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Research Article

Transforming Arithmetic Asian Option PDE to the Parabolic Equation with Constant Coefficients

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Arithmetic Asian options are difficult to price and hedge, since at present, there is no closed-form analytical solution to price them. Transforming the PDE of the arithmetic the Asian option to a heat equation with constant coefficients is found to be difficult or impossible. Also, the numerical solution of the arithmetic Asian option PDE is not very accurate since the Asian option has low volatility level. In this paper, we analyze the value of the arithmetic Asian option with a new approach using means of partial differential equations (PDEs), and we transform the PDE to a parabolic equation with constant coefficients. It has been shown previously that the PDE of the arithmetic Asian option cannot be transformed to a heat equation with constant coefficients. We, however, approach the problem and obtain the analytical solution of the arithmetic Asian option PDE.

1. Introduction

Asian options are path-dependent options whose payoff depends on the average value of the underlying assets during a specific set of dates across the life of the option. Because the payoff of the Asian options depends on the average value of the underlying asset, volatility in the average value tends to be smoother and lower than that of the plain vanilla options. Therefore, Asian options, which tend to be less expensive than comparable plain vanilla puts or calls, appear very attractive. The two basic forms of averages in Asian options (arithmetic and geometric) both can be structured as calls or puts. A geometric average Asian option is easy to price because a closed-form solution is available [1] However, the most difficult task

is arithmetic type, which is the most commonly used, though an exact analytical solution for arithmetic average rate Asian options does not existed. This solution is missing primarily because the arithmetic average of a set of lognormal random variables is not lognormally distributed.

Since no general analytical solution for the price of the arithmetic Asian option is known, a variety of techniques have been developed to analyze arithmetic average Asian options and approach the problem of valuing arithmetic average Asian option among others:

(i) Monte Carlo simulations,

(ii) binomial trees and lattices,

(iii) the PDE approach,

(iv) general numerical methods,

(v) analytical approximations.

This paper follows the PDE approach. Many authors have studied the PDE of the Asian options most recently, Geman and Yor [2] have used Laplace transform in time of the Asian option price. However, this transform is only applicable in some cases. Rogers and Shi [3] transform the problem of valuing Asian options to the problem of solving a parabolic equation in two variables from a second order. However, it is difficult to solve this equation analytically or numerically, and they derive lower-bound formulas for Asian options by computing the expectation based on some zero-mean Gaussian variable. Zhang [4] presents a theory of continuously sampled Asian option pricing; he solves the PDE with perturbation approach, and he shows that the PDE of the arithmetic the Asian option cannot be transformed to the heat equation with constant coefficients. Vecer's approach [5] is based on the Asian option as an option on a traded account. He provides a one-dimensional PDE for Asian options. Dubois and Lelievre [6] derive accurate and fast numerical methods to solve the Rogers and Shi PDE [3]. Chen and Lyuu [7] develop the lower-bound pricing formulas of the Rogers and Shi PDE [3] to include general maturities instead one year. Cruz-Báez and González -Rodrigues [8] obtain the same solution of Geman and Yor for arithmetic Asian options using partial differential equations, integral transforms, and Mathematica programming, instead of Bessel processes. Dewynne and Shaw [9] provide a simplified means of pricing arithmetic Asian options by PDE approach they derive an analytical formula for the Laplace transform in time of the Asian option, and they obtain asymptotic solutions for the Black-Scholes PDE for Asian options for low-volatility limit which is the big problem on using Laplace transform. Antoniou [10] transforms the arithmetic Asian option to the classical Black-Scholes equation PDE then to the parabolic equation using Lie symmetries. Based on general transformation techniques, we transform the PDE of the arithmetic Asian option to the simplest parabolic equation with constant coefficients.

2. Solving the Arithmetic Asian Option PDE

The Black-Scholes equation for the arithmetic Asian option is

$$\begin{gathered}\frac{\partial V}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}+S\frac{\partial V}{\partial A}-rV=0,\\ V(T,S_T,A_T)=\varphi\left(S_T,\frac{A_T}{T}\right),\end{gathered}\tag{2.1}$$

where S is a stock price, r is interest rate, σ is the asset volatility, T the expiration date, $A = \int_0^T S(t)dt$ is a running sum of the asset price, and $\varphi(S_T, A_t/T)$ is the payoff function which depends on the type of the arithmetic Asian option.

