \sum_{j=0}^k (-1)^{k-j} q^{\binom{j+1}{2}-(r+j)k} \begin{bmatrix} k \\ j \end{bmatrix}_q [r+j]_q^n \\ &= \frac{1}{(1-q)^{n-k}} \sum_{j=k}^n (-1)^{j-k} q^{r(j-k)} \begin{pmatrix} n \\ j \end{pmatrix} \begin{bmatrix} j \\ k \end{bmatrix}_q. \end{aligned} \quad (2.17)$$

Remark 2.1. Note that when $r = 0$, then $s_q(n, k; r)$ and $S_q(n, k; r)$ reduce to the q -Stirling numbers, respectively, of the first and the second kind studied by Gould, Carlitz, and Kim (see [8–11]).

Properties

The noncentral q -Stirling numbers satisfy the following properties.

- (i) For $n = 1, 2, \dots$ and $k = 1, 2, \dots, n$,

$$s_q(n, k; r) = s_q(n-1, k-1; r) - [n+r-1]_q s_q(n-1, k; r), \quad (2.18)$$

under the following conditions:

$$\begin{aligned} s_q(0, 0; r) &= 1, & s_q(n, 0; r) &= q^{\binom{n}{2} + rn} [-r]_{n,q}, & n > 0, \\ s_q(0, k; r) &= 0, & k > 0, & & s_q(n, k; r) &= 0, & k > n. \end{aligned} \quad (2.19)$$

(ii) For $n = 1, 2, \dots$ and $k = 1, 2, \dots, n$,

$$S_q(n, k; r) = S_q(n-1, k-1; r) + [r+k]_q S_q(n-1, k; r), \quad (2.20)$$

under the following conditions:

$$\begin{aligned} S_q(0, 0; r) &= 1, & S_q(n, 0; r) &= [r]_q^n, & n > 0, \\ S_q(0, k; r) &= 0, & k > 0, & & S_q(n, k; r) &= 0, & k > n. \end{aligned} \quad (2.21)$$

3. The Operator $(xD_q)^m$ and Some Related Transformations Theorems

As in the classical case (see [1]), the iterate $(xD_q)^m$, $m \in \mathbb{N}$, can be expanded in finite terms involving the q -Stirling numbers. This is the purpose of the following result.

Lemma 3.1 (see [2, 3]). *Letting f be a differentiable function, then one has*

$$(xD_q)^m f(x) = \sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k D_q^k f(x), \quad m = 1, 2, \dots, \quad (3.1)$$

where

$$\left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} = q^{k(k-1)/2} S_q(m-1, k-1; 1) = \frac{1}{[k-1]_q!} \sum_{j=0}^{k-1} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} k-1 \\ j \end{bmatrix}_q [k-j]_q^{m-1}. \quad (3.2)$$

Now, let us give three applications of the previous lemma.

Example 3.2 (q -binomial series). The q -binomial theorem asserts that

$${}_1\Phi_0(q^a; -; q, x) = \sum_{n=0}^{+\infty} \frac{(q^a; q)_n}{(q; q)_n} x^n = \frac{(q^a x; q)_\infty}{(x; q)_\infty}, \quad |x| < 1. \quad (3.3)$$

Using the fact that, for all $m \in \mathbb{N}$,

$$(xD_q)^m x^n = [n]_q^m x^n \quad (3.4)$$

and the previous lemma, we deduce that

$$\sum_{n=0}^{+\infty} \frac{(q^a; q)_n}{(q; q)_n} [n]_q^m x^n = \sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k D_q^k \left(\frac{(q^a x; q)_\infty}{(x; q)_\infty} \right), \quad |x| < 1. \quad (3.5)$$

On the other hand, the definition of q -derivative (2.9) gives

$$D_q \left(\frac{(q^a x; q)_\infty}{(x; q)_\infty} \right) = [a]_q \frac{(q^{a+1} x; q)_\infty}{(x; q)_\infty}, \quad (3.6)$$

and by iteration we have

$$D_q^k \left(\frac{(q^a x; q)_\infty}{(x; q)_\infty} \right) = \frac{\Gamma_q(a+k)}{\Gamma_q(a)} \frac{(q^{a+k} x; q)_\infty}{(x; q)_\infty}. \quad (3.7)$$

Thus,

$$\sum_{n=0}^{+\infty} \frac{(q^a; q)_n}{(q; q)_n} [n]_q^m x^n = \frac{1}{\Gamma_q(a)(x; q)_\infty} \sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k \Gamma_q(a+k) (q^{a+k} x; q)_\infty. \quad (3.8)$$

So, taking $a = 1$, we obtain

$$\sum_{n=0}^{+\infty} [n]_q^m x^n = \frac{1}{1-x} \sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k \frac{[k]_q!}{(xq; q)_k}. \quad (3.9)$$

Remark that if q tends to 1^- , we obtain the formula given in [13, page 366].

