Research Article
Left WMC2 Rings

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Throughout this paper, $R$ denotes an associative ring with identity, and all modules are unitary. For any nonempty subset $X$ of a ring $R$, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the set of right annihilators of $X$ and the set of left annihilators of $X$, respectively. We use $J(R)$, $N_*(R)$, $N(R)$, $Z_l(R)$, $E(R)$, $Soc_r(R)$, and $Soc_l(R)$ for the Jacobson radical, the prime radical, the nilpotent elements, the left singular ideal, the set of all idempotent elements, the left socle, and the right socle of $R$, respectively.

An element $k$ of $R$ is called left minimal if $Rk$ is a minimal left ideal. An element $e$ of $R$ is called left minimal idempotent if $e^2 = e$ is left minimal. We use $M_l(R)$ and $M_{El}(R)$ for the set of all left minimal elements and the set of all left minimal idempotent elements of $R$, respectively. Moreover, let $MP_l(R) = \{k \in M_l(R) \mid kR$ is projective$\}$.

A ring $R$ is called left MC2 if every minimal left ideal which is isomorphic to a summand of $kR$ is a summand. Left MC2 rings were initiated by Nicholson and Yousif in [1]. In [2–6], the authors discussed the properties of left MC2 rings. In [1], a ring $R$ is called left mininjective if $rI(k) = kR$ for every $k \in M_l(R)$, and $R$ is said to be left minsymmetric if $k \in M_l(R)$ always implies $k \in M_l(R)$. According to [1], left mininjective $\Rightarrow$ left minsymmetric $\Rightarrow$ left MC2, and no reversal holds.

A ring $R$ is called left universally mininjective [1] if $Rk$ is an idempotent left ideal of $R$ for every $k \in M_l(R)$. The work in [2] uses the term left $DS$ for the left universally mininjective. According to [1, Lemma 5.1], left $DS$ rings are left mininjective.

A ring $R$ is called left min-abel [3] if for each $e \in ME_l(R)$, $e$ is left semicentral in $R$, and $R$ is said to be strongly left min-abel [3, 7] if every element of $ME_l(R)$ is central in $R$. 

We introduce in this paper the concept of left WMC2 rings and concern ourselves with rings containing an injective maximal left ideal. Some known results for left idempotent reflexive rings and left $HI$ rings can be extended to left WMC2 rings. As applications, we are able to give some new characterizations of regular left self-injective rings with nonzero socle and extend some known results on strongly regular rings.
A ring \( R \) is called left WMC2 if \( gRe = 0 \) implies \( eRg = 0 \) for \( e \in ME_1(R) \) and \( g \in E(R) \).

Let \( F \) be a field and \( R = \{ \begin{pmatrix} a & b \\ 0 & F \end{pmatrix} \mid a, b \in F \} \). Then \( E(R) = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \} \) and \( ME_1(R) \) is empty, so \( R \) is left WMC2. Now let \( S = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} \). Then \( ME_1(S) = \{ \begin{pmatrix} u & u \\ 0 & u \end{pmatrix} \mid u \in F \} \) and \( E(S) = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mid u \in F \} \). Since \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}S\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \emptyset \), \( S \) is not left WMC2.

Let \( R \) be any ring and \( S_1 = R[x] \) and \( S_2 = R[[x]] \). Then \( ME_1(S_1) \) and \( ME_1(S_2) \) are all empty, so \( S_1 \) and \( S_2 \) are all left WMC2.

A ring \( R \) is called left idempotent reflexive [8] if \( aRe = 0 \) implies \( eRa = 0 \) for all \( a \in R \) and \( e \in E(R) \). Clearly, \( R \) is left idempotent reflexive if and only if for any \( a \in N(R) \) and \( e \in E(R) \), \( aRe = 0 \) implies \( eRa = 0 \) if and only if for any \( a \in J(R) \) and \( e \in E(R) \), \( aRe = 0 \) implies \( eRa = 0 \). Therefore, left idempotent reflexive rings are left WMC2.