In this work, we propose the following transformation, which allows us to transform the PDE (2.1) to a simpler parabolic equation:

$$V(t,S,A) = S^n f(t,x)e^{b(S/A)}, \quad x = \frac{S}{A^2},\ n = -\frac{r}{\sigma^2},\ b = \frac{2}{\sigma^2}. \tag{2.2}$$

Using the chain rule, we obtain

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \left(S^n \frac{\partial f}{\partial t}\right)e^{b(S/A)},\\
\frac{\partial V}{\partial S} &= \left(nS^{n-1}f + \frac{S^n}{A^2}\frac{\partial f}{\partial x} + b\frac{S^n}{A}f\right)e^{b(S/A)},\\
\frac{\partial^2 V}{\partial S^2} &= \left\{n(n-1)S^{n-2}f + n\frac{S^{n-1}}{A^2}\frac{\partial f}{\partial x} + bn\frac{S^{n-1}}{A}f + n\frac{S^{n-1}}{A^2}\frac{\partial f}{\partial x}\frac{S^n}{A^4}\frac{\partial^2 f}{\partial x^2} + b\frac{S^n}{A^3}\frac{\partial f}{\partial x}\right.\\
&\qquad \left. + bn\frac{S^{n-1}}{A}f + b\frac{S^n}{A^3}\frac{\partial f}{\partial x} + b^2\frac{S^n}{A^2}f\right\}e^{b(S/A)},\\
\frac{\partial^2 V}{\partial S^2} &= \left(n(n-1)S^{n-2}f + 2n\frac{S^{n-1}}{A^2}\frac{\partial f}{\partial x} + 2bn\frac{S^{n-1}}{A}f + \frac{S^n}{A^4}\frac{\partial^2 f}{\partial x^2} + 2b\frac{S^n}{A^3}\frac{\partial f}{\partial x} + b^2\frac{S^n}{A^2}f\right)e^{b(S/A)},\\
\frac{\partial V}{\partial A} &= \left(-2\frac{S^{n+1}}{A^3}\frac{\partial f}{\partial x} - b\frac{S^{n+1}}{A^2}f\right)e^{b(S/A)}.
\end{aligned} \tag{2.3}$$

Substituting (2.3) in (2.1) leads to

$$\begin{aligned}
&\left\{S^n\frac{\partial f}{\partial t} + rS\left(nS^{n-1}f + \frac{S^n}{A^2}\frac{\partial f}{\partial x} + b\frac{S^n}{A}f\right) - S\left(2\frac{S^{n+1}}{A^3}\frac{\partial f}{\partial x} + b\frac{S^{n+1}}{A^2}f\right)\right.\\
&\quad + \frac{1}{2}\sigma^2 S^2\left(n(n-1)S^{n-2}f + 2n\frac{S^{n-1}}{A^2}\frac{\partial f}{\partial x} + 2bn\frac{S^{n-1}}{A}f + \frac{S^n}{A^4}\frac{\partial^2 f}{\partial x^2}\right.\\
&\qquad \left.\left. + 2b\frac{S^n}{A^3}\frac{\partial f}{\partial x} + b^2\frac{S^n}{A^2}f\right)\right\}e^{b(S/A)} = 0.
\end{aligned} \tag{2.4}$$

Regrouping terms yields

$$\begin{aligned}
&\left\{S^n\frac{\partial f}{\partial t}+\left(rnS^n+rb\frac{S^{n+1}}{A}-b\frac{S^{n+2}}{A^2}+\frac{1}{2}\sigma^2n(n-1)S^n+\sigma^2bn\frac{S^{n+1}}{A}+\frac{1}{2}\sigma^2b^2\frac{S^{n+2}}{A^2}\right)f\right.\\
&\quad\left.+\left(r\frac{S^{n+1}}{A^2}-2\frac{S^{n+2}}{A^3}+\sigma^2n\frac{S^{n+1}}{A^2}+\sigma^2b\frac{S^{n+2}}{A^3}\right)\frac{\partial f}{\partial x}+\left(\frac{1}{2}\sigma^2\frac{S^{n+2}}{A^4}\right)\frac{\partial^2 f}{\partial x^2}\right\}e^{b(S/A)}=0,\\
&\left\{\frac{\partial f}{\partial t}+\left(rn+rb\frac{S}{A}-b\frac{S^2}{A^2}+\frac{1}{2}\sigma^2n(n-1)+\sigma^2bn\frac{S}{A}+\frac{1}{2}\sigma^2b^2\frac{S^2}{A^2}\right)f\right.\\
&\quad\left.+\left(r\frac{S}{A^2}-2\frac{S^2}{A^3}+\sigma^2n\frac{S}{A^2}+\sigma^2b\frac{S^2}{A^3}\right)\frac{\partial f}{\partial x}+\left(\frac{1}{2}\sigma^2\frac{S^2}{A^4}\right)\frac{\partial^2 f}{\partial x^2}\right\}S^ne^{b(S/A)}=0,
\end{aligned}\tag{2.5}$$

but we have $n=-r/\sigma^2$, $b=2/\sigma^2$,

$$\begin{aligned}
&\left\{\frac{\partial f}{\partial t}+\left(rn+rb\frac{S}{A}-b\frac{S^2}{A^2}+\frac{1}{2}\sigma^2n(n-1)+\sigma^2\left(\frac{-r}{\sigma^2}\right)b\frac{S}{A}+\frac{1}{2}\sigma^2\left(\frac{2}{\sigma^2}\right)b\frac{S^2}{A^2}\right)f\right.\\
&\quad\left.+\left(r\frac{S}{A^2}-2\frac{S^2}{A^3}+\sigma^2n\frac{S}{A^2}+\sigma^2\left(\frac{2}{\sigma^2}\right)\frac{S^2}{A^3}\right)\frac{\partial f}{\partial x}+\left(\frac{1}{2}\sigma^2\frac{S^2}{A^4}\right)\frac{\partial^2 f}{\partial x^2}\right\}S^ne^{b(S/A)}=0
\end{aligned}\tag{2.6}$$

