Research Article

The Semi-Difference Entire Sequence Space $cs \cap d_1$

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Let $\Gamma$ denote the space of all entire sequences. Let $\Lambda$ denote the space of all analytic sequences. In this paper, we introduce a new class of sequence space, namely, the semi-difference entire sequence space $cs \cap d_1$. It is shown that the intersection of all semi-difference entire sequence spaces $cs \cap d_1$ is $I \subset cs \cap d_1$ and $\Gamma/\Delta \subset I$.

1. Introduction

A complex sequence, whose $k$th term is $x_k$, is denoted by $\{x_k\}$ or simply $x$. Let $w$ be the set of all sequences and $\phi$ be the set of all finite sequences. Let $\ell_\infty, c, c_0$ be the classes of bounded, convergent, and null sequence, respectively. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by $\Lambda$. A sequence $x$ is called entire sequence if $\lim_{k \to \infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by $\Gamma$.

Given a sequence $x = \{x_k\}$, its $n$th section is the sequence $x^{(n)} = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$. Let $d^{(n)} = (0, 0, \ldots, 1, 0, 0, \ldots)$, 1 in the $n$th place and zeros elsewhere, $s^{(k)} = (0, 0, \ldots, 1, -1, 0, \ldots)$, 1 in the $n$th place, -1 in the $(n+1)$th place and zeros elsewhere. An FK-space (Fréchet coordinate space) is a Fréchet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k$ ($k = 1, 2, 3, \ldots$) are continuous.

We recall the following definitions (one may refer to Wilansky [1]).

An FK-space is a locally convex Fréchet space which is made up of sequences and has the property that coordinate projections are continuous. A metric space $(X, d)$ is said to have AK (or sectional convergence) if and only if $d(x^{(n)}, x) \to x$ as $n \to \infty$, (see [1]). The space is said to have AD (or) be an AD-space if $\phi$ is dense in $X$, where $\phi$ denotes the set of all finitely nonzero sequences. We note that AK implies AD (one may refer to Brown [2]).
If $X$ is a sequence space, we define

(i) $X' = \text{the continuous dual of } X$;
(ii) $X^s = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X\}$;
(iii) $X^h = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for each } x \in X\}$;
(iv) $X^i = \{a = (a_k) : \sup_n |\sum_{k=1}^{n} a_k x_k| < \infty, \text{ for each } x \in X\}$;
(v) let $X$ be an FK-space $\supset \phi$. Then, $X^f = \{f(\xi^{(n)}) : f \in X'\}$.

$X^s, X^h, X^i$ are called the $a$ (or Köthe-Töeplitz) dual of $X, \beta$—(or generalized Köthe-Töeplitz) dual of $X$, $\gamma$ dual of $X$. Note that $X^s \subset X^h \subset X^i$. If $X \subset Y$, then $Y'' \subset X'', \text{ for } \mu = \alpha, \beta, \text{ or } \gamma$.

Let $p = (p_k)$ be a sequence of positive real numbers with $\sup_k p_k = G$ and $D = \max\{1, 2^{G-1}\}$. Then, it is well known that for all $a_k, b_k \in C$, the field of complex numbers, for all $k \in N$,

$$|a_k + b_k|^p \leq D(|a_k|^{p_k} + |b_k|^p). \quad (1.1)$$

Lemma 1.1 (Wilansky [1, Theorem 7.2.7]). Let $X$ be an FK-space $\supset \phi$. Then,

(i) $X^i \subset X^f$;
(ii) if $X$ has AK, $X^h = X^f$;
(iii) if $X$ has AD, $X^h = X^i$.

2. Definitions and Preliminaries

Let $\Delta : w \rightarrow w$ be the difference operator defined by $\Delta x = (x_k - x_{k+1})_{k=1}^{\infty}$. Let

$$\Gamma = \left\{ x \in w : \lim_{k \rightarrow \infty} (|x_k|^{1/k}) = 0 \right\},$$

$$\Lambda = \left\{ x \in w : \sup_k (|x_k|^{1/k}) < \infty \right\}. \quad (2.1)$$

Define the sets $\Gamma(\Delta) = \{x \in w : \Delta x \in \Gamma\}$ and $\Lambda(\Delta) = \{x \in w : \Delta x \in \Lambda\}$.

