Research Article

A Construction of Mirror $Q$-Algebras

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We investigate how to construct mirror $Q$-algebras of a $Q$-algebra, and we obtain the necessary conditions for $M(X)$ to be a $Q$-algebra.

1. Introduction

Imai and Iséki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras [1, 2]. It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. We refer the reader for useful textbooks for $BCK/BCI$-algebra to [3–5]. Neggers et al. [6] introduced the notion of $Q$-algebras which is a generalization of $BCK/BCI/BCH$-algebras, obtained several properties, and discussed quadratic $Q$-algebras. Ahn and Kim [7] introduced the notion of $QS$-algebras, and Ahn et al. [8] studied positive implicativity in $Q$-algebras and discussed some relations between $R$–($L$–) maps and positive implicativity. Neggers and Kim introduced the notion of $d$-algebras which is another useful generalization of $BCK$-algebras and then investigated several relations between $d$-algebras and $BCK$-algebras as well as several other relations between $d$-algebras and oriented digraphs [9]. After that some further aspects were studied [10–13]. Allen et al. [14] introduced the notion of mirror image of given algebras and obtained some interesting properties: a mirror algebra of a $d$-algebra is also a $d$-algebra, and a mirror algebra of an implicative $BCK$-algebra is a left $L$-up algebra.

In this paper we introduce the notion of mirror algebras to $Q$-algebras, and we investigate how to construct mirror $Q$-algebras from a $Q$-algebra; and we also obtain the necessary conditions for $M(X)$ to be a $Q$-algebra.
2. Q-Algebras and Related Algebras

A Q-algebra [6] is a nonempty set \( X \) with a constant 0 and a binary operation \( "\ast" \) satisfying the following axioms:

(I) \( x \ast x = 0 \),
(II) \( 0 \ast x = 0 \),
(III) \( (x \ast y) \ast z = (x \ast z) \ast y \) for all \( x, y, z \in X \).

For brevity we also call \( X \) a Q-algebra. In \( X \) we can define a binary relation \( "\leq" \) by \( x \leq y \) if and only if \( x \ast y = 0 \).

Example 2.1 (see [6]). Let \( X : \{0,1,2,3\} \) be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0</td>
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<td>0</td>
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<tr>
<td>1</td>
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<td>0</td>
<td>3</td>
<td>2</td>
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<td>2</td>
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<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \( (X, \ast, 0) \) is a Q-algebra, which is not a BCK/BCI/BCH-algebra.

Ahn and Kim [7] introduced the notion of QS-algebras. They showed that the G-part of an associative QS-algebra is a group in which every element is an involution. A Q-algebra \( X \) is said to be a QS-algebra if it satisfies the following condition:

(IV) \( (x \ast y) \ast (x \ast z) = z \ast y \), for all \( x, y, z \in X \).

Proposition 2.2 (see [6]). If \( (X, \ast, 0) \) is a Q-algebra, then

(V) \( x \ast (x \ast y) \ast y = 0 \), for all \( x, y \in X \).

It was proved that every BCH-algebra is a Q-algebra and every Q-algebra satisfying some additional conditions is a BCI-algebra.

Neggers and Kim [15] introduced the notion of B-algebras which is related to several classes of algebras of interest such as BCH/BCI/BCK-algebras and which seems to have rather nice properties without being excessively complicated otherwise. And they demonstrated some interesting connections between B-algebras and groups.

Example 2.3. Let \( X := \{0,1,2,\ldots,\omega\} \) be a set. Define a binary operation \( "\ast" \) on \( X \) by

\[
x \ast y := \begin{cases} 
0, & x \leq y, \\
\omega, & y < x, x \neq 0, \\
x, & y < x, y = 0.
\end{cases}
\]

Then \( (X, \ast, 0) \) is a Q-algebra, but not a B-algebra, since \( (3 \ast \omega) \ast 0 = 0, 3 \ast (0 \ast (0 \ast \omega)) = 3 \).
Example 2.4. Let \( X := \{0, 1, \ldots, 5\} \) be a set with the following table:

\[
\begin{array}{c|cccccc}
* & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 0 & 2 & 1 & 3 & 4 & 5 \\
1 & 1 & 0 & 2 & 4 & 5 & 3 \\
2 & 2 & 1 & 0 & 5 & 3 & 4 \\
3 & 3 & 4 & 5 & 0 & 2 & 1 \\
4 & 4 & 5 & 3 & 1 & 0 & 2 \\
5 & 5 & 3 & 4 & 2 & 1 & 0 \\
\end{array}
\]

Then \( (X, \ast, 0) \) is a \( B \)-algebra, but not a \( Q \)-algebra, since \( (5 \ast 3) \ast 1 = 1, (5 \ast 1) \ast 3 = 0 \).

Example 2.5. Let \( X \) be the set of all real numbers except for a negative integer \(-n\). Define a binary operation \( \ast \) on \( X \) by

\[
x \ast y := \frac{n(x - y)}{n + y}
\]

for any \( x, y \in X \). Then \( (X, \ast, 0) \) is both a \( Q \)-algebra and \( B \)-algebra.

