Research Article

On Differential Subordinations of Multivalent Functions Involving a Certain Fractional Derivative Operator

Jae Ho Choi

*Department of Mathematics Education, Daegu National University of Education,
 1797-6 Daemyong 2 dong, Namgu, Daegu 705-715, South Korea*

Correspondence should be addressed to Jae Ho Choi, choijh@dnue.ac.kr

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We investigate several results concerning the differential subordination of analytic and multivalent functions which is defined by using a certain fractional derivative operator. Some special cases are also considered.

1. Introduction and Definitions

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} := \{1, 2, 3, \dots\}), \quad (1.1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Also let $\mathcal{A}_0$ denote the class of all analytic functions $p(z)$ with $p(0) = 1$ which are defined on $\mathbb{U}$. If f and g are analytic in $\mathbb{U}$ with $f(0) = g(0)$, then we say that f is said to be subordinate to g in $\mathbb{U}$, written $f < g$ or $f(z) < g(z)$, if there exists the Schwarz function w , analytic in $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$), and $f(z) = g(w(z))$ ($z \in \mathbb{U}$). In particular, if the function g is univalent, then the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let a , b , and c be complex numbers with $c \neq 0, -1, -2, \dots$. Then the Gaussian

hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad (1.2)$$

where $(\eta)_k$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\eta)_k = \frac{\Gamma(\eta + k)}{\Gamma(\eta)} = \begin{cases} 1 & (k = 0), \\ \eta(\eta + 1) \cdots (\eta + k - 1) & (k \in \mathbb{N}). \end{cases} \quad (1.3)$$

The hypergeometric function ${}_2F_1(a, b; c; z)$ is analytic in $\mathbb{U}$, and if a or b is a negative integer, then it reduces to a polynomial.

There are a number of definitions for fractional calculus operators in the literature (cf., e.g., [1, 2]). We use here the Saigo-type fractional derivative operator defined as follows (see [3]; see also [4]).

Definition 1.1. Let $0 \leq \lambda < 1$ and $\mu, \nu \in \mathbb{R}$. Then the generalized fractional derivative operator $\mathcal{D}_{0,z}^{\lambda,\mu,\nu}$ of a function $f(z)$ is defined by

$$\mathcal{D}_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{d}{dz} \left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\zeta)^{-\lambda} {}_2F_1\left(\mu-\lambda, 1-\nu; 1-\lambda; 1-\frac{\zeta}{z}\right) f(\zeta) d\zeta \right). \quad (1.4)$$

The function $f(z)$ is an analytic function in a simply-connected region of the z -plane containing the origin, with the order

$$f(z) = O(|z|^\epsilon) \quad (z \rightarrow 0) \quad (1.5)$$

for $\epsilon > \max\{0, \mu - \nu\} - 1$, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring that $\log(z-\zeta)$ be real when $z - \zeta > 0$.

Definition 1.2. Under the hypotheses of Definition 1.1, the fractional derivative operator $\mathcal{D}_{0,z}^{\lambda+m,\mu+m,\nu+m}$ of a function $f(z)$ is defined by

$$\mathcal{D}_{0,z}^{\lambda+m,\mu+m,\nu+m} f(z) = \frac{d^m}{dz^m} \mathcal{D}_{0,z}^{\lambda,\mu,\nu} f(z) \quad (z \in \mathbb{U}; m \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}). \quad (1.6)$$

With the aid of the above definitions, we define a modification of the fractional derivative operator $\Delta_{z,p}^{\lambda,\mu,\nu}$ by

$$\Delta_{z,p}^{\lambda,\mu,\nu} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\nu)}{\Gamma(p+1)\Gamma(p+1-\mu+\nu)} z^\mu \mathcal{D}_{0,z}^{\lambda,\mu,\nu} f(z), \quad (1.7)$$

for $f(z) \in \mathcal{A}(p)$ and $\mu - \nu - p < 1$. Then it is observed that $\Delta_{z,p}^{\lambda,\mu,\nu}$ also maps $\mathcal{A}(p)$ onto itself as follows:

$$\Delta_{z,p}^{\lambda,\mu,\nu} f(z) = z^p + \sum_{k=1}^{\infty} \frac{(p+1)_k (p+1-\mu+\nu)_k}{(p+1-\mu)_k (p+1-\lambda+\nu)_k} a_{k+p} z^{k+p} \quad (1.8)$$

$$(z \in \mathbb{U}; 0 \leq \lambda < 1; \mu - \nu - p < 1; f \in \mathcal{A}(p)).$$