The spaces $\Gamma(\Delta)$ and $\Lambda(\Delta)$ are the metric spaces with the metric

$$d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left( |\Delta x_k - \Delta y_k|^{1/k} \right) \leq 1 \right\}. \quad (2.2)$$

Because of the historical roots of summability in convergence, conservative space and matrices play a special role in its theory. However, the results seem mainly to depend on a weaker assumption, that the spaces be semi-conservative. (See Wilansky [1]).


In a similar way, in this paper, we define semi-difference entire sequence space $cs \cap d_1$, and show that semi-difference entire sequence space $cs \cap d_1$ is $I \subset cs \cap d_1$ and $\Gamma(\Delta) \subset I$. 
3. Main Results

Proposition 3.1. $\Gamma \subset \Gamma(\Delta)$ and the inclusion is strict.

Proof. Let $x \in \Gamma$. Then, we have

$$|x_k|^{1/k} \to 0, \quad \text{as } k \to \infty,$$

(3.1)

$$\frac{|\Delta x_k|^{1/k}}{2} \leq \frac{1}{2} (|x_k|^{1/k}) + \frac{1}{2} (|x_{k+1}|^{1/k}), \quad \text{by (1.1)}$$

$$\to 0, \quad \text{as } k \to \infty \quad \text{by (3.1)}.$$  

(3.2)

Let $x \in \Gamma$. Then, we have

$$\left( |x_k|^{1/k} \right) \to 0 \quad \text{as } k \to \infty.$$  

(3.3)

Then, $(x_k) \in \Gamma(\Delta)$ follows from the inequality (1.1) and (3.3).

Consider the sequence $e = (1, 1, \ldots)$. Then, $e \in \Gamma(\Delta)$ but $e \not\in \Gamma$. Hence, the inclusion $\Gamma \subset \Gamma(\Delta)$ is strict. 

Lemma 3.2. $A \in (\Gamma, c)$ if and only if

$$\lim_{n \to \infty} a_{nk} \quad \text{exists for each } k \in N,$$

(3.4)

$$\sup_{n,k} \left| \sum_{i=0}^{k} a_{ni} \right| < \infty.$$  

(3.5)

Proposition 3.3. Define the set $d_1 = \{a = (a_k) \in w : \sup_{n,k \in \mathbb{N}} |\sum_{j=0}^{k} (\sum_{i=j}^{n} a_i)| < \infty \}$. Then, $[\Gamma(\Delta)]^p = cs \cap d_1$.

Proof. Consider the equation

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \left( \sum_{j=0}^{k} y_j \right) = \sum_{k=0}^{n} \left( \sum_{j=k}^{n} a_j \right) y_k = (Cy)_{n'},$$

(3.6)

where $C = (C_{nk})$ is defined by

$$C_{nk} = \begin{cases} \sum_{j=k}^{n} a_j, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n; \ n, k \in \mathbb{N}. \end{cases}$$

(3.7)
Thus, we deduce from Lemma 3.2 with (3.6) that \( ax = (a_kx_k) \in cs \) whenever \( x = (x_k) \in \Gamma(\Delta) \) if and only if \( Cy \in c \) whenever \( y = (y_k) \in \Gamma \), that is \( C \in (\Gamma, c) \). Thus, \( (a_k) \in cs \) and \( (a_k) \in d_1 \) by Lemma 3.2 and (3.5) and (3.6), respectively. This completes the proof.

**Proposition 3.4.** \( \Gamma(\Delta) \) has AK.

**Proof.** Let \( x = \{x_k\} \in \Gamma(\Delta) \). Then, \( (|\Delta x_k|^{1/k}) \in \Gamma \). Hence,

\[
\sup_{k \geq n+1} \left( |\Delta x_k|^{1/k} \right) \longrightarrow 0, \quad \text{as } k \longrightarrow \infty, \tag{3.8}
\]

\[
d(x, x^{[n]}) = \inf \left\{ \rho > 0 : \sup_{k \geq n+1} \left( |\Delta x_k|^{1/k} \right) \leq 1 \right\} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \quad \text{by using (3.8)}
\]

\[
\Rightarrow x^{[n]} \longrightarrow x \quad \text{as } n \longrightarrow \infty,
\]

\[
\Rightarrow \Gamma(\Delta) \text{ has AK.}
\]

This completes the proof. \( \square \)

**Proposition 3.5.** \( \Gamma(\Delta) \) is not solid.

To prove Proposition 3.5, consider \( (x_k) = (1) \in \Gamma(\Delta) \) and \( a_k = \{-1\}^k \}. Then \( (a_kx_k) \notin \Gamma(\Delta) \). Hence, \( \Gamma(\Delta) \) is not solid.

