Research Article

Some Identities on the $q$-Genocchi Polynomials of Higher-Order and $q$-Stirling Numbers by the Fermionic $p$-Adic Integral on $\mathbb{Z}_p$

Seog-Hoon Rim, Jeong-Hee Jin, Eun-Jung Moon, and Sun-Jung Lee

Department of Mathematics, Kyungpook National University, Taegu 702-701, Republic of Korea

Correspondence should be addressed to Seog-Hoon Rim, shrim@knu.ac.kr

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A systemic study of some families of $q$-Genocchi numbers and families of polynomials of Nörlund type is presented by using the multivariate fermionic $p$-adic integral on $\mathbb{Z}_p$. The study of these higher-order $q$-Genocchi numbers and polynomials yields an interesting $q$-analog of identities for Stirling numbers.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C},$ and $\mathbb{C}_p$ denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$.

When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q - 1|_p < 1$. In this paper, we use the following notation:

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}, \quad (1.1)$$

see [1–10]. Hence $\lim_{q \to 1}[x]_q = x$ for all $x \in \mathbb{Z}_p$. 

The $q$-factorial is defined as $[n]_q! = [n]_q[n - 1]_q \cdots [2]_q[1]_q$, and the Gaussian binomial coefficient is defined by the standard rule

\[
\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!} = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q!}{[k]_q!}, \tag{1.2}
\]

(see [7, 9]). Note that \(\lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k} = n!/(n-k)!k! = n(n-1) \cdots (n-k+1)/k!\). It readily follows from (1.2) that

\[
\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k+1} \binom{n}{k-1}_q + \binom{n}{k}_q, \tag{1.3}
\]

(see [4, 7]).

The $q$-binomial formulas are known,

\[
(b; q)_n = (1-b)(1-bq) \cdots (1-bq^{n-1}) = \sum_{i=0}^{n} \binom{n}{i}_q q^i (-1)^i b^i,
\]

\[
\frac{1}{(b; q)_n} = \frac{1}{(1-b)(1-bq) \cdots (1-bq^{n-1})} = \sum_{i=0}^{\infty} \binom{n+i-1}{i}_q b^i. \tag{1.4}
\]

We say that $f : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and we write $f \in \text{UID}(\mathbb{Z}_p)$, if the difference quotients $\Phi_f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{C}_p$ such that $\Phi_f(x, y) = (f(x) - f(y))/(x - y)$ have a limit $f'(a)$ as $(x, y) \to (a, a)$. For $f \in \text{UID}(\mathbb{Z}_p)$, the $q$-deformed fermionic $p$-adic integral is defined as

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{p^{N-1} q} \sum_{x=0}^{p^{N-1}} f(x)(-q)^x, \tag{1.5}
\]

(see [7, 9]). Note that

\[
I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x). \tag{1.6}
\]

For $n \in \mathbb{N}$, write $f_n(x) = f(x+n)$. Then, we have

\[
I_1(f_n) = (-1)^n I_1(f) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \tag{1.7}
\]
Using (1.7), we can readily derive the Genocchi polynomials, $G_n(x)$, namely,

$$
\int_{\mathbb{R}} e^{(x+y)t} d\mu_{-1}(y) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!},
$$

(1.8)

(see [1-27]). Note that $G_n(0) = G_n$ are referred to as the $n$th Genocchi numbers. Let us now introduce the Genocchi polynomials of Nörlund type as follows:

$$
\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{(x+x_1+\cdots+x_r)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left( \frac{2t}{e^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!},
$$

(1.9)

$$
\left( \frac{e^t + 1}{2t} \right)^r e^{xt} = \sum_{n=0}^{\infty} G_n^{(-r)}(x) \frac{t^n}{n!},
$$

(1.10)

(see [7, 9]). In the special case $x = 0$, $G_n^{(-r)}(0) = G_n^{(-r)}$, and $G_n^{(r)}(0) = G_n^{(r)}$ are referred to as the Genocchi numbers of Nörlund type. Let $(Eh)(x) = h(x + 1)$ be the shift operator. Then, the $q$-difference operator $\Delta_q$ is defined as

$$
\Delta_q^n = \prod_{i=1}^{n} \left( E - q^{i-1}I \right), \quad \text{where } (Ih)(x) = h(x),
$$

