Research Article

Fine Spectra of Tridiagonal Symmetric Matrices

Muhammed Altun

Faculty of Arts and Sciences, Adıyaman University, 02040 Adıyaman, Turkey

Correspondence should be addressed to Muhammed Altun, muhammedaltun@gmail.com

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The fine spectra of upper and lower triangular banded matrices were examined by several authors. Here we determine the fine spectra of tridiagonal symmetric infinite matrices and also give the explicit form of the resolvent operator for the sequence spaces c_0 , c , ℓ_1 , and ℓ_∞ .

1. Introduction

The spectrum of an operator is a generalization of the notion of eigenvalues for matrices. The spectrum over a Banach space is partitioned into three parts, which are the point spectrum, the continuous spectrum, and the residual spectrum. The calculation of these three parts of the spectrum of an operator is called the fine spectrum of the operator.

The spectrum and fine spectrum of linear operators defined by some particular limitation matrices over some sequence spaces was studied by several authors. We introduce the knowledge in the existing literature concerning the spectrum and the fine spectrum. Wenger [1] examined the fine spectrum of the integer power of the Cesàro operator over c and, Rhoades [2] generalized this result to the weighted mean methods. Reade [3] worked on the spectrum of the Cesàro operator over the sequence space c_0 . González [4] studied the fine spectrum of the Cesàro operator over the sequence space ℓ_p . Okutoyi [5] computed the spectrum of the Cesàro operator over the sequence space bv . Recently, Rhoades and Yildirim [6] examined the fine spectrum of factorable matrices over c_0 and c . Coşkun [7] studied the spectrum and fine spectrum for the p -Cesàro operator acting over the space c_0 . Akhmedov and Başar [8, 9] have determined the fine spectrum of the Cesàro operator over the sequence spaces c_0 , ℓ_∞ , and ℓ_p . In a recent paper, Furkan, et al. [10] determined the fine spectrum of $B(r, s, t)$ over the sequence spaces c_0 and c , where $B(r, s, t)$ is a lower triangular triple-band matrix. Later, Altun and Karakaya [11] computed the fine spectra for Lacunary matrices over c_0 and c .

In this work, our purpose is to determine the fine spectra of the operator, for which the corresponding matrix is a tridiagonal symmetric matrix, over the sequence spaces c_0 , c , ℓ_1 , and ℓ_∞ . Also we will give the explicit form of the resolvent for this operator and compute the norm of the resolvent operator when it exists and is continuous.

Let X and Y be Banach spaces and $T : X \rightarrow Y$ be a bounded linear operator. By $\mathcal{R}(T)$, we denote the range of T , that is,

$$\mathcal{R}(T) = \{y \in Y : y = Tx; x \in X\}. \quad (1.1)$$

By $B(X)$, we denote the set of all bounded linear operators on X into itself. If X is any Banach space, and let $T \in B(X)$ then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$. Let $X \neq \{\theta\}$ be a complex normed space and $T : \mathfrak{D}(T) \rightarrow X$ be a linear operator with domain $\mathfrak{D}(T) \subset X$. With T , we associate the operator

$$T_\lambda = T - \lambda I, \quad (1.2)$$

where λ is a complex number and I is the identity operator on $\mathfrak{D}(T)$. If T_λ has an inverse, which is linear, we denote it by T_λ^{-1} , that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1} \quad (1.3)$$

and call it the *resolvent operator* of T_λ . If $\lambda = 0$, we will simply write T^{-1} . Many properties of T_λ and T_λ^{-1} depend on λ , and spectral theory is concerned with those properties. For instance, we will be interested in the set of all λ in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ 's the domain of T_λ^{-1} is dense in X . For our investigation of T , T_λ , and T_λ^{-1} , we need some basic concepts in spectral theory which are given as follows (see [12, pages 370-371]).

Let $X \neq \{\theta\}$ be a complex normed space, and let $T : \mathfrak{D}(T) \rightarrow X$ be a linear operator with domain $\mathfrak{D}(T) \subset X$. A *regular value* λ of T is a complex number such that

- (R1) T_λ^{-1} exists,
- (R2) T_λ^{-1} is bounded, and
- (R3) T_λ^{-1} is defined on a set which is dense in X .