In general, the existence of an injective maximal left ideal in a ring \( R \) cannot guarantee the left self-injectivity of \( R \). In [9], Osofsky proves that if \( R \) is a semiprime ring containing an injective maximal left ideal, then \( R \) is left self-injective. In [8], Kim and Baik prove that if \( R \) is left idempotent reflexive containing an injective maximal left ideal, then \( R \) is left self-injective. In [10], Wei and Li prove that if \( R \) is left MC2 containing an injective maximal left ideal, then \( R \) is left self-injective. Motivated by these results, in this paper, we show that if \( R \) is a left WMC2 ring containing an injective maximal left ideal, then \( R \) is left self-injective. As an application of this result, we show that a ring \( R \) is a semisimple Artinian ring if and only if \( R \) is a left WMC2 ring and left HI ring.

We start with the following theorem.

**Theorem 1.** The following conditions are equivalent for a ring \( R \):

1. \( R \) is left MC2;
2. For any \( a \in R \) and \( e \in ME_1(R) \), \( eaRe = 0 \) implies \( ea = 0 \);
3. For any \( e, g \in ME_1(R) \), \( (g-e)Re = 0 \) implies \( e = eg \);
4. For any \( k, l \in M_1(R) \), \( kRl = 0 \) implies \( IRk = 0 \).

**Proof.** (1)\( \Rightarrow \)(2) Assume that \( a \in R \) and \( e \in ME_1(R) \) with \( eaRe = 0 \). If \( ea \neq 0 \), then \( Rea \subseteq Re \).

By (1), \( Rea = Rg \) for some \( g \in ME_1(R) \). Hence \( Rg = RgRg = ReaRea = R(eaRe)a = 0 \), which is a contradiction. Hence \( ea = 0 \).

(2)\( \Rightarrow \)(3) Let \( e, g \in ME_1(R) \) such that \( (g-e)Re = 0 \). Then \( e(g-e)Re = 0 \). By (2), \( e(g-e) = 0 \). Hence \( e = eg \).

(3)\( \Rightarrow \)(4) Assume that \( k, l \in M_1(R) \) with \( kRl = 0 \). If \( IRk \neq 0 \), then \( IRk = Rk \). Hence \( Rk = RIRk = (Rl)^2Rk \), which implies \( Rl = Re \) for some \( e \in ME_1(R) \).

Since \( ReRk = RlRk = Rk \neq 0 \), there exists \( b \in R \) such that \( ebk \neq 0 \). Let \( g = e + ebk \). Then \( g^2 = e + ebk + ebke + ebkebk = e + ebk = g \in ME_1(R) \) because \( ebke \in RkRe = RkRl = 0 \) and \( g \neq 0 \).

Since \( (g-e)Re = ebkRe = ebkRl = 0 \), by (3), \( e = eg \). Hence \( g = eg = e \), which implies \( ebk = 0 \). It is a contradiction. Therefore \( IRk = 0 \).

(4)\( \Rightarrow \)(1) Let \( a \in M_1(R) \) and \( e \in ME_1(R) \) with \( Ra \subseteq Re \). Then there exists \( g \in ME_1(R) \) such that \( a = ga \) and \( l(a) = l(g) \).

If \( (Ra)^2 = 0 \), then \( RaR \subseteq l(a) = l(g) \), so \( aRg = 0 \), by (4), \( gRa = 0 \), which implies \( a = ga = 0 \). It is a contradiction. Hence \( (Ra)^2 \neq 0 \), so \( Ra = Rh \) for some \( h \in ME_1(R) \), which implies \( R \) is a left MC2 ring.

**Corollary 2.** Left MC2 rings are left WMC2.

**Proof.** Let \( e \in ME_1(R) \) and \( g \in E(R) \) with \( gRe = 0 \). If \( eRg \neq 0 \), then \( eb \neq 0 \) for some \( b \in R \).

