Research Article

Krammer’s Representation of the Pure Braid Group, $P_3$

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We consider Krammer’s representation of the pure braid group on three strings: $P_3 \to GL(3, \mathbb{Z}[t^\pm 1, q^\pm 1])$, where $t$ and $q$ are indeterminates. As it was done in the case of the braid group, $B_3$, we specialize the indeterminates $t$ and $q$ to nonzero complex numbers. Then we present our main theorem that gives us a necessary and sufficient condition that guarantees the irreducibility of the complex specialization of Krammer’s representation of the pure braid group, $P_3$.

1. Introduction

Let $B_n$ be the braid group on $n$ strings. There are a lot of linear representations of $B_n$. The earliest was the Artin representation, which is an embedding $B_n \to Aut(F_n)$, the automorphism group of a free group on $n$ generators. Applying the free differential calculus to elements of $Aut(F_n)$ sometimes gives rise to linear representations of $B_n$ and its normal subgroup, the pure braid group denoted by $P_n$ [1]. The Burau, Gassner, and Krammer’s representations arise this way. In a previous paper, we considered Krammer’s representation of the braid group on three strings and we specialized the indeterminates to nonzero complex numbers. We then found a necessary and sufficient condition that guarantees the irreducibility of such a representation. For more details, see [2].

In Section 2, we introduce some definitions of the pure braid group and Krammer’s representation. In Sections 3 and 4, we present our work that leads to our main theorem, Theorem 4.2, which gives a necessary and sufficient condition for the specialization of Krammer’s representation of $P_3$ to be irreducible.

2. Definitions

Definition 2.1 (see [1]). The braid group on $n$ strings, $B_n$, is the abstract group with presentation $B_n = \{ \sigma_1, \ldots, \sigma_{n-1} / \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i = 1, 2, \ldots n-2, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \}$. 


The generators $\sigma_1, \ldots, \sigma_{n-1}$ are called the standard generators of $B_n$.

**Definition 2.2.** The kernel of the group homomorphism $B_n \to S_n$ is called the pure braid group on $n$ strands and is denoted by $P_n$. It consists of those braids which connect the $i$th item of the left set to the $i$th item of the right set, for all $i$. The generators of $P_n$ are $A_{i,j}$, $1 \leq i < j \leq n$, where $A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_{i}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$.

Let us recall the Lawrence-Krammer representation of braid groups. This is a representation of $B_n$ in $GL(m, Z[t^{\pm 1}, q^{\pm 1}]) = Aut(V_0)$, where $m = n(n - 1)/2$ and $V_0$ is the free module of rank $m$ over $Z[t^{\pm 1}, q^{\pm 1}]$. The representation is denoted by $K(q,t)$. For simplicity we write $K$ instead of $K(q,t)$. What distinguishes this representation from others is that Krammer’s representation defined on the braid group, $B_n$, is a faithful representation for all $n \geq 3$ [3]. The question of whether or not a specific linear representation of an abstract group is irreducible has always been a significant question to answer, especially those representations of the braid group and its normal subgroups. In a previous result, we determined a necessary and sufficient condition for the specialization of Krammer’s representation of $B_3$ to be irreducible [2]. In our current work, we apply Krammer’s representation on the normal subgroup of $B_3$, namely, the pure braid group, $P_3$. Having done some computations, we succeed in establishing a necessary and sufficient condition for the complex specialization of Krammer’s representation of $P_3$ to be irreducible.

**Definition 2.3** (see [3]). With respect to $\{x_{i,j}\}_{1 \leq i < j \leq n}$, the free basis of $V_0$, the image of each Artin generator under Krammer’s representation is written as

\[
K(\sigma_k)(x_{i,j}) = \begin{cases} 
  tq^2x_{k,k+1}, & i = k, j = k + 1; \\
  (1-q)x_{i,k} + qx_{i,k+1}, & j = k, i < k; \\
  x_{i,k} + tq^{k-i+1}(q-1)x_{k,k+1}, & j = k + 1, i < k; \\
  tq(q-1)x_{k,k+1} + qx_{k+1,j}, & i = k, k + 1 < j; \\
  x_{k,j} + (1-q)x_{k+1,j}, & i = k + 1, k + 1 < j; \\
  x_{i,j}, & i < j < k \text{ or } k + 1 < i < j; \\
  x_{i,j} + tq^{k-i}(q-1)^2x_{k,k+1}, & i < k < k + 1 < j.
\end{cases}
\]  

(2.1)

Using the Magnus representation of subgroups of the automorphisms group of free group with $n(n - 1)/2$ generators, we determine Krammer’s representation $K(q,t) : P_3 \to GL(3, Z[t^{\pm 1}, q^{\pm 1}])$. Here $Z[t^{\pm 1}, q^{\pm 1}]$ is the ring of Laurent polynomials on two variables. The images of the generators under Krammer’s representation are given by...
Case 1

Assume that dimension of \( S \) is 1:

In this section, we find a sufficient condition for the irreducibility of Krammer’s representation of the pure braid group on three strings \( P_3 \).