Example 3.3 (q -Bessel function). We consider the function

$$C_p(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k x^k}{[k]_q! [k+p]_q!} = \left(\frac{x}{2} \right)^{-p} J_p^{(1)}((1-q)\sqrt{2}x, q), \quad (3.10)$$

where $J_p^{(1)}(\cdot, q)$ is the first Jackson's q -Bessel function of order p (see [12, 13]).

By application of the operator $(xD_q)^m$ to $C_0(x)$ and the use of relation (3.4), we obtain

$$(xD_q)^m C_0(x) = \sum_{k=0}^{+\infty} (-1)^k \frac{[k]_q^m}{([k]_q!)^2} x^k. \quad (3.11)$$

Then, using Lemma 3.1 and the fact that

$$D_q C_p(x) = -C_{p+1}(x), \quad (3.12)$$

we get

$$\sum_{k=0}^{+\infty} (-1)^k \frac{[k]_q^m}{([k]_{q!})^2} x^k = \sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} (-1)^k x^k C_k(x). \quad (3.13)$$

Example 3.4 (*q-polynomial exponential*). Take $f(x) = e_q(x)$. From relation (3.4) and Lemma 3.1, we obtain

$$(xD_q)^m e_q(x) = e_q(x) \Phi_{m,q}(x) = \sum_{n=1}^{+\infty} \frac{[n]_q^m}{[n]_{q!}} x^n, \quad (3.14)$$

where

$$\Phi_{m,q}(x) = \sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k, \quad (3.15)$$

which is called the *q-polynomial exponential*. So,

$$\sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k = E_q(-x) \sum_{n=1}^{+\infty} \frac{[n]_q^m}{[n]_{q!}} x^n = \sum_{n=1}^{+\infty} \sum_{k=1}^n (-1)^k q^{\binom{k}{2}} \frac{[n-k]_q^m}{[k]_{q!} [n-k]_{q!}} x^n. \quad (3.16)$$

In many mathematical fields there are some transformation theorems using the Stirling numbers leading one to compute certain sums (see [14]). The purpose of the following result is to give a *q*-analogue context.

Theorem 3.5. *Let $f(x)$ and $g(x)$ be two functions satisfying*

$$f(x) = \sum_{n=0}^{+\infty} a_n [x]_{q!}^n, \quad g(x) = \sum_{n=0}^{+\infty} c_n x^n. \quad (3.17)$$

Then

$$\sum_{n=0}^{+\infty} c_n f(n) x^n = \sum_{m=1}^{+\infty} a_m \sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k D_q^k g(x) \quad (3.18)$$

provided the series

$$\sum_{n=0}^{+\infty} c_n f(n) x^n \quad (3.19)$$

converges absolutely.

Proof. From Lemma 3.1 and the properties of the q -Stirling numbers of the second kind (2.21), we obtain

$$\begin{aligned} \sum_{m=1}^{+\infty} a_m \sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k D_q^k g(x) &= \sum_{m=1}^{+\infty} a_m (xD_q)^m g(x) \\ &= \sum_{m=1}^{+\infty} a_m (xD_q)^m \left(\sum_{n=0}^{+\infty} c_n x^n \right). \end{aligned} \quad (3.20)$$

The result follows, then, from relations (3.4) and (3.19). $\square$

Corollary 3.6. Let $f(x) = \sum_{n=0}^{+\infty} a_n [x]_q^n$. Then

$$\sum_{n=0}^{+\infty} f(n) x^n = \sum_{m=1}^{+\infty} a_m \sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k \frac{[k]_q!}{(x; q)_{k+1}} \quad (3.21)$$

provided the series $\sum_{n=0}^{+\infty} f(n) x^n$ converges absolutely.