Clearly, \( eb \in M_1(R) \) and \( (eb)Re = 0 \). Since \( R \) is a left MC2 ring, by Theorem 1, \( eR(eb) = 0 \),
Proof. (4)⇒(1)⇒(2) It is easy to show by the definition of left WMC2 ring.

(2)⇒(3) Let $g \in l(Re) \cap E(R)$. Then $gRe = 0$. We claim that $eRg = 0$. Otherwise, there exists $b \in R$ such that $ebg \neq 0$. Clearly, $h = ebg + g - eg \in E(R)$ and $eh = ebg \neq 0$. By (2), we have $hRe \neq 0$. But $hRe = 0$ because $gRe = 0$. This is a contradiction. Hence $eRg = 0$ and so $g \in r(Re)$. Therefore $l(Re) \cap E(R) \subseteq r(eR)$.

(3)⇒(4) Since $k \in MP_f(R)$, $gRk$ is projective. It is easy to show that $k = ek$ and $l(k) = l(e)$ for some $e \in ME_l(R)$. Since $gRk = 0$, $gR \subseteq l(k)$. Therefore $gRe = 0$, which implies $g \in l(Re) \cap E(R)$. By (3), $eRg = 0$. Hence $kRg = ekRg \subseteq eRg = 0$.

By Theorem 3, we have the following corollary.

Corollary 4. (1) Let $R$ be a left WMC2 ring. If $e \in E(R)$ satisfying $ReR = R$, then $eRe$ is left WMC2.

(2) If $R$ is a direct product of a family rings $\{R_i : i \in I\}$, then $R$ is a left WMC2 ring if and only if every $R_i$ is left WMC2.

Theorem 5. (1) If $R$ is a subdirect product of a family left WMC2 rings $\{R_i : i \in I\}$, then $R$ is a left WMC2 ring.

(2) If $R/Z_l(R)$ is a left WMC2 ring, so is $R$.

Proof. (1) Let $R_i \overset{\sim}{=}= R/A_{i}$, where $A_i$ are ideals of $R$ with $\bigcap_{i \in I} A_i = 0$. Let $e \in ME_l(R)$ and $g \in E(R)$ satisfying $gRe = 0$. For any $i \in I$, if $e \in A_i$, then $eRg \subseteq A_i$; if $e \not\in A_i$, then we can easily show that $e_i = e + A_i \in ME_l(R)$. Since $R_i$ is a left WMC2 ring and $g_i R_i e_i = 0$, where $g_i = g + A_i, e_i R_i g_i = 0$. Hence $eRg \subseteq A_i$. In any case, we have $eRg \subseteq A_i$ for all $i \in I$. Therefore $eRg \subseteq \bigcap_{i \in I} A_i = 0$ and so $eRg = 0$. This shows that $R$ is a left WMC2 ring.

(2) Let $e \in ME_l(R)$ and $g \in E(R)$ satisfying $eg \neq 0$. Clearly, in $\overline{R} = R/Z_l(R)$, $\overline{e} = e + Z_l(R) \in ME_l(\overline{R})$, $\overline{g} = g + Z_l(R) \in E(\overline{R})$. Since $\overline{R} e \overline{g} \neq \overline{R} e g \neq Z_l(R)$. Since $\overline{R}$ is a left WMC2 ring, by Theorem 3, $\overline{g}\overline{R}\overline{e} \neq 0$, which implies $gRe \neq 0$. Thus $R$ is a left WMC2 ring by Theorem 3.

Theorem 6. (1) $R$ is a strongly left min-abel ring if and only if $R$ is a left min-abel left WMC2 ring.

(2) If $R/Z_l(R)$ is a strongly left min-abel ring, then so is $R$.

