Research Article

$\mathcal{N}$ -Structures Applied to Closed Ideals in BCH-Algebras

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The notions of $\mathcal{N}$ -subalgebras and $\mathcal{N}$ -closed ideals in BCH-algebras are introduced, and the relation between $\mathcal{N}$ -subalgebras and $\mathcal{N}$ -closed ideals is considered. Characterizations of $\mathcal{N}$ -subalgebras and $\mathcal{N}$ -closed ideals are provided. Using special subsets, $\mathcal{N}$ -subalgebras and $\mathcal{N}$ -closed ideals are constructed. A condition for an $\mathcal{N}$ -subalgebra to be an $\mathcal{N}$ -closed ideal is discussed. Given an $\mathcal{N}$ -structure, the greatest $\mathcal{N}$ -closed ideal which is contained in the $\mathcal{N}$ -structure is established.

1. Introduction

In [1, 2], Hu and Li introduced the notion of BCH-algebras which are a generalization of BCK/BCI-algebras. Ahmad [3] classified BCH-algebras, and decompositions of BCH-algebras are considered by Dudek and Thomys [4]. Jun et al. [5] discussed the notion of $\mathcal{N}$ -structures and applied it to BCK/BCI-algebras. In [6], Chaudhry et al. studied closed ideals and filters in BCH-algebras. In this paper, we apply the $\mathcal{N}$ -structures to the closed ideal theory in BCH-algebras. We introduced the notion of $\mathcal{N}$ -subalgebras and $\mathcal{N}$ -closed ideals in BCH-algebras, and investigate the relation between $\mathcal{N}$ -subalgebras and $\mathcal{N}$ -closed ideals. We provide characterizations of $\mathcal{N}$ -subalgebras and $\mathcal{N}$ -closed ideals. Using special subsets, we construct $\mathcal{N}$ -subalgebras and $\mathcal{N}$ -closed ideals. We provide a condition for an $\mathcal{N}$ -subalgebra to be an $\mathcal{N}$ -closed ideal. Given an $\mathcal{N}$ -structure (X, μ) , we make the greatest $\mathcal{N}$ -closed ideal which is contained in (X, μ) .

2. Preliminaries

By a *BCH-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

- (H1) $x * x = 0$,
- (H2) $x * y = 0$ and $y * x = 0$ imply $x = y$,
- (H3) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

In a BCH-algebra X , the following conditions are valid (see [1, 4]).

- (a1) $x * 0 = x$,
- (a2) $x * 0 = 0$ implies $x = 0$,
- (a3) $0 * (x * y) = (0 * x) * (0 * y)$,
- (a4) $0 * (0 * (0 * x)) = 0 * x$.

A nonempty subset S of a BCH-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A nonempty subset A of a BCH-algebra X is called a *closed ideal* of X (see [7]) if it satisfies:

- (1) (for all $x \in X$)($x \in A \Rightarrow 0 * x \in A$),
- (2) (for all $y \in X$)(for all $x \in A$)($y * x \in A \Rightarrow y \in A$).

Note that every closed ideal is a subalgebra, but the converse is not true (see [7]). Since every closed ideal is a subalgebra, we know that any closed ideal contains the element 0. Denote by $\mathcal{S}(X)$ and $\mathcal{I}(X)$ the set of all subalgebras and closed ideals of X , respectively.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\vee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases} \quad (2.1)$$

$$\wedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases} \quad (2.2)$$

3. $\mathcal{N}$ -Closed Ideals of BCH-Algebras

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, *$\mathcal{N}$ -function* on X). By an *$\mathcal{N}$ -structure* we mean an ordered pair (X, μ) of X and an $\mathcal{N}$ -function μ on X . In what follows, let X denote a BCH-algebra and μ an $\mathcal{N}$ -function on X unless otherwise specified.

For any $\mathcal{N}$ -structure (X, μ) and $t \in [-1, 0]$, the set

$$C(\mu; t) := \{x \in X \mid \mu(x) \leq t\} \quad (3.1)$$

is called a *closed (μ, t) -cut* of (X, μ) .