Proof. By taking $g(x) = 1/(1-x) = \sum_{n=0}^{+\infty} x^n$, $|x| < 1$, in the previous theorem, and by application of relation (3.7), we obtain

$$\begin{aligned} \sum_{n=0}^{+\infty} f(n) x^n &= \sum_{m=1}^{+\infty} a_m \sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k D_q^k \left(\frac{1}{1-x} \right) \\ &= \sum_{m=1}^{+\infty} a_m \sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k \frac{\Gamma_q(k+1)}{\Gamma_q(1)} \frac{(q^{k+1}x; q)_\infty}{(x; q)_\infty} \\ &= \sum_{m=1}^{+\infty} a_m \sum_{k=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k \frac{[k]_q!}{(x; q)_{k+1}}. \end{aligned} \quad (3.22)$$

$\square$

Example 3.7. Let $f(x) = [x]_{n,q}$. Then, the fact that

$$[x]_{n,q} = q^{-\binom{n}{2}} \sum_{k=0}^n s_q(n, k; 0) [x]_q^k, \quad n = 0, 1, \dots \quad (3.23)$$

and Corollary 3.6 give

$$\sum_{k=0}^{+\infty} [k]_{n,q} x^k = \frac{q^{-\binom{n}{2}}}{1-x} \sum_{m=1}^n s_q(n, m; 0) \sum_{l=1}^m \left\{ \begin{matrix} m \\ l \end{matrix} \right\}_{q,1} x^l \frac{[l]_q!}{(qx; q)_l}. \quad (3.24)$$

Remark that when q tends to 1^- , we obtain the formula given in [1, (6.4), page 3863].

Some others summation formulas are presented in the following statements.

Corollary 3.8. For $f(x) = \sum_{n=0}^{+\infty} a_n [x]_q^n$, the transformation formulas lead to the following:

$$\begin{aligned} (1) \quad & \sum_{n=0}^{+\infty} q^{n(n-1)/2} \begin{bmatrix} \alpha \\ n \end{bmatrix}_q f(n) x^n = \sum_{n=1}^{+\infty} a_n \sum_{k=1}^n q^{k(k-1)/2} \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,1} [k]_q! \begin{bmatrix} \alpha \\ k \end{bmatrix}_q x^k (1 + q^k x)_q^{\alpha-k}; \\ (2) \quad & \sum_{n=0}^{+\infty} ((1 - q^\alpha)_q^n / (1 - q)_q^n) f(n) x^n = \sum_{n=1}^{+\infty} a_n \sum_{k=1}^n \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,1} \begin{bmatrix} \alpha+k-1 \\ k-1 \end{bmatrix}_q (x^k / (1 - x)_q^{\alpha+k}); \\ (3) \quad & \sum_{n=0}^{+\infty} (f(n) / [n]_q!) x^n = e_q(x) \sum_{n=1}^{+\infty} a_n \sum_{k=1}^n \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,1} x^k = e_q(x) \sum_{n=1}^{+\infty} a_n \Phi_{n,q}(x); \\ (4) \quad & \sum_{n=0}^{+\infty} q^{n(n-1)/2} (f(n) / [n]_q!) x^n = \sum_{n=1}^{+\infty} a_n \sum_{k=1}^n \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,1} q^{k(k-1)/2} (x^k / (-(1 - q)q^k x; q)_\infty) \end{aligned}$$

provided the series converge absolutely.

Proof. The results are direct consequences of Theorem 3.5 by putting the following:

$$\begin{aligned} (1) \quad & g(x) = (1 + x)_q^\alpha = \sum_{n=0}^{+\infty} q^{n(n-1)/2} \begin{bmatrix} \alpha \\ n \end{bmatrix}_q x^n \text{ and remark that } D_q^k (1 + x)_q^\alpha = \\ & q^{k(k-1)/2} [\alpha]_q [\alpha - 1]_q \cdots [\alpha - k + 1]_q (1 + q^k x)_q^{\alpha-k}; \\ (2) \quad & g(x) = 1 / (1 - x)_q^\alpha = \sum_{n=0}^{+\infty} ((1 - q^\alpha)_q^n / (1 - q)_q^n) x^n \text{ and remark that } D_q^k (1 / (1 - x)_q^\alpha) = \\ & [\alpha]_q [\alpha + 1]_q \cdots [\alpha + k - 1]_q / (1 - x)_q^{\alpha+k}; \\ (3) \quad & g(x) = e_q(x); \\ (4) \quad & g(x) = E_q(x) \text{ and remark that } D_q^k E_q(x) = q^{k(k-1)/2} E_q(q^k x). \quad \square \end{aligned}$$

Remark 3.9. The last formulas coincide with some of the ones given in [15] when q tends to 1.