Proof. (1) Theorem 1.8 in [3] shows that $R$ is a strongly left min-abel ring if and only if $R$ is a left min-abel left MC2 ring, so by Corollary 2, we obtain that strongly left min-abel ring is left min-abel left WMC2.
Conversely, let $R$ be a left min-abel left $WMC_2$ ring. Let $e \in ME_i(R)$ and $a \in R$ satisfying $eaRe = 0$. Set $g = 1 - e + ea$. Then, clearly, $g \in E(R)$ and $eg = ea$. Since $R$ is a left min-abel ring, $(1 - e)Re = (1 - e)eRe = 0$, so $gRe = 0$. Since $R$ is a left $WMC_2$ ring, $eRg = 0$, which implies $ea = eg = 0$, by Theorem 1, $R$ is a left MC2 ring. Hence $R$ is a strongly left min-abel ring.

(2) It is an immediate corollary of (1), [3, Corollary 1.5(2)] and Theorem 5(2).

A ring $R$ is called left idempotent reflexive [8] if $aRe = 0$ implies $eRa = 0$ for all $a \in R$ and $e \in E(R)$. Clearly, left idempotent reflexive rings are left $WMC_2$.

In general, the existence of an injective maximal left ideal in a ring $R$ cannot guarantee the left self-injectivity of $R$. Proposition 5 in [8] proves that if $R$ is a left idempotent reflexive ring containing an injective maximal left ideal, then $R$ is a left self-injective ring. Theorem 4.1 in [10] proves that if $R$ is a left $MC_2$ ring containing an injective maximal left ideal, then $R$ is a left self-injective ring. We can generalize the results as follows.

**Theorem 7.** Let $R$ be a left $WMC_2$ ring. If $R$ contains an injective maximal left ideal, then $R$ is a left self-injective ring.

**Proof.** Let $M$ be an injective maximal left ideal of $R$. Then $R = M \oplus N$ for some minimal left ideal $N$ of $R$. Hence we have $M = Re$ and $N = R(1 - e)$ for some $e^2 = e \in R$. If $MN = 0$, then $eR(1 - e) = 0$. Since $R$ is left $WMC_2$ and $1 - e \in ME_i(R)$, $(1 - e)Re = 0$. So $e$ is central. Now let $L$ be any proper essential left ideal of $R$ and $f : L \to N$ any nonzero left $R$-homomorphism. Then $L/U \cong N$, where $U = \ker f$ is a maximal submodule of $L$. Now $L = U \oplus V$, where $V \cong N = R(1 - e)$ is a minimal left ideal of $R$. Since $e$ is central, $V = R(1 - e)$. For any $z \in L$, let $z = x + y$, where $x \in U$, $y \in V$. Then $f(z) = f(x) + f(y) = f(y)$. Since $y = y(1 - e) = (1 - e)y$, $f(z) = f(y) = f(y(1 - e)) = yf(1 - e)$. Since $x(1 - e) = (1 - e)x \in V \cap U = 0$, $xf(1 - e) = f(x(1 - e) = f(0) = 0$. Thus $f(z) = yf(1 - e) = yf(1 - e) + xf(1 - e) = (y + x)f(1 - e) = zf(1 - e)$. Hence $\ker N$ is injective. If $MN \neq 0$, by the proof of [11, Proposition 5], we have $\ker N$ is injective. Hence $R = M \oplus N$ is left self-injective.

A ring $R$ is called strongly left $DS$ [3] if $k^2 \neq 0$ for all $k \in M_i(R)$. Since strongly left $DS \Rightarrow$ left $DS \Rightarrow$ left mininjective $\Rightarrow$ left minsymmetric $\Rightarrow$ left $MC_2 \Rightarrow$ left $WMC_2$, we have the following corollary.

**Corollary 8.** Let $R$ contain an injective maximal left ideal. If $R$ satisfies one of the following conditions, then $R$ is a left self-injective ring.

1. $R$ is a strongly left $DS$ ring.
2. $R$ is a left $DS$ ring.
3. $R$ is a left mininjective ring.
4. $R$ is a left minsymmetric ring.
5. $R$ is a strongly left min-abel ring.
6. $R$ is a left $MC_2$ ring.