(1.11)

(see [4, 7, 9]). It follows from (1.11) that

$$
f(x) = \sum_{n=0}^{\infty} \binom{x}{n} \Delta_q^n f(0),
$$

(1.12)

where $\Delta_q^n f(0) = \sum_{k=0}^{n} \binom{n}{k} q^{\frac{k}{2}} f(n - k)$ (see [5, 6, 10]). The $q$-Stirling number of the second kind (as defined by Carlitz) is given by

$$
S_2(n, k; q) = \frac{q^{\frac{k}{2}}}{[k]_q!} \sum_{j=0}^{n} (-1)^j q^{\frac{j}{2}} \binom{k}{j} \binom{k-j}{q} [k-j]_q^n,
$$

(1.13)

(see [7, 10]). By (1.12) and (1.13), we see that

$$
S_2(n, k; q) = \frac{q^{-\frac{k}{2}}}{[k]_q!} \Delta_q^k 0^n,
$$

(1.14)

(see [6, 10]).

In this paper, the $q$-extensions of (1.9) are considered in several ways. Using these $q$-extensions, we derive some interesting identities and relations for Genocchi polynomials and
numbers of Nörlund type. The purpose of this paper is to present a systemic study of some families $q$-Genocchi numbers and polynomials of Nörlund type by using the multivariate fermionic $p$-adic integral on $\mathbb{Z}_p$.

2. $q$-Extensions of Genocchi Numbers and Polynomials of Nörlund Type

In this section, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. We first consider the $q$-extensions of (1.8) given by the rule

$$
\sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} = t \int_{\mathbb{Z}_p} e^{[x+y]_q} \frac{d\mu_1(y)}{1 + q^i}.
$$

(2.1)

Thus, we obtain the following lemma.

**Lemma 2.1.** If $n \geq 0$, then

$$
\frac{G_{n+1,q}(x)}{n+1} = 2 \sum_{m=0}^{\infty} (-1)^m [m + x]_q^n = 2 \sum_{i=0}^{n} \left( \frac{(-1)^i q^i}{1 + q^i} \right) \sum_{l=0}^{k} \left( \begin{array}{c} k \\ l \end{array} \right)_q (x - k)_q^{l}. 
$$

(2.2)

By (1.14),

$$
[x]_q^n = \sum_{k=0}^{n} \left( \begin{array}{c} x \\ k \end{array} \right)_q [k]_q! S_2(k, n-k; q) q^{(k)}_q
$$

$$
= \sum_{k=0}^{n} [x]_q [x-1]_q \cdots [x-k+1]_q \frac{q^{(k)}_q \cdots (n-k)_q}{(n-k)_q!} \Delta_q^{n-k} 0^k
$$

(2.3)

Thus, we have

$$
\frac{G_{n+1,q}(x)}{n+1} = \sum_{k=0}^{n} \frac{q^{(k)}_q S_2(k, n-k; q) k^k}{(1-q)^k} \sum_{l=0}^{k} \left( \frac{k}{l} \right)_q q^{(l)}_q (-1)^l \sum_{m=0}^{l} \left( \begin{array}{c} l \\ m \end{array} \right)_q (q-1)^m \frac{G_{m+1,q}(1-k)}{m+1},
$$

(2.4)

and we obtain the following theorem.
Theorem 2.2. If \( n \geq 0 \), then

\[
\frac{G_{n+1,q}}{n+1} = \sum_{k=0}^{n} q^{\frac{k}{2}} S_2(k, n-k; q) \sum_{l=0}^{k} \binom{k}{l} q^{\frac{l}{2}} (-1)^l \sum_{m=0}^{l} \binom{l}{m} (q-1)^m \frac{G_{m+1,q}(1-k)}{m+1}, \tag{2.5}
\]

where \( G_{n,q} = G_{n,q}(0) \) stand for the \( n \)th Genocchi numbers.