It is easily verified from (1.8) that

$$z \left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \right)' = (p - \mu) \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) + \mu \Delta_{z,p}^{\lambda,\mu,\nu} f(z). \quad (1.9)$$

Note that $\Delta_{z,p}^{0,0,\nu} f = f$, $\Delta_{z,p}^{1,1,\nu} f = zf'/p$, and $\Delta_{z,p}^{\lambda,\lambda,\nu} f = \Omega_z^{(\lambda,p)} f$, where $\Omega_z^{(\lambda,p)}$ is the fractional derivative operator defined by Srivastava and Aouf [5, 6].

In this manuscript, we will use the method of differential subordination to derive certain properties of multivalent functions defined by fractional derivative operator $\Delta_{z,p}^{\lambda,\mu,\nu}$.

2. Main Results

In order to establish our results, we require the following lemma due to Miller and Mocanu [7].

Lemma 2.1. Let $q(z)$ be univalent in $\mathbb{U}$ and let $\theta(w)$ and $\phi(w)$ be analytic in a domain $\mathfrak{D}$ containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that

- (1) $Q(z)$ is starlike (univalent) in $\mathbb{U}$,
- (2) $\operatorname{Re}\{zh'(z)/Q(z)\} = \operatorname{Re}\{\theta'(q(z))/\phi(q(z)) + zQ'(z)/Q(z)\} > 0$ ($z \in \mathbb{U}$).

If $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathfrak{D}$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \quad (2.1)$$

then $p(z) < q(z)$ and $q(z)$ is the best dominant.

We begin by proving the following

Theorem 2.2. Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta \neq 0$, and let $0 \leq \lambda < 1$, $\mu, \nu \in \mathbb{R}$, $\mu \neq p-1$, $\mu - \nu - p < 1$, and $\gamma(p - \mu - 1)/\beta < 2$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $\mathbb{U}$ and satisfies

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} \frac{\gamma(p - \mu - 1) - \beta}{\beta} & \text{if } \frac{\gamma(p - \mu - 1)}{\beta} \geq 1, \\ 0 & \text{if } \frac{\gamma(p - \mu - 1)}{\beta} \leq 1. \end{cases} \quad (2.2)$$

If $f(z) \in \mathcal{A}(p)$ and

$$\begin{aligned} & \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} \left\{ \alpha \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + \beta \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} + \gamma \right\} \\ & < \frac{1}{p-\mu-1} \{ (p-\mu)(\alpha+\beta) - \alpha + [\gamma(p-\mu-1) - \beta]q(z) - \beta zq'(z) \}, \end{aligned} \quad (2.3)$$

then

$$\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} < q(z) \quad (2.4)$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} \quad (z \in \mathbb{U}). \quad (2.5)$$

Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. A simple computation using (2.5) gives

$$\frac{zp'(z)}{p(z)} = \frac{z \left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \right)'}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} - \frac{z \left(\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) \right)'}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}. \quad (2.6)$$

By applying the identity (1.9) in (2.6), we obtain

$$\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} = \frac{1}{p-\mu-1} \left\{ \frac{p-\mu}{p(z)} - 1 - \frac{zp'(z)}{p(z)} \right\}. \quad (2.7)$$

Making use of (2.5) and (2.7), we have

$$\begin{aligned} & \left\{ \alpha \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + \beta \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} + \gamma \right\} \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} \\ & = \left\{ \frac{\alpha}{p(z)} + \frac{\beta}{p-\mu-1} \left(\frac{p-\mu}{p(z)} - 1 - \frac{zp'(z)}{p(z)} \right) + \gamma \right\} p(z) \\ & = \frac{1}{p-\mu-1} \{ (p-\mu)(\alpha+\beta) - \alpha + [\gamma(p-\mu-1) - \beta]p(z) - \beta zp'(z) \}. \end{aligned} \quad (2.8)$$

In view of (2.8), the subordination (2.3) becomes

$$[\gamma(p - \mu - 1) - \beta]p(z) - \beta zp'(z) < [\gamma(p - \mu - 1) - \beta]q(z) - \beta zq'(z) \quad (2.9)$$

and this can be written as (2.1), where

$$\theta(w) = [\gamma(p - \mu - 1) - \beta]w, \quad \phi(w) = -\beta. \quad (2.10)$$