The *resolvent set* $\rho(T)$ of T is the set of all regular values λ of T . Its complement $\sigma(T) = \mathbb{C} \setminus \rho(T)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T . Furthermore, the spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows: the *point spectrum* $\sigma_p(T)$ is the set such that T_λ^{-1} does not exist. A $\lambda \in \sigma_p(T)$ is called an *eigenvalue* of T . The *continuous spectrum* $\sigma_c(T)$ is the set such that T_λ^{-1} exists and satisfies (R3) but not (R2). The *residual spectrum* $\sigma_r(T)$ is the set such that T_λ^{-1} exists but does not satisfy (R3).

A triangle is a lower triangular matrix with all of the principal diagonal elements nonzero. We shall write ℓ_∞ , c , and c_0 for the spaces of all bounded, convergent, and null sequences, respectively. And by ℓ_p , we denote the space of all p -absolutely summable sequences, where $1 \leq p < \infty$. Let μ and γ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk} , where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from μ into γ , and we denote it by writing $A : \mu \rightarrow \gamma$, if for every sequence

$x = (x_k) \in \mu$ the sequence $Ax = \{(Ax)_n\}$, the A -transform of x , is in γ , where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \quad (1.4)$$

By $(\mu : \gamma)$, we denote the class of all matrices A such that $A : \mu \rightarrow \gamma$. Thus, $A \in (\mu : \gamma)$ if and only if the series on the right side of (1.4) converges for each $n \in \mathbb{N}$ and every $x \in \mu$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \gamma$ for all $x \in \mu$.

A tridiagonal symmetric infinite matrix is of the form

$$S = S(q, r) = \begin{bmatrix} q & r & 0 & 0 & 0 & 0 & \cdots \\ r & q & r & 0 & 0 & 0 & \cdots \\ 0 & r & q & r & 0 & 0 & \cdots \\ 0 & 0 & r & q & r & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (1.5)$$

where $q, r \in \mathbb{C}$. The spectral results are clear when $r = 0$, so for the sequel we will have $r \neq 0$.

Theorem 1.1 (cf. [13]). *Let T be an operator with the associated matrix $A = (a_{nk})$.*

(i) $T \in B(c)$ if and only if

$$\|A\| := \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty, \quad (1.6)$$

$$a_k := \lim_{n \rightarrow \infty} a_{nk} \text{ exists for each } k, \quad (1.7)$$

$$a := \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} \text{ exists.} \quad (1.8)$$

(ii) $T \in B(c_0)$ if and only if (1.6) and (1.7) with $a_k = 0$ for each k .

(iii) $T \in B(\ell_\infty)$ if and only if (1.6).

In these cases, the operator norm of T is

$$\|T\|_{(\ell_\infty: \ell_\infty)} = \|T\|_{(c:c)} = \|T\|_{(c_0:c_0)} = \|A\|. \quad (1.9)$$

(iv) $T \in B(\ell_1)$ if and only if

$$\|A^t\| = \sup_k \sum_{n=1}^{\infty} |a_{nk}| < \infty. \quad (1.10)$$

In this case, the operator norm of T is $\|T\|_{(\ell_1: \ell_1)} = \|A^t\|$.

Corollary 1.2. *Let $\mu \in \{c_0, c, \ell_1, \ell_\infty\}$. $S(q, r) : \mu \rightarrow \mu$ is a bounded linear operator and $\|S(q, r)\|_{(\mu: \mu)} = |q| + 2|r|$.*

2. The Spectra and Point Spectra

Theorem 2.1. $\sigma_p(S, \mu) = \emptyset$ for $\mu \in \{\ell_1, c_0, c\}$.