and $S^ne^{b(S/A)}\neq 0$,

$$\begin{aligned}
&\frac{\partial f}{\partial t}+\left(rn+\frac{1}{2}\sigma^2n(n-1)\right)f+\left(\left(r+\sigma^2n\right)\frac{S}{A^2}\right)\frac{\partial f}{\partial x}+\left(\frac{1}{2}\sigma^2\frac{S^2}{A^4}\right)\frac{\partial^2 f}{\partial x^2}=0,\\
&\frac{\partial f}{\partial t}+\left(r\left(-\frac{r}{\sigma^2}\right)+\frac{1}{2}\sigma^2\left(\left(-\frac{r}{\sigma^2}\right)^2-\left(-\frac{r}{\sigma^2}\right)\right)\right)f\\
&\quad+\left(\left(r+\sigma^2\left(-\frac{r}{\sigma^2}\right)\right)\frac{S}{A^2}\right)\frac{\partial f}{\partial x}+\left(\frac{1}{2}\sigma^2\frac{S^2}{A^4}\right)\frac{\partial^2 f}{\partial x^2}=0,\\
&\frac{\partial f}{\partial t}+\frac{1}{2}\left(r-\frac{r^2}{\sigma^2}\right)f+\frac{1}{2}\sigma^2x^2\frac{\partial^2 f}{\partial x^2}=0.
\end{aligned}\tag{2.7}$$

This is a parabolic equation with only second derivation with respect to x, to further simplify the above equation, assume

$$\begin{aligned}
&f(t,x)=e^{(-1/2)(r-r^2/\sigma^2)t}g(t,x),\\
&\frac{\partial g}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 g}{\partial x^2}=0.
\end{aligned}\tag{2.8}$$

For constant coefficients, assume

$$\tau = T - t, \qquad z = \ln x,$$

$$x\frac{\partial g}{\partial x} = \frac{\partial g}{\partial z}, \qquad x^2\frac{\partial^2 g}{\partial x^2} = \frac{\partial^2 g}{\partial z^2} - \frac{\partial g}{\partial z}, \tag{2.9}$$

$$\frac{\partial g}{\partial \tau} = \frac{1}{2}\sigma^2\left(\frac{\partial^2 g}{\partial z^2} - \frac{\partial g}{\partial z}\right).$$

Applying the following change,

$$\xi = z - \frac{1}{2}\sigma^2\tau,$$

$$\frac{\partial g}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial \xi^2}. \tag{2.10}$$

Assume

$$\xi = \frac{\sqrt{2}}{\sigma}\eta,$$

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial \eta^2}. \tag{2.11}$$

The above equation is the simplest heat equation, so the solution of the arithmetic Asian option PDE is

$$V(t,S,A) = S^n f(t,x)e^{b(S/A)}, \quad x = \frac{S}{A^2}, \quad n = -\frac{r}{\sigma^2}, \; b = \frac{2}{\sigma^2}, \tag{2.12}$$

where $f(t,x)$ satisfied the following PDE:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\left(r - \frac{r^2}{\sigma^2}\right)f + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 f}{\partial x^2} = 0,$$

$$V(t,S,A) = S^n g(t,x)e^{(-1/2)(r-r^2/\sigma^2)t}e^{b(S/A)},$$

$$V(t,S,A) = S^n g(\tau,z)e^{(-1/2)(r-r^2/\sigma^2)t}e^{b(S/A)}, \tag{2.13}$$

$$V(t,S,A) = S^n g(\tau,\xi)e^{(-1/2)(r-r^2/\sigma^2)t}e^{b(S/A)},$$

$$V(t,S,A) = S^n g(\tau,\eta)e^{(-1/2)(r-r^2/\sigma^2)t}e^{b(S/A)},$$

where $g(\tau,\eta)$ satisfied the heat equation (2.11), and $\tau = T - t$, $\eta = (\sigma/\sqrt{2})(\ln(S/A^2) - (1/2)\sigma^2\tau)$.

For full details of the solution of the heat equation, see [11].

3. Conclusion

Solving the Black-Scholes PDE of the arithmetic Asian option has been an outstanding issue in mathematical finance for several decades, because the equation is a degenerate partial differential equation in three dimensions, also the numerical solution of the arithmetic the Asian option PDE is not very accurate since the Asian option has low volatility level. In this paper we reached the problem of solving the PDE by transforming the arithmetic Asian option PDE to the heat equation with constant coefficients, so we obtained closed-form analytical solution for the arithmetic Asian option PDE.

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