**Proposition 3.6.** \( (\Gamma(\Delta))^\mu = cs \cap d_1 \) for \( \mu = \alpha, \beta, \gamma, f \).

**Proof.**

**Step 1.** \( \Gamma(\Delta) \) has AK by Proposition 3.4. Hence, by Lemma 1.1(ii), we get \( (\Gamma(\Delta))^\beta = (\Gamma(\Delta))^f \). But \( (\Gamma(\Delta))^\beta = cs \cap d_1 \). Hence,

\[
(\Gamma(\Delta))^f = cs \cap d_1. \tag{3.10}
\]

**Step 2.** Since AK \( \Rightarrow \) AD. Hence, by Lemma 1.1(iii), we get \( (\Gamma(\Delta))^\beta = (\Gamma(\Delta))^f \). Therefore,

\[
(\Gamma(\Delta))^f = cs \cap d_1. \tag{3.11}
\]

**Step 3.** \( \Gamma(\Delta) \) is not normal by Proposition 3.5. Hence by Proposition 2.7 of Kamthan and Gupta [6], we get

\[
(\Gamma(\Delta))^\alpha \neq (\Gamma(\Delta))^f \neq cs \cap d_1. \tag{3.12}
\]

From (3.10) and (3.11), we have

\[
(\Gamma(\Delta))^\beta = (\Gamma(\Delta))^f = (\Gamma(\Delta))^f = cs \cap d_1. \tag{3.13}
\]

\( \square \)
Lemma 3.7 (Wilansky [1, Theorem 8.6.1]). \( Y \supset X \Leftrightarrow Y^f \subset X^f \) where \( X \) is an AD-space and \( Y \) an FK-space.

Proposition 3.8. Let \( Y \) be any FK-space \( \supset \phi \). Then, \( Y \supset \Gamma(\Delta) \) if and only if the sequence \( \delta^{(k)} \) is weakly converges in \( cs \cap d_1 \).

Proof. The following implications establish the result.
\[
Y \supset \Gamma(\Delta) \Leftrightarrow Y^f \subset (\Gamma(\Delta))^f, \text{ since } \Gamma(\Delta) \text{ has AD by Lemma 3.7.}
\]
\[
\Leftrightarrow Y^f \subset cs \cap d_1, \text{ since } (\Gamma(\Delta))^f = cs \cap d_1.
\]
\[
\Leftrightarrow \text{for each } f \in Y^{'}, \text{ the topological dual of } Y.
\]
\[
\Leftrightarrow f(\delta^{(k)}) \in cs \cap d_1.
\]
\[
\Leftrightarrow \delta^{(k)} \text{ is weakly converges in } cs \cap d_1.
\]

This completes the proof. \( \square \)

4. Properties of Semi-Difference Entire Sequence Space \( cs \cap d_1 \)

Definition 4.1. An FK-space \( \Delta X \) is called “semi-difference entire sequence space \( cs \cap d_1 \)” if its dual \( (\Delta X)^f \subset cs \cap d_1 \).

In other words \( \Delta X \) is semi-difference entire sequence space \( cs \cap d_1 \) if \( f(\delta^{(k)}) \in cs \cap d_1 \) for all \( f \in (\Delta X)^f \) for each fixed \( k \).

Example 4.2. \( \Gamma(\Delta) \) is semi-difference entire sequence space \( cs \cap d_1 \). Indeed, if \( \Gamma(\Delta) \) is the space of all difference of entire sequences, then by Lemma 4.3, \( (\Gamma(\Delta))^f = cs \cap d_1 \).

Lemma 4.3 (Wilansky [1, Theorem 4.3.7]). Let \( z \) be a sequence. Then \( (z^\phi, P) \) is an AK space with \( P = (P_k : k = 0, 1, 2, \ldots) \), where \( P_0(x) = \sup_n |\sum_{k=1}^n z_k x_k| \), and \( P_n(x) = |x_n| \). For any \( k \) such that \( z_k \neq 0, P_k \) may be omitted. If \( z \in \phi, P_n \) may be omitted.