Using the similar method to the transfer principle in fuzzy theory (see [8, 9]), we can consider transfer principle in $\mathcal{N}$ -structures. Let A be a subset of X and satisfy the following property $\mathcal{P}$ expressed by a first-order formula:

$$\mathcal{P} : \frac{t_1(x, \dots, y) \in A, \dots, t_n(x, \dots, y) \in A}{t(x, \dots, y) \in A}, \quad (3.2)$$

where $t_1(x, \dots, y), \dots, t_n(x, \dots, y)$ and $t(x, \dots, y)$ are terms of X constructed by variables $x, \dots, y$. We note that the subset A satisfies the property $\mathcal{P}$ if, for all elements $a, \dots, b \in X$, $t(a, \dots, b) \in A$ whenever $t_1(a, \dots, b), \dots, t_n(a, \dots, b) \in A$. For the subset A we define an $\mathcal{N}$ -structure (X, μ_A) which satisfies the following property

$$\bar{\mathcal{P}} : \mu_A(t(x, \dots, y)) \leq \vee \{ \mu_A(t_1(x, \dots, y)), \dots, \mu_A(t_n(x, \dots, y)) \}. \quad (3.3)$$

We establish a statement without proof, and we call it $\mathcal{N}$ -transfer principle in $\mathcal{N}$ -structures.

Theorem 3.1. (*$\mathcal{N}$ -transfer principle*) An $\mathcal{N}$ -structure (X, μ) satisfies the property $\bar{\mathcal{P}}$ if and only if for all $\alpha \in [-1, 0]$,

$$C(\mu; \alpha) \neq \emptyset \implies C(\mu; \alpha) \text{ satisfies the property } \mathcal{P}. \quad (3.4)$$

Definition 3.2. By an $\mathcal{N}$ -subalgebra of X we mean an $\mathcal{N}$ -structure (X, μ) in which μ satisfies:

$$(\forall x, y \in X) \quad (\mu(x * y) \leq \vee \{ \mu(x), \mu(y) \}). \quad (3.5)$$

Theorem 3.3. For an $\mathcal{N}$ -structure (X, μ) , the following are equivalent:

- (1) (X, μ) is an $\mathcal{N}$ -subalgebra of X ;
- (2) (for all $t \in [-1, 0]$) $(C(\mu; t) \in \mathcal{S}(X) \cup \{\emptyset\})$.

Proof. It follows from the $\mathcal{N}$ -transfer principle. □

Definition 3.4. By an $\mathcal{N}$ -closed ideal of X we mean an $\mathcal{N}$ -structure (X, μ) in which μ satisfies:

$$(\forall x, y \in X) \quad (\mu(0 * x) \leq \mu(x) \leq \vee \{ \mu(x * y), \mu(y) \}). \quad (3.6)$$

It is clear that if (X, μ) is an $\mathcal{N}$ -closed ideal or an $\mathcal{N}$ -subalgebra, then $\mu(0) \leq \mu(x)$ for all $x \in X$.

Table 1: Cayley table.

*	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Theorem 3.5. *Every $\mathcal{N}$ -closed ideal is an $\mathcal{N}$ -subalgebra.*

Proof. Let (X, μ) be an $\mathcal{N}$ -closed ideal of X . For any $x, y \in X$, we have

$$\begin{aligned}
 \mu(x * y) &\leq \vee \{ \mu((x * y) * x), \mu(x) \} \\
 &= \vee \{ \mu((x * x) * y), \mu(x) \} \\
 &= \vee \{ \mu(0 * y), \mu(x) \} \\
 &\leq \vee \{ \mu(x), \mu(y) \}.
 \end{aligned} \tag{3.7}$$

Hence (X, μ) is an $\mathcal{N}$ -subalgebra of X . □

The converse of Theorem 3.5 may not be true as seen in the following example.