Proof. Since $\ell_1 \subset c_0 \subset c$, it is enough to show that $\sigma_p(S, c) = \emptyset$. Let λ be an eigenvalue of the operator S . An eigenvector $x = (x_0, x_1, \dots) \in c$ corresponding to this eigenvalue satisfies the linear system of equations:

$$\begin{aligned} qx_0 + rx_1 &= \lambda x_0 \\ rx_0 + qx_1 + rx_2 &= \lambda x_1 \\ rx_1 + qx_2 + rx_3 &= \lambda x_2 \\ &\vdots \end{aligned} \tag{2.1}$$

If $x_0 = 0$, then $x_k = 0$ for all $k \in \mathbb{N}$. Hence $x_0 \neq 0$. Without loss of generality we can suppose $x_0 = 1$. Then $x_1 = (\lambda - q)/r$ and the system of equations turn into the linear homogeneous recurrence relation

$$x_n + px_{n-1} + x_{n-2} = 0 \quad \text{for } n \geq 2, \tag{2.2}$$

where $p = (q - \lambda)/r$. The characteristic polynomial of the recurrence relation is

$$x^2 + px + 1 = 0. \tag{2.3}$$

There are three cases here.

Case 1 ($p = -2$). Then characteristic polynomial has only one root: $\alpha = 1$. Hence, the solution of the recurrence relation is of the form

$$x_n = (A + Bn)(\alpha)^n = A + Bn, \tag{2.4}$$

where A and B are constants which can be determined by the first two terms x_0 and x_1 . $1 = x_0 = A + B0$, so we have $A = 1$. And $-p = x_1 = A + B1$, so we have $B = 1$. Then $x_n = n + 1$. This means $(x_n) \notin c$. So, we conclude that there is no eigenvalue in this case.

Case 2 ($p = 2$). Then characteristic polynomial has only one root: $\alpha = -1$. The solution of the recurrence relation, found as in Case 1, is $x_n = (n + 1)(-1)^n$. So, there is no eigenvalue in this case.

Case 3 ($p \neq \pm 2$). Then the characteristic polynomial has two distinct roots $\alpha_1 \neq \pm 1$ and $\alpha_2 \neq \pm 1$ with $\alpha_1 \alpha_2 = 1$. Let $|\alpha_1| \geq 1 \geq |\alpha_2|$. The solution of the recurrence relation is of the form

$$x_n = A(\alpha_1)^n + B(\alpha_2)^n. \tag{2.5}$$

Using the first two terms and the fact that $p = -(\alpha_1 + \alpha_2)$, we get $A = \alpha_1/(\alpha_1 - \alpha_2)$ and $B = \alpha_2/(\alpha_2 - \alpha_1)$. So we have

$$x_n = \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}. \quad (2.6)$$

If $|\alpha_1| > 1 > |\alpha_2|$, then

$$|x_n| \geq \frac{1}{|\alpha_1 - \alpha_2|} (|\alpha_1|^{n+1} - |\alpha_2|^{n+1}). \quad (2.7)$$

So $\lim_n |x_n| = \infty$, which means $(x_n) \notin c$. Now, if $|\alpha_1| = |\alpha_2| = 1$, then there exists $\theta \in (0, \pi)$ such that $\alpha_1 = e^{i\theta}$ and $\alpha_2 = e^{-i\theta}$. So, $x_n = [\sin(n+1)\theta] / \sin \theta$. Again we have $(x_n) \notin c$. Hence there is no eigenvalue also in this case. $\square$

Repeating all the steps in the proof of this theorem for ℓ_∞ , we get to the following.

Theorem 2.2. $\sigma_p(S, \ell_\infty) = (q - 2r, q + 2r)$.

Theorem 2.3. Let $p = (q - \lambda)/r$. Let α_1 and α_2 be the roots of the polynomial $P(x) = x^2 + px + 1$, with $|\alpha_2| > 1 > |\alpha_1|$. Then the resolvent operator over c_0 is $S_\lambda^{-1} = (s_{nk})$, where

$$s_{nk} = \frac{1}{r(\alpha_1^2 - 1)} \cdot \begin{cases} \alpha_1^{n-k+1} - \alpha_1^{n+k+3} & \text{if } n \geq k \\ \alpha_1^{-n+k+1} - \alpha_1^{n+k+3} & \text{if } n < k. \end{cases} \quad (2.8)$$

Moreover, this operator is continuous and the domain of the operator is the whole space c_0 .