Since $\beta \neq 0$, we find from (2.10) that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}$ with $\phi(w) \neq 0$. Let the functions $Q(z)$ and $h(z)$ be defined by

$$\begin{aligned} Q(z) &= zq'(z)\phi(q(z)) = -\beta zq'(z), \\ h(z) &= \theta(q(z)) + Q(z) = [\gamma(p - \mu - 1) - \beta]q(z) - \beta zq'(z). \end{aligned} \quad (2.11)$$

Then, by virtue of (2.2), we see that $Q(z)$ is starlike and

$$\operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \operatorname{Re} \left\{ \frac{\beta - \gamma(p - \mu - 1)}{\beta} + \left(1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0. \quad (2.12)$$

Hence, by using Lemma 2.1, we conclude that $p(z) < q(z)$, which completes the proof of Theorem 2.2. $\square$

Remark 2.3. If we put $\lambda = \mu$ in Theorem 2.2, then we get new subordination result for the fractional derivative operator $\Omega_z^{(\lambda, p)}$ due to Srivastava and Aouf [5, 6].

Theorem 2.4. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha, \delta \neq 0$, and let $0 \leq \lambda < 1$, $\mu, \nu \in \mathbb{R}$, $\mu \neq p$, $\mu - \nu - p < 1$, and $1 + \delta(p - \mu)(\alpha + \gamma)/\alpha > 0$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $\mathbb{U}$ and satisfies

$$\operatorname{Re} \left(1 + \frac{zq''(z)}{q'(z)} \right) > \begin{cases} \frac{\delta(\mu - p)(\alpha + \gamma)}{\alpha} & \text{if } \frac{\delta(p - \mu)(\alpha + \gamma)}{\alpha} \leq 0, \\ 0 & \text{if } \frac{\delta(p - \mu)(\alpha + \gamma)}{\alpha} \geq 0. \end{cases} \quad (2.13)$$

If $f(z) \in \mathcal{A}(p)$ and

$$\begin{aligned} & \left\{ \alpha \frac{\Delta_{z,p}^{\lambda+1, \mu+1, \nu+1} f(z)}{\Delta_{z,p}^{\lambda, \mu, \nu} f(z)} + \beta \left(\frac{z^p}{\Delta_{z,p}^{\lambda, \mu, \nu} f(z)} \right)^\delta + \gamma \right\} \left(\frac{\Delta_{z,p}^{\lambda, \mu, \nu} f(z)}{z^p} \right)^\delta \\ & < \frac{\alpha}{\delta(p - \mu)} zq'(z) + (\alpha + \gamma)q(z) + \beta. \end{aligned} \quad (2.14)$$

then

$$\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p} \right)^\delta < q(z) \quad (2.15)$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p} \right)^\delta \quad (z \in \mathbb{U}). \quad (2.16)$$

Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. By a simple computation, we find from (2.16) that

$$\frac{zp'(z)}{p(z)} = \frac{\delta z \left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \right)'}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} - p\delta. \quad (2.17)$$

By using the identity (1.9) in (2.17), we obtain

$$\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} = \frac{1}{\delta(p-\mu)} \frac{zp'(z)}{p(z)} + 1. \quad (2.18)$$

Applying (2.16) and (2.18), we have

$$\begin{aligned} & \left\{ \alpha \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + \beta \left(\frac{z^p}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \right)^\delta + \gamma \right\} \left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p} \right)^\delta \\ &= \left\{ \alpha \left(\frac{1}{\delta(p-\mu)} \frac{zp'(z)}{p(z)} + 1 \right) + \frac{\beta}{p(z)} + \gamma \right\} p(z) \\ &= \frac{\alpha}{\delta(p-\mu)} zp'(z) + (\alpha + \gamma)p(z) + \beta. \end{aligned} \quad (2.19)$$

In view of (2.19), the subordination (2.14) becomes

$$\delta(p-\mu)(\alpha+\gamma)p(z) + \alpha zp'(z) < \delta(p-\mu)(\alpha+\gamma)q(z) + \alpha zq'(z) \quad (2.20)$$

and this can be written as (2.1), where

$$\theta(w) = \delta(p-\mu)(\alpha+\gamma)w, \quad \phi(w) = \alpha. \quad (2.21)$$