**Theorem 3.1.** For \((q,t) \in \mathbb{C} \times \mathbb{C}^*\), Krammer’s representation \( K(q,t) : P_3 \to GL(3,C) \) is irreducible if \( t^2q^3 \neq 1, tq^3 \neq 1, t \neq -1, q \neq 1, tq \neq 1, \) and \( tq^2 \neq -1 \).

**Proof.** For simplicity, we write \( K(\alpha) \) instead of \( K(q,t)(\alpha) \), where \( \alpha \in P_3 \). Suppose, to get contradiction, that \( K(q,t) : P_3 \to GL(3,C) \) is reducible; then there exists a proper nonzero invariant subspace \( S \), where the dimension of \( S \) is either 1 or 2. We will show that a contradiction is obtained in each of these cases.

Assume that dimension of \( S \) is 1:

The subspace \( S \) has to be one of the following subspaces: \( \langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle, \langle e_1 + ue_2 \rangle, \langle e_2 + ue_3 \rangle, \langle e_1 + ue_2 + ve_3 \rangle \), where \( u, v \) are non zero complex numbers.

**Case 1** \((S = \langle e_1 \rangle)\). Since \( e_1 \in S \), it follows that \( A_{1,2}(e_1) \in S \) which implies that \( t^2q^3(q - 1) = 0 \), a contradiction.
Case 2 ($S = \langle e_2 \rangle$). Since $e_2 \in S$, it follows that $A_{1,2}(e_2) \in S$ which implies that $1 - q = 0$, a contradiction.

Case 3 ($S = \langle e_3 \rangle$). Since $e_3 \in S$, it follows that $A_{1,2}(e_3) \in S$ which implies that $q(1 - q) = 0$, a contradiction.

Case 4 ($S = \langle e_1 + ue_2 \rangle, \ u \neq 0$). Since $e_1 + ue_2 \in S$, it follows that $A_{1,2}(e_1 + ue_2) \in S$. This implies that

\[
\begin{pmatrix}
 t^2 q^4 \\
 t^2 q^3 (q - 1) + qu \\
tq(q - 1) + (1 - q)u
\end{pmatrix} = m \begin{pmatrix}
 1 \\
u \\
0
\end{pmatrix},
\]

(3.1)

where $m$ is a complex number. Solving this system of equations implies that $(tq - 1)(tq^2 + 1) = 0$, which is a contradiction to the hypothesis.

Case 5 ($S = \langle e_2 + ue_3 \rangle, \ u \neq 0$). Since $e_2 + ue_3 \in S$, it follows that $A_{2,3}(e_2 + ue_3) \in S$. This implies that

\[
\begin{pmatrix}
 q(1 - q) + tq^3(q - 1)u \\
 q + t^2 q^4(q - 1)u \\
t^2 q^4 u
\end{pmatrix} = m \begin{pmatrix}
 0 \\
u \\
1
\end{pmatrix},
\]

(3.2)

where $m$ is a complex number. By solving this system of equations, we get that $(tq - 1)(tq^2 + 1) = 0$, which is a contradiction.

Case 6 ($S = \langle e_1 + ue_3 \rangle, \ u \neq 0$). Since $e_1 + ue_3 \in S$, it follows that $A_{1,2}(e_1 + ue_3) \in S$. This implies that

\[
\begin{pmatrix}
 t^2 q^4 \\
 t^2 q^3 (q - 1) + q(1 - q)u \\
tq(q - 1) + (1 - q + q^2)u
\end{pmatrix} = m \begin{pmatrix}
 1 \\
0 \\
u
\end{pmatrix},
\]

(3.3)

where $m$ is a complex number. By solving this system of equations, we get that $(tq - 1)(tq^2 + 1)(tq^2 + q - 1) = 0$.