Proof. Let α_1 and α_2 be as it is stated in the theorem. From $(1/r)S_\lambda x = y$ we get to the system of equations:

$$\begin{aligned} px_0 + x_1 &= y_0 \\ x_0 + px_1 + x_2 &= y_1 \\ x_1 + px_2 + x_3 &= y_2 \\ &\vdots \end{aligned} \quad (2.9)$$

This is a nonhomogenous linear recurrence relation. Using the fact that $(x_n), (y_n) \in c_0$, for (2.9) we can reach a solution with generating functions. This solution can be given by

$$x_n = \frac{1}{\alpha_1^2 - 1} \sum_{k=0}^{\infty} t_{nk} y_k, \quad (2.10)$$

where

$$t_{nk} = \begin{cases} \alpha_1^{n+1-k} - \alpha_1^{n+3+k} & \text{if } n \geq k \\ \alpha_1^{k+1-n} - \alpha_1^{k+3+n} & \text{if } n < k. \end{cases} \quad (2.11)$$

Let $T = (t_{nk})$. We can see that by using Theorem 1.1, $T \in B(c_0)$. So $(1/\alpha_1^2 - 1)T$ is the resolvent operator of $(1/r)S_\lambda$ and is continuous. $\square$

If $T : \mu \rightarrow \mu$ (μ is ℓ_1 or c_0) is a bounded linear operator represented by the matrix A , then it is known that the adjoint operator $T^* : \mu^* \rightarrow \mu^*$ is defined by the transpose A^t of the matrix A . It should be noted that the dual space c_0^* of c_0 is isometrically isomorphic to the Banach space ℓ_1 and the dual space ℓ_1^* of ℓ_1 is isometrically isomorphic to the Banach space ℓ_∞ .

Corollary 2.4. $\sigma(S, \mu) \subset [q - 2r, q + 2r]$ for $\mu \in \{\ell_1, c_0, c, \ell_\infty\}$.

Proof. $\sigma(S, c_0) = \sigma(S^*, c_0^*) = \sigma(S, \ell_1) = \sigma(S^*, \ell_1^*) = \sigma(S, \ell_\infty)$. And by Cartlidge [14], if a matrix operator A is bounded on c , then $\sigma(A, c) = \sigma(A, \ell_\infty)$. Hence we have $\sigma(S, c_0) = \sigma(S, \ell_1) = \sigma(S, \ell_\infty) = \sigma(S, c)$. What remains is to show that $\sigma(S, c_0) \subset [q - 2r, q + 2r]$. By Theorem 2.3, there exists a resolvent operator of S_λ which is continuous and the whole space c_0 is the domain if the roots of the polynomial $P(x) = x^2 + px + 1$ satisfy

$$|\alpha_2| > 1 > |\alpha_1|. \quad (2.12)$$

So, if $\lambda \in \sigma(S, c_0)$ then (2.12) is not satisfied. Since $\alpha_1 \alpha_2 = 1$, (2.12) is not satisfied means, the roots can be only of the form

$$\alpha_1 = \frac{1}{\alpha_2} = e^{i\theta} \quad (2.13)$$

for some $\theta \in [0, 2\pi)$. Then $(q - \lambda)/r = p = -(\alpha_1 + \alpha_2) = -(e^{i\theta} + e^{-i\theta}) = -2 \cos \theta$. Hence $\lambda = q + 2r \cos \theta$, which means λ can be only on the line segment $[q - 2r, q + 2r]$. $\square$

Theorem 2.5. $\sigma(S, \mu) = [q - 2r, q + 2r]$ for $\mu \in \{\ell_1, c_0, c, \ell_\infty\}$.