4. The Operator $((x; q)_1 D_q)^m$ and Related Transformation Theorem

Lemma 4.1. For a suitable function f , one has for $m = 1, 2, \dots$

$$[(x; q)_1 D_q]^m f(x) = \sum_{k=1}^m (-1)^{m-k} S_q(m-1, k-1; 1)(x; q)_k D_q^k f(x). \quad (4.1)$$

Proof. The formula can be obtained by induction with respect to m . Indeed, for $m = 1$, we have

$$[(x; q)_1 D_q] f(x) = (x; q)_1 D_q f(x) = S_q(0, 0; 1)(x; q)_1 D_q f(x). \quad (4.2)$$

Assuming that formula (4.1) is true for m , then

$$\begin{aligned}
 [(x; q)_1 D_q]^{m+1} f(x) &= (x; q)_1 D_q \left[\sum_{k=1}^m (-1)^{m-k} S_q(m-1, k-1; 1) (x; q)_k D_q^k f(x) \right] \\
 &= (x; q)_1 \sum_{k=1}^m (-1)^{m-k} S_q(m-1, k-1; 1) (qx; q)_k D_q^{k+1} f(x) \\
 &\quad - (x; q)_1 \sum_{k=1}^m (-1)^{m-k} S_q(m-1, k-1; 1) [k]_q (qx; q)_{k-1} D_q^k f(x) \\
 &= \sum_{k=2}^{m+1} (-1)^{m-k+1} S_q(m-1, k-2; 1) (x; q)_k D_q^k f(x) \\
 &\quad - \sum_{k=1}^m (-1)^{m-k} [k]_q S_q(m-1, k-1; 1) (x; q)_k D_q^k f(x) \\
 &= \sum_{k=2}^m (-1)^{m-k+1} \left[S_q(m-1, k-2; 1) - [k]_q S_q(m-1, k-1; 1) \right] (x; q)_k D_q^k f(x) \\
 &\quad + S_q(m-1, m-1; 1) (x; q)_{m+1} D_q^{m+1} f(x) \\
 &\quad - (-1)^{m-1} S_q(m-1, 0; 1) (x; q)_1 D_q f(x).
 \end{aligned} \tag{4.3}$$

The result is easily deduced by formulas (2.20), and (2.21). $\square$

Theorem 4.2. Let $f(x)$ and $g(x)$ be two functions defined by

$$f(x) = \sum_{n=0}^{+\infty} (-1)^n \alpha_n [x]_q^n, \quad g(x) = \sum_{n=0}^{+\infty} c_n (x; q)_n. \tag{4.4}$$

If the series

$$\sum_{n=0}^{+\infty} c_n f(n)(x; q)_n \tag{4.5}$$

converges absolutely, then

$$\sum_{n=0}^{+\infty} c_n f(n)(x; q)_n = \sum_{m=1}^{+\infty} \alpha_m \sum_{k=1}^m (-1)^{m-k} S_q(m-1, k-1; 1) (x; q)_k D_q^k g(x). \tag{4.6}$$

Proof. From the previous lemma, we obtain for $m = 1, 2, \dots$,

$$\sum_{m=0}^{+\infty} \alpha_m [(x; q)_1 D_q]^m g(x) = \sum_{m=1}^{+\infty} \alpha_m \sum_{k=1}^m (-1)^{m-k} S_q(m-1, k-1; 1) (x; q)_k D_q^k g(x). \tag{4.7}$$

So, the absolute convergence of the series (4.5) and the fact that

$$[(x; q)_1 D_q]^m (x; q)_n = (-1)^m [n]_q^m (x; q)_n, \quad m \in \mathbb{N} \quad (4.8)$$

achieve the proof. $\square$

Corollary 4.3. Let $f(x) = \sum_{m=0}^{+\infty} (-1)^m \alpha_m [x]_q^m$. Then

$$\sum_{m=1}^{+\infty} \alpha_m \sum_{k=1}^m (-1)^{m-k} S_q(m-1, k-1; 1)(x; q)_k [n]_{k,q} x^{n-k} = \sum_{k=0}^n (-q)^k s_q(k, 0, -n) f(k)(x; q)_k. \quad (4.9)$$

Proof. Put $g(x) = x^n$.

Using the representation (see [5])

$$x^n = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-(n-1)k} q^{\binom{k}{2}} (x; q)_k = \sum_{k=0}^n (-q)^k s_q(k, 0, -n) (x; q)_k, \quad (4.10)$$

relation (4.6) and the fact that

$$D_q^k (x^n) = [n]_{k,q} x^{n-k} \quad (4.11)$$

give the desired result. $\square$

Remark 4.4. Note that recently Liu in his paper (see [16]) has obtained some interesting q -identities in showing that the solutions of two difference equations involve some series of q -operators D_q^n of q -Cauchy type.

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