Consider a \( q \)-extension in (1.9) such that \( G_{0,q}(x) = G_{1,q}(x) = \cdots = G_{r-1,q}(x) = 0 \) and

\[
\frac{G_{n+r,q}(x)}{r!(n+r)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_r]_q d\mu_1(x_1) \cdots d\mu_1(x_r)
\]

\[
= \frac{2^r}{(1-q)^r} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\frac{l}{2}} \left( \frac{1}{1+q^l} \right)^r = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m [m+x]_q^n.
\]  

Let \( F_q^{(r)}(t,x) = \sum_{n=0}^{\infty} G_{n,q}^{(r)}(x) (t^n/n!) \). Then,

\[
F_q^{(r)}(t,x) = 2^r t^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m e^{[m+x]_q^{t}}.
\]  

In the special case \( x = 0 \), the numbers \( G_{n,q}^{(r)}(0) = G_{n,q}^{(r)} \) are referred to as \( q \)-extension of the Genocchi numbers of order \( r \). In the sense of the \( q \)-extension in (1.10), consider the \( q \)-extension of Genocchi polynomials of Nörlund type given by

\[
G_q^{(r)}(t,x) = F_q^{(-r)}(t,x) = \frac{1}{2^r} \sum_{m=0}^{r} \binom{r}{m} e^{[m+x]_q^{t}} = \sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!}. \tag{2.8}
\]

By (2.8), \( G_{0,q}^{(-r)}(x) = G_{1,q}^{(-r)}(x) = \cdots = G_{r-1,q}^{(-r)}(x) = 0 \) and \( r!(r)G_{n-r,q}^{(-r)}(x) = (1/2^r) \sum_{m=0}^{r} \binom{r}{m} [m+x]_q^n \).

Therefore, we obtain the following theorem.

Theorem 2.3. For \( r \in \mathbb{N} \), and, \( n \geq 0 \), write

\[
2^r t^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m e^{[m+x]_q^{t}} = \sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!}. \tag{2.9}
\]

Then,

\[
\frac{G_n^{(r)}(x)}{r!(n+r)} = \frac{2^r}{(1-q)^r} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\frac{l}{2}} \left( \frac{1}{1+q^l} \right)^r = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m [m+x]_q^n,
\]  

\[
r!(\binom{n}{r}) G_n^{(-r)}(x) = \frac{1}{2^r} \sum_{m=0}^{r} \binom{r}{m} [m+x]_q^n.
\]  

The numbers $G_{n,q}^{(h,r)}(0) = G_{n,q}^{(h,r)}$ are referred to as the $q$-extension of Genocchi numbers of Nörlund type. For $h \in \mathbb{Z}$ and $r \in \mathbb{N}$, introduce the extended higher-order $q$-Genocchi polynomials as follows:

$$G_{n+r,q}^{(h,r)}(x) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} q^{\sum_{j=1}^{r}(h-j)x_j}[x + x_1 + \cdots + x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r). \quad (2.11)$$

Then,

$$G_{n+r,q}^{(h,r)}(x) = \frac{2^r}{r!} \sum_{r \neq 0} \frac{n!}{n!} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) (-1)^i q^i x^n \frac{2^r}{(1-q)^n} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) (-1)^i q^i x^n \left( \begin{array}{c} m + r - 1 \\ m \end{array} \right) q^{(h-r)m} [x + m]_q^n. \quad (2.12)$$

Let $F_{q}^{(h,r)}(t, x) = \sum_{n=0}^{\infty} C_{n,q}^{(h,r)}(x)(t^n/n!)$. Then, we can readily see that

$$F_{q}^{(h,r)}(t, x) = 2^r t \sum_{m=0}^{\infty} \left( \begin{array}{c} m + r - 1 \\ m \end{array} \right) q^{(h-r)m} e^{x+m}t. \quad (2.13)$$

Therefore, we obtain the following theorem.