Proof. By Theorem 2.2 and Corollary 2.4 $(q - 2r, q + 2r) \subset \sigma(S, \ell_\infty) \subset [q - 2r, q + 2r]$. Since the spectrum of a bounded linear operator over a complex Banach space is closed, we have $\sigma(S, \ell_\infty) = [q - 2r, q + 2r]$. And from the proof of Corollary 2.4 we have $\sigma(S, \ell_1) = \sigma(S, c_0) = \sigma(S, c) = \sigma(S, \ell_\infty)$. $\square$

3. The Continuous Spectra and Residual Spectra

Lemma 3.1 (see [15, page 59]). T has a dense range if and only if T^* is one to one.

Corollary 3.2. If $T \in (\mu : \mu)$ then $\sigma_r(T, \mu) = \sigma_p(T^*, \mu^*) \setminus \sigma_p(T, \mu)$.

Theorem 3.3. $\sigma_r(S, c_0) = \emptyset$.

Proof. $\sigma_p(S, \ell_1) = \emptyset$ by Theorem 2.1. Now using Corollary 3.2, we have $\sigma_r(S, c_0) = \sigma_p(S^*, c_0^*) \setminus \sigma_p(S, c_0) = \sigma_p(S, \ell_1) \setminus \sigma_p(S, c_0) = \emptyset$. $\square$

Theorem 3.4. $\sigma_r(S, \ell_1) = (q - 2r, q + 2r)$.

Proof. Similarly as in the proof of the previous theorem, we have $\sigma_r(S, \ell_1) = \sigma_p(S^*, \ell_1^*) \setminus \sigma_p(S, \ell_1) = \sigma_p(S, \ell_\infty) \setminus \sigma_p(S, \ell_1) = (q - 2r, q + 2r)$. $\square$

If $T : c \rightarrow c$ is a bounded matrix operator represented by the matrix A , then $T^* : c^* \rightarrow c^*$ acting on $\mathbb{C} \oplus \ell_1$ has a matrix representation of the form

$$\begin{bmatrix} \chi & 0 \\ b & A^t \end{bmatrix}, \quad (3.1)$$

where χ is the limit of the sequence of row sums of A minus the sum of the limits of the columns of A , and b is the column vector whose k th entry is the limit of the k th column of A for each $k \in \mathbb{N}$. For $S_\lambda : c \rightarrow c$, the matrix S_λ^* is of the form

$$\begin{bmatrix} 2r + q - \lambda & 0 \\ 0 & S_\lambda \end{bmatrix}. \quad (3.2)$$

Theorem 3.5. $\sigma_r(S, c) = \{q + 2r\}$.

Proof. Let $x = (x_0, x_1, \dots) \in \mathbb{C} \oplus \ell_1$ be an eigenvector of S^* corresponding to the eigenvalue λ . Then we have $(2r + q)x_0 = \lambda x_0$ and $Sx' = \lambda x'$ where $x' = (x_1, x_2, \dots)$. By Theorem 2.1, $x' = (0, 0, \dots)$. Then $x_0 \neq 0$. And $\lambda = 2r + q$ is the only value that satisfies $(2r + q)x_0 = \lambda x_0$. Hence $\sigma_p(S^*, c^*) = \{2r + q\}$. Then $\sigma_r(S, c) = \sigma_p(S^*, c^*) \setminus \sigma_p(S, c) = \{2r + q\}$. $\square$

Now, since the spectrum σ is the disjoint union of σ_p , σ_r , and σ_c , we can find σ_c over the spaces ℓ_1 , c_0 , and c . So we have the following.

Theorem 3.6. For the operator S , one has the following:

$$\begin{aligned} \sigma_c(S, \ell_1) &= \{q - 2r, q + 2r\}, \\ \sigma_c(S, c_0) &= [q - 2r, q + 2r], \\ \sigma_c(S, c) &= [q - 2r, q + 2r]. \end{aligned} \quad (3.3)$$

4. The Resolvent Operator

The following theorem is a generalization of Theorem 2.3.