Example 3.6. Consider a BCH-algebra $X = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 1 (see [7]). Let (X, μ) be an $\mathcal{N}$ -structure in which μ is given by

$$\mu = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -0.8 & -0.3 & -0.3 & -0.3 & -0.8 \end{pmatrix}. \tag{3.8}$$

It is easy to check that (X, μ) is an $\mathcal{N}$ -subalgebra of X but it is not an $\mathcal{N}$ -closed ideal of X since $\mu(3) = -0.3 > -0.8 = \vee \{ \mu(3 * 4), \mu(4) \}$.

In order to discuss the converse of Theorem 3.5 we need to strengthen some conditions. We first consider the following lemma.

Lemma 3.7. *Every $\mathcal{N}$ -subalgebra (X, μ) of X satisfies the following inequality:*

$$(\forall x \in X) \quad (\mu(x) \geq \mu(0 * x)). \tag{3.9}$$

Proof. For any $x \in X$, we get

$$\begin{aligned}
 \mu(0 * x) &\leq \vee \{ \mu(0), \mu(x) \} = \vee \{ \mu(x * x), \mu(x) \} \\
 &= \vee \{ \vee \{ \mu(x), \mu(x) \}, \mu(x) \} = \mu(x),
 \end{aligned} \tag{3.10}$$

which is the desired result. □

Theorem 3.8. *If an $\mathcal{N}$ -subalgebra (X, μ) satisfies*

$$(\forall x, y \in X) \quad (\mu(y) \leq \vee \{\mu(y * x), \mu(x)\}), \quad (3.11)$$

then (X, μ) is an $\mathcal{N}$ -closed ideal of X .

Proof. It is straightforward by Lemma 3.7. □

Proposition 3.9. *Let (X, μ) be an $\mathcal{N}$ -closed ideal of X that satisfies the following inequality*

$$(\forall x \in X) \quad (\mu(x) \leq \mu(0 * x)). \quad (3.12)$$

Then (X, μ) satisfies the inequality

$$(\forall x, y \in X) \quad (\mu(y * x) \leq \mu(x * y)). \quad (3.13)$$

Proof. Using (3.12), (3.6), (a3), (H1), and (H3), we have

$$\begin{aligned} \mu(y * x) &\leq \mu(0 * (y * x)) \\ &\leq \vee \{\mu((0 * (y * x)) * (x * y)), \mu(x * y)\} \\ &= \vee \{\mu(((0 * y) * (0 * x)) * (x * y)), \mu(x * y)\} \\ &= \vee \{\mu(((0 * y) * (x * y)) * (0 * x)), \mu(x * y)\} \\ &= \vee \{\mu(((0 * (x * y)) * y) * (0 * x)), \mu(x * y)\} \\ &= \vee \{\mu(((0 * x) * (0 * y)) * (0 * x)) * y, \mu(x * y)\} \\ &= \vee \{\mu((0 * (0 * y)) * y), \mu(x * y)\} \\ &= \vee \{\mu(0), \mu(x * y)\} = \mu(x * y) \end{aligned} \quad (3.14)$$

for all $x, y \in X$. □

Using the $\mathcal{N}$ -transfer principle, we have a characterization of an $\mathcal{N}$ -closed ideal.

Theorem 3.10. *For an $\mathcal{N}$ -structure (X, μ) , the following are equivalent:*

- (1) (X, μ) is an $\mathcal{N}$ -closed ideal of X .
- (2) (for all $t \in [-1, 0]$) $(C(\mu; t) \in \mathcal{O}(X) \cup \{\emptyset\})$.

Consider two subsets of X as follows:

$$D_1 := \{x \in X \mid 0 * x = 0\}, \quad D_2 := \{x \in X \mid 0 * (0 * x) = x\}. \quad (3.15)$$

Since D_1 and D_2 are a closed ideal and a subalgebra, respectively, the following theorems are direct results of the $\mathcal{N}$ -transfer principle.

Theorem 3.11. Let (X, μ) be an $\mathcal{N}$ -structure in which μ is given by

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in D_1, \\ \beta & \text{otherwise} \end{cases} \quad (3.16)$$

for all $x \in X$ where $\alpha < \beta$. Then (X, μ) is an $\mathcal{N}$ -closed ideal of X .