**Theorem 2.4.** For $h \in \mathbb{Z}$ and $n \geq 0$, let

$$2^r t \sum_{m=0}^{\infty} \left( \begin{array}{c} m + r - 1 \\ m \end{array} \right) q^{(h-r)m} e^{x+m}t = \sum_{n=0}^{\infty} C_{n,q}^{(h,r)}(x) \frac{t^n}{n!}. \quad (2.14)$$

Then,

$$C_{n+r,q}^{(h,r)}(x) = \frac{2^r}{r!} \sum_{r \neq 0} \frac{n!}{n!} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) (-1)^i q^i x^n \frac{2^r}{(1-q)^n} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) (-1)^i q^i x^n \left( \begin{array}{c} m + r - 1 \\ m \end{array} \right) q^{(h-r)m} [x + m]_q^n. \quad (2.15)$$

Let us now define the extended higher-order Nörlund type $q$-Genocchi polynomials as follows:

$$r! \binom{n}{r} C_{n-r,q}^{(h,r)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \sum_{x_i} q^{\sum_{j=1}^{r}(h-j)x_j} \frac{(-1)^i q^i x^n}{(1-q)^n} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) (-1)^i q^i x^n \left( \begin{array}{c} m + r - 1 \\ m \end{array} \right) q^{(h-r)m} [x + m]_q^n. \quad (2.16)$$
By (2.16),
\[
\begin{align*}
\frac{r!}{r^r} \binom{n}{r} G_{n-r,q}^{(h-r)}(x) &= \frac{1}{2^r (1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x^{(h-r+l)q} \\
&= \frac{1}{2^r} \sum_{m=0}^{r} \binom{r}{m} q^{m/2} q^{(h-r)m} [m + x]_q^n.
\end{align*}
\]

(2.17)

Let \( F_q^{(h-r)}(t, x) = \sum_{m=0}^{\infty} G_{n,q}^{(h-r)}(x) (t^n/n!) \). Then, we have
\[
F_q^{(h-r)}(t, x) = \frac{1}{2^r} \sum_{m=0}^{r} \binom{r}{m} q^{m/2} q^{(h-r)m} e^{[m+x]_q^t},
\]

where, \( G_{0,q}^{(h-r)}(x) = G_{1,q}^{(h-r)}(x) = \cdots = G_{r-1,q}^{(h-r)}(x) = 0 \). Therefore, we obtain the following theorem.

**Theorem 2.5.** For \( h \in \mathbb{Z}, n \geq 0, \) and \( r \in \mathbb{N} \), write
\[
\frac{1}{2^r} \sum_{m=0}^{r} \binom{r}{m} q^{m/2} q^{(h-r)m} e^{[m+x]_q^t} + \sum_{n=0}^{\infty} G_{n,q}^{(h-r)}(x) \frac{t^n}{n!}.
\]

(2.19)

Then,
\[
\begin{align*}
\frac{r!}{r^r} \binom{n}{r} G_{n-r,q}^{(h-r)}(x) &= \frac{1}{2^r (1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x^{(h-r+l)q} \\
&= \frac{1}{2^r} \sum_{m=0}^{r} \binom{r}{m} q^{m/2} q^{(h-r)m} [m + x]_q^n,
\end{align*}
\]

(2.20)

where, \( G_{0,q}^{(h-r)}(x) = G_{1,q}^{(h-r)}(x) = \cdots = G_{r-1,q}^{(h-r)}(x) = 0 \).

For \( h = r \),
\[
\begin{align*}
\frac{G_{n+r,q}^{(h-r)}(x)}{r! \binom{n+r}{r}} &= \frac{2^r}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x^{(m+r-l)q} \\
&= \frac{1}{2^r} \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1) [m]_q^n,
\end{align*}
\]

(2.21)
\[
\begin{align*}
\frac{r! \binom{n}{r} G_{n-r,q}^{(h-r)}(x)}{r! \binom{n-r}{r}} &= \frac{1}{2^r (1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x^{(m-r-l)q} \\
&= \frac{1}{2^r} \sum_{m=0}^{r} \binom{r}{m} q^{m/2} [m + x]_q^n.
\end{align*}
\]