By our hypothesis, $(tq - 1)(tq^2 + 1) \neq 0$. This implies that $tq^2 + q - 1 = 0$. That is, $tq^2 = 1 - q$. Also, we have that $A_{2,3}(e_1 + ue_3) \in S$. This implies that

\[
\begin{pmatrix}
 1 - q + q^2 + tq^3(q - 1)u \\
 1 - q + t^2 q^4(q - 1)u \\
t^2 q^4 u
\end{pmatrix} = n \begin{pmatrix}
 1 \\
0 \\
u
\end{pmatrix},
\]

(3.4)
where \( n \) is a complex number. By solving this system of equations, we get that \( t^2q^3 = -1 \). This means that

\[
t^2q^3 = tq(tq^2) = tq(1 - q) = tq - tq^2 = tq - 1 + q.
\] (3.5)

This implies that \( q(t + 1) = 0 \), which contradicts the hypothesis.

**Case 7** \((S = (e_1 + ue_2 + ve_3), \ u, v \neq 0)\). Since \( e_1 + ue_2 + ve_3 \in S \), it follows that \( A_{1,2}(e_1 + ue_2 + ve_3) \in S \). This implies that

\[
\begin{pmatrix}
t^2q^4 \\
t^2q^3(q - 1) + qu + q(1 - q)v \\
tq(q - 1) + (1 - q)u + (1 - q + q^2)v
\end{pmatrix} = m \begin{pmatrix} 1 \\ u \\ v \end{pmatrix},
\] (3.6)

where \( m \) is a complex number. Since \( A_{2,3}(e_1 + ue_2 + ve_3) \in S \), it follows that

\[
\begin{pmatrix}
1 - q + q^2 + q(1 - q)u + tq^3(q - 1)v \\
1 - q + qu + t^2q^4(q - 1)v \\
t^2q^4v
\end{pmatrix} = n \begin{pmatrix} 1 \\ u \\ v \end{pmatrix},
\] (3.7)

where \( n \) is a complex number. Solving these two system of equations, we get that \( m = n = t^2q^4 \). Also, we have that

\[
q(t^2q^3 - 1)u + q(q - 1)v = t^2q^3(q - 1),
\] (3.8)

\[
(q - 1)u + \left(t^2q^4 - q^2 + q - 1\right)v = tq(q - 1),
\] (3.9)

\[
q(1 - q)u + tq^3(q - 1)v = t^2q^4 - q^2 + q - 1,
\] (3.10)

\[
q(t^2q^3 - 1)u - t^2q^4(q - 1)v = 1 - q.
\] (3.11)

Subtracting (3.11) from (3.8), we get that \( q(1 + t^2q^3)v = 1 + t^2q^3 \). Here, we have 2 cases whether or not \((1 + t^2q^3)\) is zero.

If \(1 + t^2q^3 = 0\), then we rewrite (3.8), (3.9), (3.10), and (3.11) to become as follows:

\[
2qu - q(q - 1)v = q - 1,
\] (3.12)

\[
(q - 1)u - \left(q^3 + 1\right)v = tq(q - 1),
\] (3.13)

\[
q(1 - q)u + tq^3(q - 1)v = -\left(q^2 + 1\right).
\] (3.14)
Multiplying (3.13) by $q$ and adding it to (3.14) we get that

$$q(tq^3 - tq^2 - q^2 - 1)v = tq^3 - tq^2 - q^2 - 1. \quad (3.15)$$

A simple computation shows that $tq^3 - tq^2 - q^2 - 1 \neq 0$. Thus $v = 1/q$. Substituting $v = 1/q$ in (3.12), we get that $u = (q - 1)/q$. Substituting $u$ and $v$ in (3.14), we get that $tq^2 = tq - 2$. Having that $tq^2 = t(q - 1)/q$ implies that $tq = tq(q - 1)/q = tq(tq - 2)$.

This means that $1 + t^2q^3 \neq 0$. Then $v = 1/q$ and $u = (q - 1)/q$ by (3.8). Substituting $u$ and $v$ in (3.9), we get that $(tq - 1)(tq^2 + 1) = 0$, which contradicts the hypothesis.

**Assume that dimension of $S$ is 2:**

Easy computations show that the subspace $S$ cannot be in the form $S = \langle e_i, e_j \rangle$ or $S = \langle e_i + ue_j, e_k \rangle$ for $i \neq j \neq k$.

It suffices to consider only the case $S = \langle e_1 + ue_2, e_1 + ve_3 \rangle$, where $u, v \neq 0$.