Theorem 3.12. Let (X, μ) be an $\mathcal{N}$ -structure in which μ is given by

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in D_2, \\ \beta & \text{otherwise} \end{cases} \quad (3.17)$$

for all $x \in X$ where $\alpha < \beta$. Then (X, μ) is an $\mathcal{N}$ -subalgebra of X .

We provide a condition for an $\mathcal{N}$ -subalgebra to be an $\mathcal{N}$ -closed ideal.

Theorem 3.13. Let (X, μ) be an $\mathcal{N}$ -subalgebra of X in which μ satisfies

$$(\forall x, y \in X) \quad (\mu(y * x) \geq \mu(x * y)). \quad (3.18)$$

Then (X, μ) is an $\mathcal{N}$ -closed ideal of X .

Proof. Taking $x = 0$ in (3.18) induces $\mu(0 * y) \leq \mu(y * 0) = \mu(y)$ for all $y \in X$. Using (a1), (3.18), (H1), (H3), and (3.5), we have

$$\begin{aligned} \mu(y) &= \mu(y * 0) \leq \mu(0 * y) \\ &= \mu((x * x) * y) = \mu((x * y) * x) \\ &\leq \vee \{ \mu(x * y), \mu(x) \} \leq \vee \{ \mu(y * x), \mu(x) \} \end{aligned} \quad (3.19)$$

for all $x, y \in X$. Therefore (X, μ) is an $\mathcal{N}$ -closed ideal of X . $\square$

For any $\mathcal{N}$ -structure (X, μ) and any element $w \in X$, we consider the set

$$X_w := \{x \in X \mid \mu(x) \leq \mu(w)\}. \quad (3.20)$$

Then X_w is nonempty subset of X .

Theorem 3.14. *If an $\mathcal{N}$ -structure (X, μ) is an $\mathcal{N}$ -closed ideal of X , then X_w is a closed ideal of X for all $w \in X$.*

Proof. If $x \in X_w$, then $\mu(x) \leq \mu(w)$ which implies from (3.6) that $\mu(0 * x) \leq \mu(x) \leq \mu(w)$. Thus $0 * x \in X_w$. Let $x, y \in X$ be such that $y \in X_w$ and $x * y \in X_w$. Then $\mu(y) \leq \mu(w)$ and $\mu(x * y) \leq \mu(w)$. Using (3.6), we have

$$\mu(x) \leq \vee \{ \mu(x * y), \mu(y) \} \leq \mu(w), \quad \text{i.e., } x \in X_w. \quad (3.21)$$

Therefore X_w is a closed ideal of X . $\square$

Proposition 3.15. *Let (X, μ) be an $\mathcal{N}$ -structure such that X_w is a closed ideal of X for all $w \in X$. Then (X, μ) satisfies the following assertion:*

$$\mu(x) \geq \vee \{ \mu(y * z), \mu(z) \} \implies \mu(x) \geq \mu(y) \quad (3.22)$$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$ be such that $\mu(x) \geq \vee \{ \mu(y * z), \mu(z) \}$. Then $y * z \in X_x$ and $z \in X_x$. Since X_x is a closed ideal of X , it follows that $y \in X_x$ so that $\mu(y) \leq \mu(x)$. This completes the proof. $\square$

Theorem 3.16. *If an $\mathcal{N}$ -structure (X, μ) satisfies (3.22) and $\mu(0 * x) \leq \mu(x)$ for all $x \in X$, then X_w is a closed ideal of X for all $w \in X$.*

Proof. For each $w \in X$, let $x, y \in X$ be such that $x * y \in X_w$ and $y \in X_w$. Then $\mu(x * y) \leq \mu(w)$ and $\mu(y) \leq \mu(w)$, which imply that $\vee \{ \mu(x * y), \mu(y) \} \leq \mu(w)$. It follows from (3.22) that $\mu(x) \leq \mu(w)$ so that $x \in X_w$. If $x \in X_w$, then $\mu(0 * x) \leq \mu(x) \leq \mu(w)$ by assumption. Hence $0 * x \in X_w$. Therefore X_w is a closed ideal of X . $\square$