(2.22)
It can readily be seen that

\[
\frac{q^{mx}2^r}{(-q^{m-r}; q)_r} = \int_{z_p} \cdots \int_{z_p} q^{\sum_{j=1}^r (m-j)x_j + mx} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
\]

\[
= \int_{z_p} \cdots \int_{z_p} (x + x_1 + \cdots + x_r)_q (q-1)^m q^{\sum_{j=1}^r jx_j} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
\]

\[
= \sum_{l=0}^m {m \choose l} (q-1)^l \int_{z_p} \cdots \int_{z_p} [x + x_1 + \cdots + x_r]^l q^{\sum_{j=1}^r jx_j} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
\]

\[
= \sum_{l=0}^m {m \choose l} (q-1)^l c_{l+r,d}^{(0,r)}(x) \int_{l+r}^{(i_+ r)}.
\]

(2.23)

By (2.23), \(q^{mx}2^r/(-q^{m-r}; q)_r = \sum_{l=0}^m {m \choose l} (q-1)^l (G_{l+r,d}^{(0,r)}(x)/r!(i_+ r))\). As is known,

\[
\text{L}_1(f_1) + \text{L}_1(f) = 2f(0), \quad \text{where} \quad f_1(x) = f(x + 1).
\]

(2.24)

It follows from (2.24) that

\[
q^{h-1} \int_{z_p} \cdots \int_{z_p} [x + x_1 + \cdots + x_r]^n q^{\sum_{j=1}^r (h-j)x_j} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
\]

\[
= - \int_{z_p} \cdots \int_{z_p} [x + x_1 + \cdots + x_r]^n q^{\sum_{j=1}^r (h-j)x_j} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
\]

\[
+ 2 \int_{z_p} \cdots \int_{z_p} [x + x_2 + \cdots + x_r]^n q^{\sum_{j=1}^r (h-j)x_j} d\mu_{-1}(x_2) \cdots d\mu_{-1}(x_r).
\]

(2.25)

By (2.25),

\[
q^{h-1} \frac{C_{n+r,d}^{(h,r)}(x + 1)}{n + r} + \frac{C_{n+r,d}^{(h,r)}(x)}{n + r} = 2C_{n+r-1,d}^{(h-1,r-1)}(x).
\]

(2.26)

A simple manipulation shows that

\[
q^x \int_{z_p} \cdots \int_{z_p} [x + x_1 + \cdots + x_r]^n q^{\sum_{j=1}^r (h-j+1)x_j} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
\]

\[
= (q-1) \int_{z_p} \cdots \int_{z_p} [x + x_1 + \cdots + x_r]^{n+1} q^{\sum_{j=1}^r (h-j)x_j} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
\]

\[
+ \int_{z_p} \cdots \int_{z_p} [x + x_1 + \cdots + x_r]^n q^{\sum_{j=1}^r (h-j)x_j} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).
\]

(2.27)

By (2.27), \(q^x(G_{n+r,d}^{(h,r)}(x)/(n + 1)) = (q-1)(G_{n+r+1,d}^{(h,r)}(x)/(n + r + 1)) + (G_{n+r,d}^{(h,r)}(x)/(n + 1)).\)
Therefore, we obtain the following proposition.

**Proposition 2.6.** For \( h \in \mathbb{Z} \), \( r \in \mathbb{N} \) and \( n \geq 0 \), the following equations

\[
q^{h-1} \frac{G_{n+r,q}^{(h,r)}(x+1)}{n+r} + \frac{G_{n+r,q}^{(h,r)}(x)}{n+r} = 2G_{n+r-1,q}^{(h+1,r-1)}(x),
\]
\[
q^x \frac{G_{n+r,q}^{(h+1,r)}(x)}{n+1} = (q-1) \frac{G_{n+r+1,q}^{(h,r)}(x)}{n+r+1} + \frac{G_{n+r,q}^{(h,r)}(x)}{n+1} + \frac{G_{n+r,q}^{(h,r)}(x)}{n+1}
\]

hold. Moreover, \((q^{mx^2})/((q^{m-x}; q)_r) = \sum_{l=0}^{m} \binom{m}{l} (q-1)^l (G_{n+q}^{(r)}(x)/r^l)\).