Since $e_1 + ue_2 \in S$, it follows that $A_{1,2}(e_1 + ve_3) \in S$ and so

$$
\begin{pmatrix}
  t^2q^4 \\
  t^2q^3(q - 1) + qu \\
  tq(q - 1) + (1 - q)u
\end{pmatrix}
\in S. \quad (3.16)
$$

Also, we have that $e_1 + ve_3 \in S$, then $A_{1,2}(e_1 + ve_3) \in S$, and so

$$
\begin{pmatrix}
  t^2q^4 \\
  t^2q^3(q - 1) + q(1 - q)v \\
  tq(q - 1) + (1 - q + q^2)v
\end{pmatrix}
\in S. \quad (3.17)
$$

This implies that $((q - q^2)v - qu)e_2 + ((1 - q + q^2)v + (q - 1)u)e_3 \in S$. Note that $((q - q^2)v - qu)$ and $((1 - q + q^2)v + (q - 1)u)$ cannot be both zeros. Assume then that $(q - q^2)v - qu \neq 0$.

Having that $ue_2 - ve_3 \in S$, we get that $\{u((1 - q + q^2)v + (q - 1)u) + v((q - q^2)v - qu)\}e_3 \in S$ and so

$$
(u + qv)(u - v)e_3 \in S. \quad (3.18)
$$

If $(u + qv)(u - v) \neq 0$, then $e_3 \in S$ and thus $S$ is the whole space. Now if $(u + qv)(u - v) = 0$, then we have 2 cases: $u = -qv$ and $u = v$:

Let $u = -qv$. Since $\begin{pmatrix} 0 \\ q \\ 1 \end{pmatrix} \in S$, it follows that $\begin{pmatrix} 0 \\ t^2q^3 \\ t^2q^2 \end{pmatrix} \in S. \quad (3.19)$
On the other hand, we have that

\[ q^{-2} A_{2,3} \begin{pmatrix} 0 \\ q \\ 1 \end{pmatrix} = \begin{pmatrix} (q-1)(tq-1) \\ 1 + t^2q^2(q-1) \\ t^2q^2 \end{pmatrix} \in S. \]  \hspace{1cm} (3.20)

Subtracting (3.19) from (3.20) we get that

\[ \begin{pmatrix} 0 \\ (q-1)(tq-1) \\ 1+t^2q^2 \\ (q-1) \\ 0 \end{pmatrix} \in S. \]

This means that

\[ (q-1)(tq-1)e_1 + (1-t^2q^2)e_2 \in S. \] \hspace{1cm} (3.21)

We also have that

\[ e_1 - qve_2 \in S. \] \hspace{1cm} (3.22)

Solving (3.21) and (3.22), we get that \((1 + tq + q(1 - q)v)e_2 \in S.\)

If \((1 + tq) + q(1 - q)v \neq 0,\) we are done. Otherwise, we have that \(v = (tq + 1)/q(q - 1)\)

and \(u = -qv = (tq + 1)/(1 - q).\) On the other hand, we have that

\[ \begin{pmatrix} 1 \\ u \\ 0 \end{pmatrix} \in S \quad \text{then} \quad \begin{pmatrix} 1 - q \\ 1 + tq \\ 0 \end{pmatrix} \in S. \] \hspace{1cm} (3.23)

Also, we have that

\[ A_{2,3} \begin{pmatrix} 1 - q \\ 1 + tq \\ 0 \end{pmatrix} = \begin{pmatrix} (1-q)(1+q^2+tq^2) \\ 1 + q^2 + tq^2 - q \\ 0 \end{pmatrix} \in S. \] \hspace{1cm} (3.24)

Solving (3.23) and (3.24) implies that \(q(1 + t)(1 + tq^2)e_2 \in S\) and thus \(e_2 \in S.\) Hence \(S = \mathbb{C}^3.\)

Let \(u = v.\) Since \(e_2 - e_3 \in S,\) it follows that \(A_{2,3}(e_2 - e_3) \in S.\) That is, we have that

\[ \begin{pmatrix} (q-1)(-1-tq^2) \\ 1 - t^2q^4(q-1) \\ -t^2q^3 \end{pmatrix} \in S. \] \hspace{1cm} (3.25)
We also have that
\[
\begin{pmatrix}
0 \\
-t^2 q^3 \\
-t^2 q^3
\end{pmatrix} \in S. 
\] (3.26)

Subtracting (3.26) from (3.25), we get that
\[
(q - 1)(-1 - tq^3)e_1 + (1 - t^2 q^4)e_2 \in S. 
\] (3.27)

Also we have that
\[
e_1 + ve_2 \in S. 
\] (3.28)

Solving (3.27) and (3.28), we get that \([(1 + t q^3) [(1 - t q^2) + v(q - 1)] e_2 \in S.\] If \([(1 - t q^2) + v(q - 1)] = 0, then we get that \(u = v = (t q^2 - 1)/(q - 1).\)