Since $\alpha \neq 0$, it follows from (2.21) that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}$ with $\phi(w) \neq 0$. Let the functions $Q(z)$ and $h(z)$ be defined by

$$\begin{aligned} Q(z) &= zq'(z)\phi(q(z)) = \alpha zq'(z), \\ h(z) &= \theta(q(z)) + Q(z) = \delta(p - \mu)(\alpha + \gamma)q(z) + \alpha zq'(z). \end{aligned} \quad (2.22)$$

Then, by virtue of (2.13), we see that $Q(z)$ is starlike and

$$\operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \operatorname{Re} \left\{ \frac{\delta(p - \mu)(\alpha + \gamma)}{\alpha} + \left(1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0. \quad (2.23)$$

Hence, by using Lemma 2.1, we conclude that $p(z) < q(z)$, which proves Theorem 2.4. $\square$

If we put $\lambda = \mu = 0$ in Theorem 2.4, then we have the following.

Corollary 2.5. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha, \delta \neq 0$, and let $1 + p\delta(\alpha + \gamma)/\alpha > 0$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $\mathbb{U}$ and satisfies*

$$\operatorname{Re} \left(1 + \frac{zq''(z)}{q'(z)} \right) > \begin{cases} \frac{-p\delta(\alpha + \gamma)}{\alpha} & \text{if } \frac{\delta(\alpha + \gamma)}{\alpha} \leq 0, \\ 0 & \text{if } \frac{\delta(\alpha + \gamma)}{\alpha} \geq 0. \end{cases} \quad (2.24)$$

If $f(z) \in \mathcal{A}(p)$ and

$$\left\{ \alpha \frac{zf'(z)}{f(z)} + \beta \left(\frac{z^p}{f(z)} \right)^\delta + \gamma \right\} \left(\frac{f(z)}{z^p} \right)^\delta < \frac{\alpha}{p\delta} zq'(z) + (\alpha + \gamma)q(z) + \beta, \quad (2.25)$$

then $(f(z)/z^p)^\delta < q(z)$ and $q(z)$ is the best dominant.

By putting $\delta = \alpha$ in Corollary 2.5, we obtain the following.

Corollary 2.6. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha \neq 0$, and let $1 + p(\alpha + \gamma) > 0$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $\mathbb{U}$ and satisfies*

$$\operatorname{Re} \left(1 + \frac{zq''(z)}{q'(z)} \right) > \begin{cases} -p(\alpha + \gamma) & \text{if } \alpha + \gamma \leq 0, \\ 0 & \text{if } \alpha + \gamma \geq 0. \end{cases} \quad (2.26)$$

If $f(z) \in \mathcal{A}(p)$ and

$$\left\{ \alpha \frac{zf'(z)}{f(z)} + \beta \left(\frac{z^p}{f(z)} \right)^\alpha + \gamma \right\} \left(\frac{f(z)}{z^p} \right)^\alpha < \frac{zq'(z)}{p} + (\alpha + \gamma)q(z) + \beta, \quad (2.27)$$

then $(f(z)/z^p)^\alpha < q(z)$ and $q(z)$ is the best dominant.

By using Lemma 2.1, we obtain the following.

Theorem 2.7. Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta \neq 0$, and let $0 \leq \lambda < 1$, $\mu, \nu \in \mathbb{R}$, $\mu \neq 0$, $\mu - \nu - p < 1$, and $1 + \gamma/\beta > 0$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $\mathbb{U}$ and satisfies

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} -\frac{\gamma}{\beta} & \text{if } \frac{\gamma}{\beta} \leq 0, \\ 0 & \text{if } \frac{\gamma}{\beta} \geq 0. \end{cases} \quad (2.28)$$

If $f(z) \in \mathcal{A}(p)$ and

$$\left\{ \alpha\beta \left[(p - \mu - 1) \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} - (p - \mu) \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + 1 \right] + \gamma \right\} \cdot \left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \right)^\alpha < \beta zq'(z) + \gamma q(z), \quad (2.29)$$

then

$$\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \right)^\alpha < q(z) \quad (2.30)$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \right)^\alpha \quad (z \in \mathbb{U}). \quad (2.31)$$

Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. A simple computation using (1.9) and (2.31) gives

$$\frac{1}{\alpha} \frac{zp'(z)}{p(z)} = (p - \mu - 1) \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} - (p - \mu) \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + 1. \quad (2.32)$$