It is well known that if $R$ is a left self-injective ring, then $J(R) = Z_i(R)$. Therefore by [2, Theorem 5.1] and Corollary 8, we have the following corollary.

**Corollary 9.** Let $R$ contain an injective maximal left ideal. Then $R$ is left self-injective if and only if $J(R) = Z_i(R)$. 

A ring $R$ is called left nil-injective [5] if for any $a \in N(R)$, $rl(a) = aR$, and $R$ is said to be left NC2 [5] if for any $a \in N(R)$, $rRa$ is projective implies that $Ra = Re$ for some $e \in E(R)$. By [5, Theorem 2.22], left nil-injective rings are left NC2 and left NC2 rings are left MC2. A ring $R$ is right Kasch if every simple right $R$-module can be embedded in $RkR$, and $R$ is said to be left C2 [12] if every left ideal that is isomorphic to a direct summand of $RkR$ is itself a direct summand. Clearly, left self-injective rings are left C2 [13] and left C2 rings are left NC2 and by [14, Lemma 1.15], right Kasch rings are left C2. Hence, we have the following corollary.

**Corollary 10.** (1) Let $R$ contain an injective maximal left ideal. Then the following conditions are equivalent:

(a) $R$ is a left self-injective ring;
(b) $R$ is a left nil-injective ring;
(c) $R$ is a left C2 ring;
(d) $R$ is a left NC2 ring.

(2) If $R$ is a right Kasch ring containing an injective maximal left ideal, then $R$ is a left self-injective ring.

A ring $R$ is called left min-AP-injective if for any $k \in M_1(R)$, $rl(k) = kR \oplus X_k$, where $X_k$ is a right ideal of $R$. Clearly, left mininjective rings are left min AP-injective.

**Lemma 11.** (1) If $R$ is a left min - AP-injective ring, then $R$ is left WMC2.

(2) If $Soc(qR) \subseteq Soc(R_R)$, then $R$ is left WMC2.

**Proof.** (1) Let $e \in ME_1(R)$ and $g \in E(R)$ satisfying $eg \neq 0$. Since $R$ is a left min-AP-injective ring and $l(e) = l(eg)$, $eR = rl(e) = rl(eg) = egR \oplus X_{eg}$, where $X_{eg}$ is a right ideal of $R$. Set $e = egb + x, b \in R$ and $x \in X_{eg}$. Then $eg = e(eg) = egbeg + xeg$, so $xeg = eg - egbeg \in egR \cap X_{eg}$, which implies $xeg = 0$, so $eg = egbeg$. Let $h = egb$. Then $h \in ME_1(R)$ and $egR = hR$. Therefore $hR = hRkR = egRegR$ which implies $gRe \neq 0$. By Theorem 3, $R$ is a left WMC2 ring.

(2) Assume that $e \in ME_1(R)$ and $a \in R$ satisfying $eaRe = 0$. If $ea \neq 0$, then $ea \in Soc(qR) \subseteq Soc(R_R)$. Thus there exists a minimal right ideal $kR$ of $R$ such that $kR \subseteq eaR$. Clearly, $l(k) = l(ea) = l(e)$ and $kRkR \subseteq eaReR = 0$. Hence $RkR \subseteq l(k)$. Let $I$ be a complement right ideal of $RkR$ in $R$. Then $I \subseteq l(k)$ and $e \in Soc(qR) \subseteq Soc(R_R) \subseteq RkR \oplus I \subseteq l(k) = l(e)$, which is a contradiction. Hence $ea = 0$. By Theorem 1, $R$ is a left MC2 ring, so $R$ is left WMC2 by Corollary 2.

Since left mininjective rings are left min-AP-injective and $Soc(qR) \subseteq Soc(R_R)$. Hence by Theorem 7, Corollary 8 and Lemma 11, we have the following theorem.