By (2.21),

\[
\frac{G_{n+r,q}^{(r,r)}(r-x)}{r!(\binom{n+r}{r})} = \frac{2^r}{1-q^{-1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{l(r-x)} \sum_{i=0}^{l} \frac{1}{(-q^{-1}; q^{-1})} \]
\[
= (-1)^n q^{n+x(r)} \frac{2^r}{1-q^{-1}} \sum_{i=0}^{n} \binom{n}{l} (-1)^l q^x \frac{(-1)^n q^{n+x(r)}}{r!(\binom{n+r}{r})}.
\]

Hence,

\[
\int_{x_1} \cdots \int_{x_r} [r-x+x_1+\ldots+x_r]^{n+x(r)} q^{-\sum_{i=1}^{r} (r-x)_i} d\mu_1(x_1) \cdots d\mu_r(x_r)
\]
\[
= (-1)^n q^{n+x(r)} \int_{x_1} \cdots \int_{x_r} [x+x_1+\ldots+x_r]^{n+r} q^{\sum_{i=1}^{r} (r-x)_i} d\mu_1(x_1) \cdots d\mu_r(x_r).
\]

For \( h = r \), \( G_{n+r,q}^{(r,r)}(0) = (-1)^n q^{n+x(r)} C_{n+r,q}^{(r,r)}(r) \). It also follows from (2.26) that

\[
q^{r-1} \frac{G_{n+r,q}^{(r,r)}(x+1)}{n+r} + \frac{G_{n+r,q}^{(r,r)}(x)}{n+r} = 2G_{n+r-1,q}^{(r+1,r-1)}(x).
\]

The Stirling numbers of the first kind are defined as

\[
\prod_{k=1}^{n} (1 + [k]_q) = \sum_{k=0}^{n} S_1(n, k; q) z^k,
\]

(see [6, 9]),

\[
q^{\frac{m}{m+1}} \binom{r}{m}_q = \frac{q^{\frac{m}{m+1}} [m]_q \cdots [r-m+1]_q}{[m]_q!} = \frac{1}{[m]_q!} \prod_{k=0}^{m-1} ([r]_q - [k]_q).
\]
It can readily be seen that
\[
\prod_{k=0}^{n-1} (z - [k]_q) = z^n \prod_{k=0}^{n-1} \left( 1 - \frac{[k]_q}{z} \right) = \sum_{k=0}^{n} S_1(n-1, k; q) (-1)^k z^{n-k}. \tag{2.34}
\]

By (2.33) and (2.34),
\[
\prod_{k=0}^{m-1} ([r]_q - [k]_q) = \sum_{k=0}^{m} S_1(m-1, k; q) (-1)^k [r]_q^{m-k}. \tag{2.35}
\]

Formulas (2.22) and (2.35) imply the following assertion.

**Proposition 2.7.** For \( r \in \mathbb{N} \) and \( n \in \mathbb{Z}_+ \),
\[
r! \binom{n}{r} G_{n-r,q}^{(r)}(x) = \frac{1}{2^r [m]_q^r} \sum_{m=0}^{r} \sum_{k=0}^{m} S_1(m-1, k; q) (-1)^k [r]_q^{m-k} [m + x]_q^n. \tag{2.36}
\]