If we consider several families of abstract algebras including the well-known \( BCK \)-algebras and several larger classes including the class of \( d \)-algebras which is a generalization of \( BCK \)-algebras, then it is usually difficult and often impossible to obtain a complementation operation and the associated “de Morgan’s laws.” In the sense of this point of view it is natural to construct a “mirror image” of a given algebra which when adjoined to the original algebra permits a natural complementation to take place. The class of \( BCK \)-algebras is not closed under this operation but the class of \( d \)-algebras is, thus explaining why it may be better to work with this class rather than the class of \( BCK \)-algebras. Allen et al. [14] introduced the notion of mirror algebras of a given algebra.

Let \( (X, \ast, 0) \) be an algebra. Let \( M(X) := X \times \{0, 1\} \), and define a binary operation “\( \ast \)” on \( M(X) \) as follows:

\[
(x, 0) \ast (y, 0) := (x \ast y, 0),
\]
\[
(x, 1) \ast (y, 1) := (y \ast x, 0),
\]
\[
(x, 0) \ast (y, 1) := (x \ast (x \ast y), 0),
\]
\[
(x, 1) \ast (y, 0) := \begin{cases} (y, 1) & \text{when } x \ast y = 0, \\
(x, 1) & \text{when } x \ast y \neq 0. \end{cases}
\]

Then we say that \( M(X) := (M(X), \ast, (0,0)) \) is a left mirror algebra of the algebra \( (X, \ast, 0) \). Similarly, if we define

\[
(x, \ast) \ast (y, 1) := (y \ast (y \ast x), 0),
\]

then \( M(X) := (M(X), \ast, (0,0)) \) is a right mirror algebra of the algebra \( (X, \ast, 0) \).
It was shown [14] that the mirror algebra of a $d$ (resp., $d$-$BH$)-algebra is also a $d$ (resp., $d$-$BH$)-algebra, but the mirror algebra of a $BCK$-algebra need not be a $BCK$-algebra.

3. A Construction of Mirror $Q$-Algebras

In [14] Allen et al. defined (left, right) mirror algebras of an algebra, but it is not known how to construct mirror algebras of any given algebra. In this paper, we investigate a construction of a mirror algebra in $Q$-algebras.

Let $(X, \ast, 0)$ be a $Q$-algebra, and let $M(X) := X \times \{0, 1\}$. Define a binary operation “$\oplus$” on $M(X)$ by

(M1) $(x, 0) \oplus (y, 0) = (x \ast y, 0)$,
(M2) $(x, 1) \oplus (y, 1) = (y \ast x, 0)$,
(M3) $(x, 0) \oplus (y, 1) = (\alpha(x, y), 0)$,
(M4) $(x, 1) \oplus (y, 0) = (\beta(x, y), 1)$,

where $\alpha, \beta : X \times X \to X$ are mappings.

Consider condition (I). If we let $x = y$ in (1) and (2), then (I) holds trivially. Consider condition (II). For any $(x, 0) \in M(X)$, we have $(x, 0) \oplus (0, 0) = (x \ast 0, 0) = (x, 0)$. For any $(x, 1) \in M(X)$, we have $(x, 1) = (x, 1) \oplus (0, 0) = (\beta(x, 0), 1)$, which shows that the required condition is $\beta(x, 0) = x$. Consider condition (III). There are 8 cases to check that condition (III) holds.

Case 1 ($(x, 0), (y, 0), (z, 0)$). It holds trivially.

Case 2 ($(x, 0), (y, 1), (z, 0)$). Since $((x, 0) \oplus (y, 1)) \oplus (z, 0) = (\alpha(x, y), 0) \oplus (z, 0) = (\alpha(x, y) \ast z, 0)$ and $((x, 0) \oplus (z, 0)) \oplus (y, 1) = (x \ast z, 0) \oplus (y, 1) = (\alpha(x \ast z, y), 0)$, we obtain the requirement that $\alpha(x, y) \ast z = \alpha(x \ast z, y)$.

Case 3 ($(x, 0), (y, 0), (z, 1)$). It is the same as Case 2.

Case 4 ($(x, 0), (y, 1), (z, 1)$). Since $((x, 0) \oplus (y, 1)) \oplus (z, 1) = (\alpha(x, y), 0) \oplus (z, 1) = (\alpha(x, y), z, 0)$ and $((x, 0) \oplus (z, 1)) \oplus (y, 1) = (\alpha(x, z), 0) \oplus (y, 1) = (\alpha(x, z), y, 0)$, we obtain the requirement that $\alpha(\alpha(x, y), z) = \alpha(\alpha(x, z), y)$.

Case 5 ($(x, 1), (y, 0), (z, 0)$). Since $((x, 1) \oplus (y, 0)) \oplus (z, 0) = (\beta(x, y), 1) \oplus (z, 0) = (\beta(x, y), z, 0)$ and $((x, 1) \oplus (z, 0)) \oplus (y, 0) = (\beta(x, z), 1) \oplus (y, 0) = (\beta(x, z), y, 0)$, we obtain the requirement that $\beta(\beta(x, y), z) = \beta(\beta(x, z), y)$.