By using (2.29), (2.31), and (2.32), we get

$$\left\{ \alpha\beta \left[(p - \mu - 1) \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} - (p - \mu) \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + 1 \right] + \gamma \right\} \cdot \left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \right)^\alpha = \beta zp'(z) + \gamma p(z). \quad (2.33)$$

And this can be written as (2.1) when $\theta(w) = \gamma w$ and $\phi(w) = \beta$. Note that $\phi(w) \neq 0$ and $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}$. Let the functions $Q(z)$ and $h(z)$ be defined by

$$\begin{aligned} Q(z) &= zq'(z)\phi(q(z)) = \beta zq'(z), \\ h(z) &= \theta(q(z)) + Q(z) = \gamma q(z) + \beta zq'(z). \end{aligned} \quad (2.34)$$

Then, by virtue of (2.28), we see that $Q(z)$ is starlike and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\gamma}{\beta} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0. \quad (2.35)$$

Hence, by applying Lemma 2.1, we observe that $p(z) < q(z)$, which evidently proves Theorem 2.7. $\square$

Finally, we prove

Theorem 2.8. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha, \delta \neq 0$, and let $0 \leq \lambda < 1$, $\mu, \nu \in \mathbb{R}$, $\mu \neq p$, $p+1-\mu+\nu > 0$ and $1 - \delta(p-\mu)(\alpha+\gamma)/\alpha > 0$. Suppose that $q(z) \in \mathcal{A}_0$ be univalent in $\mathbb{U}$ and satisfies

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} \frac{\delta(p-\mu)(\alpha+\gamma)}{\alpha} & \text{if } \frac{\delta(p-\mu)(\alpha+\gamma)}{\alpha} \geq 0, \\ 0 & \text{if } \frac{\delta(p-\mu)(\alpha+\gamma)}{\alpha} \leq 0. \end{cases} \quad (2.36)$$

If $f(z) \in \mathcal{A}(p)$ and

$$\begin{aligned} &\left\{\alpha \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + \beta \left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p}\right)^\delta + \gamma\right\} \left(\frac{z^p}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}\right)^\delta \\ &< \beta + (\alpha + \gamma)q(z) - \frac{\alpha}{\delta(p-\mu)}zq'(z), \end{aligned} \quad (2.37)$$

then

$$\left(\frac{z^p}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}\right)^\delta < q(z) \quad (2.38)$$

and $q(z)$ is the best dominant.

Proof. If we define the function $p(z)$ by

$$p(z) = \left(\frac{z^p}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}\right)^\delta \quad (z \in \mathbb{U}), \quad (2.39)$$

then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Hence, by using the same techniques as detailed in the proof of Theorem 2.2, we obtain the desired result. $\square$

By taking $\lambda = \mu = 0$ in Theorem 2.8 and after a suitable change in the parameters, we have the following.

Corollary 2.9. *Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $p\alpha < 1/2$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $\mathbb{U}$ and satisfies*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} 2p\alpha & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha < 0. \end{cases} \quad (2.40)$$

If $f(z) \in \mathcal{A}(p)$ and

$$\alpha\left(1 + \frac{zf'(z)}{f(z)}\right)\left(\frac{z^p}{f(z)}\right)^\alpha < 2\alpha q(z) - \frac{1}{p}zq'(z), \quad (2.41)$$

then $(z^p/f(z))^\alpha < q(z)$ and $q(z)$ is the best dominant.

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References

- [1] H. M. Srivastava and R. G. Buschman, *Theory and Applications of Convolution Integral Equations*, vol. 79 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.
- [2] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, New York, NY, USA, 1993.
- [3] R. K. Raina and H. M. Srivastava, "A certain subclass of analytic functions associated with operators of fractional calculus," *Computers & Mathematics with Applications*, vol. 32, no. 7, pp. 13–19, 1996.
- [4] R. K. Raina and J. H. Choi, "On a subclass of analytic and multivalent functions associated with a certain fractional calculus operator," *Indian Journal of Pure and Applied Mathematics*, vol. 33, no. 1, pp. 55–62, 2002.
- [5] H. M. Srivastava and M. K. Aouf, "A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I," *Journal of Mathematical Analysis and Applications*, vol. 171, no. 1, pp. 1–13, 1992.
- [6] H. M. Srivastava and M. K. Aouf, "A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. II," *Journal of Mathematical Analysis and Applications*, vol. 192, no. 3, pp. 673–688, 1995.
- [7] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Application*, vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.