The generalized Genocchi numbers and polynomials of Nörlund type are defined by
\[
\frac{2^r t^r}{(e^{w_1 t} + 1)(e^{w_2 t} + 1) \cdots (e^{w_r t} + 1)} e^{xt} = \sum_{n=0}^{\infty} G_{n}^{(r)}(x \mid w_1, \ldots, w_r) \frac{t^n}{n!}, \tag{2.37}
\]
and \( G_{n}^{(r)}(w_1, \ldots, w_r) = G_{n}^{(r)}(0 \mid w_1, \ldots, w_r) \). We can now also define a \( q \)-extension of (2.37) as follows. For \( w_1, \ldots, w_r \in \mathbb{Z}_p \) and \( \delta_1, \ldots, \delta_r \in \mathbb{Z} \), write
\[
G_{n+r,q}^{(r)}(x \mid w_1, \ldots, w_r; \delta_1, \ldots, \delta_r) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 w_1 + \cdots + x_r w_r + x]_q^n d\mu_{-q^{\delta_1}}(x_1) \cdots d\mu_{-q^{\delta_r}}(x_r), \tag{2.38}
\]
and \( G_{n+r,q}^{(r)}(w_1, \ldots, w_r; \delta_1, \ldots, \delta_r) = G_{n+r,q}^{(r)}(0 \mid w_1, \ldots, w_r; \delta_1, \ldots, \delta_r) \). Thus,
\[
G_{n+r,q}^{(r)}(x \mid w_1, \ldots, w_r; \delta_1, \ldots, \delta_r) = \frac{[2]_{q^{\delta_1}} \cdots [2]_{q^{\delta_r}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx}}{(1-q)^n (1+q^{\delta_1+w_1}) \cdots (1+q^{\delta_r+w_r})}. \tag{2.39}
\]

Another \( q \)-extension of Nörlund type generalized Genocchi numbers and polynomials is also of interest, namely,
\[
G_{n+r,q}^{(r)}(x \mid w_1, \ldots, w_r; \delta_1, \ldots, \delta_r) \quad \begin{align*}
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 w_1 + \cdots + x_r w_r + x]_q^{\delta_1+w_1+\cdots+\delta_r} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r), \tag{2.40}
\end{align*}
\]
and $G^{(r)}_{ntq}(w_1, \ldots, w_r; \delta_1, \ldots, \delta_r) = G^{(r)}_{ntq}(0 \mid w_1, \ldots, w_r; \delta_1, \ldots, \delta_r)$. By (2.40),

$$
G^{(r)}_{ntq}(x \mid w_1, \ldots, w_r; \delta_1, \ldots, \delta_r) = \frac{2^r \sum_{i=0}^{n} \binom{n}{i}(-1)^i q^i}{(1-q)^n(1 + q^{n_1+lw_1}) \cdots (1 + q^{n_r+lw_r})}.
$$

(2.41)

3. Further Remarks

For $h = 0$, consider the following polynomials $G^{(0,r)}_{ntq}(x)/r!(n_r)$ and $r!(n_r)G^{(0,-r)}_{ntq}(x)$:

$$
\frac{G^{(0,r)}_{ntq}(x)}{r!(n_r)} = \int_{\mathbb{R}^r} \cdots \int_{\mathbb{R}^r} [x + x_1 + \cdots + x_r]^n q^{-\sum_{j=1}^{r} jx_j} d\mu_1(x_1) \cdots d\mu_r(x_r),
$$

(3.1)

Then,

$$
\frac{G^{(0,r)}_{n+q}(x)}{r!(n_r)} = \frac{2^r \sum_{m=0}^{\infty} \binom{m + r - 1}{m} q^{-m}(-1)^m [x + m]^n}{2^r \sum_{m=0}^{\infty} \binom{m + r - 1}{m} q^{-m}(-1)^m [x + m]^n}.
$$

(3.2)

Let $F^{(0,r)}_q(t, x) = \sum_{n=0}^{\infty} G^{(0,r)}_{ntq}(x)(t^n/n!)$ and let $F^{(0,-r)}_q(t, x) = \sum_{n=0}^{\infty} G^{(0,-r)}_{ntq}(x)(t^n/n!)$. Then,

$$
F^{(0,r)}_q(t, x) = 2^r t^r \sum_{m=0}^{\infty} \binom{m + r - 1}{m} q^{-m}(-1)^m e^{[x+m]_q t},
$$

$$
F^{(0,-r)}_q(t, x) = \frac{1}{2^r t^r} \sum_{m=0}^{\infty} \binom{m + r - 1}{m} q^{-m} e^{[x+m]_q t}. \tag{3.3}
$$