Case 6 ($(x, 1), (y, 0), (z, 1)$). Since $((x, 1) \oplus (y, 0)) \oplus (z, 1) = (\beta(x, y), 1) \oplus (z, 1) = (z \ast \beta(x, y), 0)$ and $((x, 1) \oplus (z, 1)) \oplus (y, 0) = (z \ast x, 0) \oplus (y, 0) = ((z \ast x) \ast y, 0)$, we obtain the requirement that $z \ast \beta(x, y) = (z \ast x) \ast y$.

Case 7 ($(x, 1), (y, 1), (z, 0)$). It is the same as Case 6.

Case 8 ($(x, 1), (y, 1), (z, 1)$). Since $((x, 1) \oplus (y, 1)) \oplus (z, 1) = (\beta(x, y), 0) \oplus (z, 1) = (\alpha(\beta(x, y), z), 0)$ and $((x, 1) \oplus (z, 1)) \oplus (y, 1) = (\alpha(\beta(x, z), y), 0)$ by exchanging $y$ with $z$, we obtain the requirement that $\alpha(\beta(x, y), z) = \alpha(\beta(x, z), y)$. If we summarize this discussion, we obtain the following theorem.
Theorem 3.1. Let \((X, *, 0)\) be a \(Q\)-algebra, and let \(M(X) := X \times \{0, 1\}\) be a set with a binary operation “\(\oplus\)” on \(M(X)\) with \((M1) \sim (M4)\). Then the necessary conditions for \((M(X), \oplus, (0, 0))\) to be a \(Q\)-algebra are the following:

(i) \(\beta(x, 0) = x\),
(ii) \(\alpha(x, y, z) = \alpha(x, z, y)\),
(iii) \(\alpha(x, y) \ast z = \alpha(x \ast z, y)\),
(iv) \(\beta(\beta(x, y), z) = \beta(\beta(x, z), y)\),
(v) \(z \ast \beta(x, y) = (z \ast x) \ast y\),
(vi) \(\alpha(\beta(x, y), z) = \alpha(\beta(x, z), y)\)

for any \(x, y, z \in X\).

Remark 3.2. By condition \((M1)\), if we identify \((x, 0) \equiv x\) for any \(x \in X\), then \(X\) is a subalgebra of \(M(X)\). By applying Theorem 3.1, we obtain many (mirror) \(Q\)-algebras: \(X \subseteq M(X) \subseteq M(M(X)) = M^2(X) \subseteq M^3(X) \subseteq M^4(X) \subseteq \cdots\).

Example 3.3. Let \(Z\) be the set of all integers. Then \((Z, -, 0)\) is a \(Q\)-algebra where “\(-\)” is the usual subtraction in \(Z\). If we define mappings \(\alpha, \beta : Z \times Z \rightarrow Z\) by \(\alpha(x, y) = \beta(x, y) = x + y\) for any \(x, y \in Z\), then the mirror algebra \((M(Z), \oplus, (0, 0))\) is also a \(Q\)-algebra, that is, \((x, 0) \oplus (y, 0) = (x - y, 0), (x, 1) \oplus (y, 1) = (y - x, 0), (x, 0) \oplus (y, 1) = (x + y, 0), \text{and} (x, 1) \oplus (y, 0) = (x + y, 1)\).

Example 3.4. Let \(X := \{0, 1\}\) be a set with the following table:

\[
\begin{array}{c|cc}
* & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 1 & 0 \\
\end{array}
\]  

Then \((X, *, 0)\) is a \(Q\)-algebra. Using the same method we obtain its mirror algebra as follows: \(M(X) = \{0, \alpha, \beta, \gamma\}\) with the following table:

\[
\begin{array}{c|cccc}
\oplus & 0 & \alpha & \beta & \gamma \\
\hline
0 & 0 & 0 & 0 & 0 \\
\alpha & \alpha & 0 & \alpha & 0 \\
\beta & \beta & \beta & 0 & 0 \\
\gamma & \gamma & \gamma & \beta & \alpha \\
\end{array}
\]  

where \(0 := (0, 0), \ \alpha := (0, 1), \ \beta := (1, 0), \text{and} \ \gamma := (1, 1)\). It is easy to see that \((M(X), \oplus, 0)\) is a \(Q\)-algebra.

Problems

(1) Find necessary conditions for \(M(X)\) to be a QS-algebra if \((X, *, 0)\) is a QS-algebra.

(2) Given a homomorphism \(f : X \rightarrow Y\) of \(Q\)-algebras, construct a homomorphism \(\tilde{f} : M(X) \rightarrow M(Y)\) of \(Q\)-algebras which is an extension of \(f\).

(3) Given \(Q\)-algebras \(X, Y\), are the mirror algebras \(M(M(X \times Y))\) and \(M(X) \times M(Y)\) isomorphic?
References