Now we have that \(e_1 + ue_2 \in S\) and so
\[
\begin{pmatrix}
(q - 1)(1 + q^2 - tq^3) \\
(tq^2 - 1)(1 + q^2 - tq^3) \\
0
\end{pmatrix} \in S. 
\] (3.29)

We also have that
\[
A_{2,3}
\begin{pmatrix}
q - 1 \\
tq^2 - 1 \\
0
\end{pmatrix} = 
\begin{pmatrix}
(q - 1)(1 + q^2 - tq^3) \\
-q^2 + q - 1 + tq^3 \\
0
\end{pmatrix} \in S. 
\] (3.30)

Subtracting (3.30) from (3.29), we get that \(q(1 - tq)(tq^3 - 1)e_2 \in S\) and so \(e_2 \in S.\) Thus \(S = C^3.\)

Next, we find a necessary condition that guarantees the irreducibility of the complex specialization of Krammer’s representation of \(P_3.\)

**4. Necessary Condition for Irreducibility**

We present the following theorem.

**Theorem 4.1.** For \((q, t) \in (C^*)^2,\) Krammer’s representation \(K(q, t) : P_3 \to GL(3, C)\) is reducible if one of the following conditions is satisfied:

1. \(t^2 q^3 = 1,\)
2. \(tq^3 = 1,\)
\( t = -1, \)
\( q = 1, \)
\( tq = 1, \)
\( tq^2 = -1. \)

**Proof.** Notice that the first three conditions followed from the reducibility on \( B_3. \) Under each of the last three conditions of our hypothesis, we find a proper nonzero invariant subspace under the action of complex specialization of Krammer’s representation of \( P_3. \) Recall that the matrices \( K(A_{1,2}), K(A_{2,3}), \) and \( K(A_{1,3}) \) that will be used in the proof are those given in Definition 2.3.

**Proof of 4 \((q = 1)\).** We have that

\[
K(A_{1,2}) = \begin{pmatrix} t^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K(A_{2,3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^2 \end{pmatrix}, \tag{4.1}
\]

\[
K(A_{1,3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

We take the invariant subspace as the one generated by \( e_1 = (1, 0, 0). \)

**Proof of 5 \((tq = 1)\).** We have that

\[
K(A_{1,2}) = \begin{pmatrix} q^2 & 0 & 0 \\ q(q - 1) & q & q(1 - q) \\ q - 1 & 1 - q & 1 - q + q^2 \end{pmatrix},
K(A_{2,3}) = \begin{pmatrix} 1 - q + q^2 & q(1 - q) & q^2(q - 1) \\ 1 - q & q & q^2(q - 1) \\ 0 & 0 & q^2 \end{pmatrix}, \tag{4.2}
K(A_{1,3}) = \begin{pmatrix} q & q(q - 1) & (1 - q)q^2 \\ -(q - 1)^2 & 1 + 2q(q - 1) & -q(q - 1)^2 \\ 1 - q & q - 1 & q \end{pmatrix}.
\]

We take the invariant subspace as the one generated by \( m = (0, q, 1)^T. \) More precisely, we have that

\[
K(A_{1,2})(m) = m, \quad K(A_{2,3})(m) = q^2m, \quad K(A_{1,3})(m) = q^2m. \tag{4.3}
\]
Proof of 6 ($tq^2 = -1$). We have that

\[ K(A_{1,2}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 + tq & q & q(1 - q) \\ -1 - tq & 1 - q & 1 - q + q^2 \end{pmatrix}, \]

\[ K(A_{2,3}) = \begin{pmatrix} 1 - q + q^2 & q(1 - q) & q(1 - q) \\ 1 - q & q & q - 1 \\ 0 & 0 & 1 \end{pmatrix}, \] (4.4)

\[ K(A_{1,3}) = \begin{pmatrix} q & q(q - 1) & q(q - 1) \\ q - 2 - tq & q^2 - 2q + 2 & (q - 1)^2 \\ tq + 1 & q - 1 & q \end{pmatrix}. \]

We take the invariant subspace as the one generated by $m = (-q, 1, 0)^T$. More precisely, we have that

\[ K(A_{1,2})(m) = m, \quad K(A_{2,3})(m) = q^2 m, \quad K(A_{1,3})(m) = m. \] (4.5)

Combining Theorems 3.1 and 4.1, we obtain our main theorem.

**Theorem 4.2.** For $(q, t) \in (C^*)^2$, Krammer’s representation $K(q, t) : P_3 \to GL(3, C)$ is irreducible if and only if $t^2q^3 \neq 1$, $tq^3 \neq 1, t \neq -1, q \neq 1, tq \neq 1$, and $tq^2 \neq -1$.

**References**


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