Consider the following polynomials:

$$
\frac{G^{(h,1)}_{ntq}(x)}{n + 1} = \int_{\mathbb{R}^r} q^{x_1/(h-1)} [x + x_1]^n d\mu_1(x_1) = \frac{2^r \sum_{i=0}^{n} \binom{n}{i}(-1)^i q^i}{(1-q)^n(1 + q^{n_1+h-1})}. \tag{3.4}
$$
A simple calculation of the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) show that

\[
q^x \int_{\mathbb{Z}_p} [x + x_1]_q^n q^{x_1 (h-1)} d\mu_{-1}(x_1) \\
= (q - 1) \int_{\mathbb{Z}_p} [x + x_1]_q^{n+1} q^{x_1 (h-2)} d\mu_{-1}(x_1) + \int_{\mathbb{Z}_p} [x + x_1]_q^n q^{x_1 (h-2)} d\mu_{-1}(x_1). 
\] (3.5)

By (3.5), \( q^x G^{(h,1)}_{n+1,q}(x) = (q - 1)(G^{(h-1,1)}_{n+2,q}(x)/2(n + 2)) + G^{(h-1,1)}_{n+1,q}(x) \). It can readily be proved that

\[
\int_{\mathbb{Z}_p} [x + x_1]_q^n q^{x_1 (h-1)} d\mu_{-1}(x_1) = \sum_{j=0}^n \binom{n}{j} [x]_q^{n-j} q^{jx} \int_{\mathbb{Z}_p} [x_1]_q^j q^{x_1 (h-1)} d\mu_{-1}(x_1). 
\] (3.6)

By (3.6), \( G^{(h,1)}_{n+1,q}(x)/(n+1) = \sum_{j=0}^n \binom{n}{j} [x]_q^{n-j} q^{jx} (G^{(h,1)}_{j+1,q}/(j+1)) \). Using (2.24), we can also prove that

\[
\int_{\mathbb{Z}_p} [x + x_1 + 1]_q^n q^{x_1 (h-1)} d\mu_{-1}(x_1) + \int_{\mathbb{Z}_p} [x + x_1]_q^n q^{x_1 (h-1)} d\mu_{-1}(x_1) = 2[x]_q^n. 
\] (3.7)

Thus, \( q^{h-1}(G^{(h,1)}_{n+1,q}(x)/(n+1)) + (G^{(h,1)}_{n+1,q}(x)/(n+1)) = 2[x]_q^n \). For \( x = 0 \), we have \( q^{h-1}(G^{(h,1)}_{n+1,q}(1)/(n+1)) + (G^{(h,1)}_{n+1,q}(1)/(n+1)) = 2\delta_{n,0} \), where \( \delta_{n,0} \) is the Kronecker delta.

It is easy to see that \( G^{(h,1)}_{1,q} = \int_{\mathbb{Z}_p} q^{x_1 (h-1)} d\mu_{-1}(x_1) = 2/(1 + q^{h-1}) = 2/[2 q^{h-1}] \). By (3.4),

\[
\frac{G^{(h,1)}_{n+1,q}(1-x)}{n+1} = \int_{\mathbb{Z}_p} [1 - x + x_1]_q^n q^{-x_1 (h-1)} d\mu_{-1}(x_1) \\
= (-1)^n q^{n+h-1} \frac{2}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l}(-1)^l q^{lx} \\
= (-1)^n q^{n+h-1} \frac{G^{(h,1)}_{n+1,q}(x)}{n+1}. 
\] (3.8)

In particular, if \( x = 1 \), then \( G^{(h,1)}_{n+1,q-1}(0)/(n+1) = (-1)^n q^{n+h-1}(G^{(h,1)}_{n+1,q}(1)/(n+1)) = (-1)^n q^n(G^{(h,1)}_{n+1,q}/(n+1)) \) for \( n \geq 1 \).

Recently, Kim has studied \( p \)-adic fermionic integral on \( \mathbb{Z}_p \) connected with the problems of mathematical physics (see [6, 10, 11]), and our result are closely related to his results. In the future, we will try to study \( p \)-adic stochastic problems associated with our theorems. For example, \( p \)-adic \( q \)-Bernstein polynomials seem to be closely related to our results (see [6, 14, 20]).
References


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