Theorem 4.1. Let $\mu \in \{c_0, c, \ell_1, \ell_\infty\}$. The resolvent operator S^{-1} over μ exists and is continuous, and the domain of S^{-1} is the whole space μ if and only if $0 \notin [q - 2r, q + 2r]$. In this case, S^{-1} has a matrix representation (s_{nk}) defined by

$$s_{nk} = \frac{1}{r(\alpha_1^2 - 1)} \cdot \begin{cases} \alpha_1^{n-k+1} - \alpha_1^{n+k+3} & \text{if } n \geq k \\ \alpha_1^{-n+k+1} - \alpha_1^{n+k+3} & \text{if } n < k, \end{cases} \quad (4.1)$$

where α_1 is the root of the polynomial $P(x) = rx^2 + qx + r$ with $|\alpha_1| < 1$.

Proof. Let μ be one of the sequence spaces in $\{c_0, c, \ell_1, \ell_\infty\}$. Suppose S has a continuous resolvent operator where the domain of the resolvent operator is the whole space μ . Then $\lambda = 0$ is not in $\sigma(S, \mu) = [q - 2r, q + 2r]$. Conversely if $0 \notin [q - 2r, q + 2r]$, then S has a continuous resolvent operator, and since S is bounded by Lemma 7.2-7.3 of [12] the domain of this resolvent operator is the whole space μ .

Now, suppose $0 \notin [q - 2r, q + 2r]$. Let α_1 and α_2 be the roots of the polynomial $P(x) = rx^2 + qx + r$ where $|\alpha_1| \leq |\alpha_2|$. Since $0 \notin [q - 2r, q + 2r]$, by the proof of Corollary 2.4 $|\alpha_1| \neq |\alpha_2|$. Then $|\alpha_1| < 1 < |\alpha_2|$. So S satisfies the conditions of Theorem 2.3. Hence the resolvent operator of S is represented by the matrix $S^{-1} = (s_{nk})$ defined by

$$s_{nk} = \frac{1}{r(\alpha_1^2 - 1)} \cdot \begin{cases} \alpha_1^{n-k+1} - \alpha_1^{n+k+3} & \text{if } n \geq k \\ \alpha_1^{-n+k+1} - \alpha_1^{n+k+3} & \text{if } n < k, \end{cases} \quad (4.2)$$

when $\mu = c_0$ by that theorem. The matrix S^{-1} is already a left inverse of the matrix S . Observe that S^{-1} satisfies also the corresponding conditions of Theorem 1.1, which means $S^{-1} \in (\mu, \mu)$ for $\mu \in \{c, \ell_1, \ell_\infty\}$. So, the matrix S^{-1} is the representation of the resolvent operator also for the spaces in $\{c, \ell_1, \ell_\infty\}$. $\square$

Remark 4.2. If a matrix A is a triangle, we can see that the resolvent (when it exists) is the unique lower triangular left hand inverse of A . In our case, S is far away from being a triangle. The matrix S^{-1} of this theorem is not the unique left inverse of the matrix S for $0 \notin [q - 2r, q + 2r]$. For example, the matrix $T = (t_{nk})$ defined by

$$t_{nk} = \frac{1}{r(\alpha_1^2 - 1)} \cdot \begin{cases} \alpha_1^{-n+k+1} - \alpha_1^{n-k+1} & \text{if } n < k \\ 0 & \text{if } n \geq k \end{cases} \quad (4.3)$$

is another left inverse of S . Then $\lambda S^{-1} + (1 - \lambda)T$ is also a left inverse of S for any $\lambda \in \mathbb{C}$, which means there exist infinitely many left inverses for S .

Theorem 4.3. Let $0 \notin [q - 2r, q + 2r]$, and, α_1 be the root of $P(x) = rx^2 + qx + r$ with $|\alpha_1| < 1$. Then for $\mu \in \{c_0, c, \ell_1, \ell_\infty\}$ we have

$$\|S^{-1}\|_{(\mu, \mu)} = \frac{|\alpha_1| + |\alpha_1|^2}{|r||1 - \alpha_1^2|(1 - |\alpha_1|)}. \quad (4.4)$$