**Theorem 12.** Let $R$ contain an injective maximal left ideal. Then the following conditions are equivalent:

(1) $R$ is left self-injective;
(2) $R$ is left min-AP-injective;
(3) $Soc(qR) \subseteq Soc(R_R)$.

A ring $R$ is called
Proposition 13. (1) The following conditions are equivalent for a ring \( R \):

(a) \( R \) is semiprime;

(b) \( R \) is strongly reflexive and every proper essential right ideal of \( R \) contains no nonzero nilpotent ideal;

(c) \( R \) is reflexive and every proper essential right ideal of \( R \) contains no nonzero nilpotent ideal;

(d) \( R \) is strongly reflexive and \( N_*(R) \cap Z_l(R) = 0 \);

(e) \( R \) is reflexive and \( N_*(R) \cap Z_l(R) = 0 \).

(2) \( R \) is symmetric if and only if \( R \) is \( ZI \) and strongly reflexive.

(3) \( R \) is reversible if and only if \( R \) is \( ZI \) and reflexive.

(4) Strongly reflexive \( \Rightarrow \) reflexive \( \Rightarrow \) left idempotent reflexive.

It is well known that if \( R \) is a left self-injective ring, then \( Z_l(R) = J(R) \), so \( R/Z_l(R) \) is semiprimitive. Hence \( R/Z_l(R) \) is left \( WMC2 \) by Proposition 13. Thus, by Theorems 5 and 7, we have the following theorem.

Theorem 14. Let \( R \) contain an injective maximal left ideal. Then

(1) \( R \) is a left self-injective ring if and only if \( R/Z_l(R) \) is a left \( WMC2 \) ring.

(2) If \( R \) satisfies one of the following conditions, then \( R \) is a left self-injective:

(a) \( R \) is a semiprime ring;

(b) \( R \) is a strongly reflexive;

(c) \( R \) is reflexive;

(d) \( R \) is a left idempotent reflexive.

Recall that a ring \( R \) is left \( pp \) if every principal left ideal of \( R \) is projective as left \( R \)-module. As an application of Theorem 7, we have the following result.

Theorem 15. The following conditions are equivalent for a ring \( R \):

(1) \( R \) is a von Neumann regular left self-injective ring with \( Soc_1(R) \neq 0 \);

(2) \( R \) is a left \( WMC2 \) left \( pp \) ring containing an injective maximal left ideal;

(3) \( R \) is a left \( pp \) ring containing an injective maximal left ideal and \( R/Z_l(R) \) is a left \( WMC2 \) ring.

Proof. (1)\( \Rightarrow \)(3) is trivial.

(3)\( \Rightarrow \)(2) is a direct result of Theorem 5(2).
The following conditions are equivalent for a ring $R$:

1. $R$ is semisimple Artinian;
2. $R$ is left $WMC_2$ left $HI$;
3. $R$ is left $NC_2$ left $HI$;
4. $R$ is left min-$AP$-injective left $HI$;
5. $R$ is left idempotent reflexive left $HI$.

Corollary 16. The following conditions are equivalent for a ring $R$:

By Theorem 7, $R$ is left self-injective. Hence, by [13, Theorem 1.2], $R$ is left C2, so $R$ is von Neumann regular because $R$ is left pp. In addition, we have $\text{Soc}(R) \neq 0$ since there is an injective maximal left ideal.

By [17], a ring $R$ is said to be left $HI$ if $R$ is left hereditary containing an injective maximal left ideal. Osofsky [9] proves that a left self-injective left hereditary ring is semisimple Artinian. We can generalize the result as follows.

Corollary 16. The following conditions are equivalent for a ring $R$:

1. $R$ is semisimple Artinian;
2. $R$ is left $WMC_2$ left $HI$;
3. $R$ is left $NC_2$ left $HI$;
4. $R$ is left min-$AP$-injective left $HI$;
5. $R$ is left idempotent reflexive left $HI$.

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