Theorem 3.17. *Given an $\mathcal{N}$ -structure (X, μ) , let (X, μ^*) be an $\mathcal{N}$ -structure in which μ^* is defined by*

$$\mu^*(x) = \wedge \{ t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle \} \quad (3.23)$$

for all $x \in X$. Then (X, μ^*) is the greatest $\mathcal{N}$ -closed ideal of X such that $(X, \mu^*) \subseteq (X, \mu)$, where $\langle C(\mu; t) \rangle$ is a closed ideal of X generated by $C(\mu; t)$.

Proof. For any $s \in \text{Im}(\mu^*)$, let $s_n = s + (1/n)$ for any $n \in \mathbb{N}$. Let $x \in C(\mu^*; s)$. Then $\mu^*(x) \leq s$, and so

$$\wedge \{ t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle \} \leq s < s + \frac{1}{n} = s_n \quad (3.24)$$

for all $n \in \mathbb{N}$. Hence there exists $t^* \in \{ t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle \}$ such that $t^* < s_n$. Thus $C(\mu; t^*) \subseteq C(\mu; s_n)$, and so $x \in \langle C(\mu; t^*) \rangle \subseteq \langle C(\mu; s_n) \rangle$ for all $n \in \mathbb{N}$. Consequently

$x \in \bigcap_{n \in \mathbb{N}} \langle C(\mu; s_n) \rangle$. On the other hand, if $x \in \bigcap_{n \in \mathbb{N}} \langle C(\mu; s_n) \rangle$, then $s_n \in \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle\}$ for any $n \in \mathbb{N}$. Therefore

$$\mu^*(x) = \bigwedge \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle\} \leq s_n = s + \frac{1}{n} \quad (3.25)$$

for all $n \in \mathbb{N}$. Since n is arbitrary, it follows that $\mu^*(x) \leq s$ so that $x \in C(\mu^*; s)$. Thus $C(\mu^*; s) = \bigcap_{n \in \mathbb{N}} \langle C(\mu; s_n) \rangle$, which is a closed ideal of X . Using Theorem 3.10, we conclude that (X, μ^*) is an $\mathcal{N}$ -closed ideal of X . For any $x \in X$, let

$$s \in \{t \in [-1, 0] \mid x \in C(\mu; t)\}. \quad (3.26)$$

Then $x \in C(\mu; s)$ and thus $x \in \langle C(\mu; s) \rangle$. It follows that

$$s \in \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle\} \quad (3.27)$$

so that $\{t \in [-1, 0] \mid x \in C(\mu; t)\} \subseteq \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle\}$. Hence

$$\begin{aligned} \mu(x) &= \bigwedge \{t \in [-1, 0] \mid x \in C(\mu; t)\} \\ &\geq \bigwedge \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle\} \\ &= \mu^*(x), \end{aligned} \quad (3.28)$$

and so $(X, \mu^*) \subseteq (X, \mu)$. Finally, let (X, ν) be an $\mathcal{N}$ -closed ideal of X such that $(X, \nu) \subseteq (X, \mu)$. Let $x \in X$. If $\mu^*(x) = 0$, then clearly $\nu(x) \leq \mu^*(x)$. Assume that $\mu^*(x) = s \neq 0$. Then $x \in C(\mu^*; s) = \bigcap_{n \in \mathbb{N}} \langle C(\mu; s_n) \rangle$, and so $x \in \langle C(\mu; s_n) \rangle$ for all $n \in \mathbb{N}$. It follows that $\nu(x) \leq \mu(x) \leq s_n = s + (1/n)$ for all $n \in \mathbb{N}$ so that $\nu(x) \leq s = \mu^*(x)$ since n is arbitrary. This shows that $(X, \nu) \subseteq (X, \mu^*)$. This completes the proof. $\square$

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