Proof. Since S^{-1} is a symmetric matrix, the supremum of the ℓ_1 norms of the rows is equal to the supremum of the ℓ_1 norms of the columns. So, according to Theorem 1.1, what we need is to calculate the supremum of the ℓ_1 norms of the rows of S^{-1} . Denote the n th row S^{-1} by S_n^{-1} for $n = 0, 1, \dots$. Now, let us fix the row n and calculate the ℓ_1 norm for this row. Let $\rho = 1/|r(1 - \alpha_1^2)|$. By using Theorem 4.1, we have

$$\begin{aligned}
 \|S_n^{-1}\|_{\ell_1} &= \rho \left(\sum_{k=0}^n |\alpha_1^{n-k+1} - \alpha_1^{n+k+3}| + \sum_{k=n+1}^{\infty} |\alpha_1^{-n+k+1} - \alpha_1^{n+k+3}| \right) \\
 &\leq \rho \left(\sum_{k=0}^n (|\alpha_1|^{n-k+1} + |\alpha_1|^{n+k+3}) + \sum_{k=n+1}^{\infty} (|\alpha_1|^{-n+k+1} + |\alpha_1|^{n+k+3}) \right) \\
 &= \rho \left(\sum_{k=0}^n |\alpha_1|^{n-k+1} + \sum_{k=n+1}^{\infty} |\alpha_1|^{-n+k+1} + \sum_{k=0}^{\infty} |\alpha_1|^{n+k+3} \right) \\
 &= \rho \left(\sum_{k=1}^{n+1} |\alpha_1|^k + \sum_{k=2}^{\infty} |\alpha_1|^k + \sum_{k=n+3}^{\infty} |\alpha_1|^k \right) \\
 &= \rho \left(2 \sum_{k=0}^{\infty} |\alpha_1|^k - 2 - |\alpha_1| - |\alpha_1|^{n+2} \right) \\
 &\leq \rho \left(2 \sum_{k=0}^{\infty} |\alpha_1|^k - 2 - |\alpha_1| \right) = \frac{|\alpha_1| + |\alpha_1|^2}{|r| |1 - \alpha_1^2| (1 - |\alpha_1|)}.
 \end{aligned} \tag{4.5}$$

Hence

$$\|S^{-1}\|_{(\mu;\mu)} = \sup_n \|S_n^{-1}\|_{\ell_1} \leq \frac{|\alpha_1| + |\alpha_1|^2}{|r| |1 - \alpha_1^2| (1 - |\alpha_1|)}. \tag{4.6}$$

On the other hand

$$\begin{aligned}
 \|S_n^{-1}\|_{\ell_1} &= \rho \left(\sum_{k=0}^n |\alpha_1^{n-k+1} - \alpha_1^{n+k+3}| + \sum_{k=n+1}^{\infty} |\alpha_1^{-n+k+1} - \alpha_1^{n+k+3}| \right) \\
 &\geq \rho \left(\sum_{k=0}^n (|\alpha_1|^{n-k+1} - |\alpha_1|^{n+k+3}) + \sum_{k=n+1}^{\infty} (|\alpha_1|^{-n+k+1} - |\alpha_1|^{n+k+3}) \right) \\
 &= \rho \left(\sum_{k=0}^n |\alpha_1|^{n-k+1} + \sum_{k=n+1}^{\infty} |\alpha_1|^{-n+k+1} - \sum_{k=0}^{\infty} |\alpha_1|^{n+k+3} \right) \\
 &= \rho \left(\sum_{k=1}^{n+1} |\alpha_1|^k + \sum_{k=2}^{\infty} |\alpha_1|^k - \sum_{k=n+3}^{\infty} |\alpha_1|^k \right).
 \end{aligned} \tag{4.7}$$

Then

$$\|S^{-1}\|_{(\mu;\mu)} = \sup_n \|S_n^{-1}\|_{\ell_1} \geq \lim_n \|S_n^{-1}\|_{\ell_1} = \rho \left(\sum_{k=1}^{\infty} |\alpha_1|^k + \sum_{k=2}^{\infty} |\alpha_1|^k \right) = \frac{|\alpha_1| + |\alpha_1|^2}{|r| |1 - \alpha_1^2| (1 - |\alpha_1|)}. \tag{4.8}$$

□

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