Proposition 4.4. Let \( z \) be a sequence. \( z^\phi \) is a semi-difference entire sequence space \( cs \cap d_1 \) if and only if \( z \) is in \( cs \cap d_1 \).

Proof. Suppose that \( z^\phi \) is a semi-difference entire sequence space \( cs \cap d_1 \). \( z^\phi \) has AK by Lemma 4.3. Therefore \( z^\phi^\phi = (z^\phi)^f \) by Lemma 1 [1]. So \( z^\phi \) is semi-difference entire sequence space \( cs \cap d_1 \) if and only if \( z^\phi^\phi \subset cs \cap d_1 \). But then \( z \in z^\phi^\phi \subset cs \cap d_1 \). Hence, \( z \) is in \( cs \cap d_1 \).

Conversely, suppose that \( z \) is in \( cs \cap d_1 \). Then \( z^\phi \supset \{ cs \cap d_1 \}^\phi \) and \( z^\phi^\phi \subset \{ cs \cap d_1 \}^\phi^\phi = cs \cap d_1 \). But \( (z^\phi)^f = z^\phi^\phi \). Hence, \( (z^\phi)^f \subset cs \cap d_1 \). Therefore \( z^\phi \) is semi-difference entire sequence space \( cs \cap d_1 \). This completes the proof. \( \square \)

Proposition 4.5. Every semi-difference entire sequence space \( cs \cap d_1 \) contains \( \Gamma \).

Proof. Let \( \Delta X \) be any semi-difference entire sequence space \( cs \cap d_1 \). Hence, \( (\Delta X)^f \subset cs \cap d_1 \). Therefore \( f(\delta^{(k)}) \in cs \cap d_1 \) for all \( f \in (\Delta X)^f \). So, \( \{ \delta^{(k)} \} \) is weakly converges in \( cs \cap d_1 \) with respect to \( \Delta X \). Hence, \( \Delta X \supset \Gamma(\Delta) \) by Proposition 3.8. But \( \Gamma(\Delta) \supset \Gamma \). Hence, \( \Delta X \supset \Gamma \). This completes the proof. \( \square \)

Proposition 4.6. \( \Delta X \) is semi-difference entire sequence space \( cs \cap d_1 \).
Proof. Let $\Delta X = \bigcap_{n=1}^{\infty} \Delta X_n$. Then $\Delta X$ is an FK-space which contains $\phi$. Also every $f \in (\Delta X)'$ can be written as $f = g_1 + g_2 + \ldots + g_m$, where $g_k \in (\Delta X_n)'$ for some $n$ and for $1 \leq k \leq m$. But then $f(\delta^k) = g_1(\delta^k) + g_2(\delta^k) + \ldots + g_m(\delta^k)$. Since $\Delta X_n (n = 1, 2, \ldots)$ are semi-difference entire sequence space $cs \cap d_1$, it follows that $g_i(\delta^k) \in cs \cap d_1$ for all $i = 1, 2, \ldots m$. Therefore $f(\delta^k) \in cs \cap d_1$ for all $k$ and for all $f$. Hence, $\Delta X$ is semi-difference entire sequence space $cs \cap d_1$. This completes the proof.

Proposition 4.7. The intersection of all semi-difference entire sequence space $cs \cap d_1$ is $I \subseteq (cs \cap d_1)^{\phi}$ and $\Gamma(\Delta) \subseteq I$.

Proof. Let $I$ be the intersection of all semi-difference entire sequence space $cs \cap d_1$. By Proposition 4.4, we see that the intersection

$$I \subseteq \bigcap \{z^\phi : z \in cs \cap d_1\} = (cs \cap d_1)^{\phi}. \tag{4.1}$$

By Proposition 4.6 it follows that $I$ is semi-difference entire sequence space $cs \cap d_1$. By Proposition 4.5, consequently

$$\Gamma_M = \Gamma(\Delta) \subseteq I. \tag{4.2}$$

From (4.1) and (4.2), we get $I \subseteq (cs \cap d_1)^{\phi}$ and $\Gamma(\Delta) \subseteq I$. This completes the proof.

Corollary 4.8. The smallest semi-difference entire sequence space $cs \cap d_1$ is $I \subseteq (cs \cap d_1)^{\phi}$ and $\Gamma(\Delta) \subseteq I$.

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References

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