Research Article

On Maximal Subsemigroups of Partial Baer-Levi Semigroups

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1. Introduction

Suppose that $X$ is an infinite set with $|X| \geq q \geq \aleph_0$ and $I(X)$ is the symmetric inverse semigroup defined on $X$. In 1984, Levi and Wood determined a class of maximal subsemigroups $M_A$ (using certain subsets $A$ of $X$) of the Baer-Levi semigroup $BL(q) = \{\alpha \in I(X) : \text{dom} \alpha = X \text{ and } |X \setminus X\alpha| = q\}$. Later, in 1995, Hotzel showed that there are many other classes of maximal subsemigroups of $BL(q)$, but these are far more complicated to describe. It is known that $BL(q)$ is a subsemigroup of the partial Baer-Levi semigroup $PS(q) = \{\alpha \in I(X) : |X \setminus X\alpha| = q\}$. In this paper, we characterize all maximal subsemigroups of $PS(q)$ when $|X| > q$, and we extend $M_A$ to obtain maximal subsemigroups of $PS(q)$ when $|X| = q$.

1.1

We also write $g(\alpha) = |X \setminus \text{dom} \alpha|$, $d(\alpha) = |X \setminus \text{ran} \alpha|$, and $r(\alpha) = |\text{ran} \alpha|$, respectively. Let $I(X)$ denote the symmetric inverse semigroup on $X$: that is, the set of all injective mappings in $P(X)$. If $|X| = p \geq q \geq \aleph_0$, we write

$$BL(q) = \{\alpha \in I(X) : g(\alpha) = 0, \ d(\alpha) = q\}, \ PS(q) = \{\alpha \in I(X) : d(\alpha) = q\},$$

and refer to these cardinals as the gap, the defect, and the rank of $\alpha$, respectively.
where \( BL(q) \) is the Baer-Levi semigroup of type \( (p, q) \) defined on \( X \) (see [1, 2, vol 2, Section 8.1]). It is wellknown that this semigroup is right simple, right cancellative, and idempotent-free. On the other hand, in [3] the authors showed that \( PS(q) \), the partial Baer-Levi semigroup on \( X \), does not have these properties but it is right reductive in the sense that for every \( \alpha, \beta \in PS(q) \), if \( \alpha \gamma = \beta \gamma \) for all \( \gamma \in PS(q) \), then \( \alpha = \beta \). Also, they showed that \( PS(q) \) satisfies the dual property, that is, it is left reductive (see [1, 2, vol 1, p 9]). The authors also characterized Green’s relations and ideals of \( PS(q) \) and, in [3, Corollary 1], they proved that \( PS(q) \) contains an inverse subsemigroup: namely, the set \( R(q) \) defined by

\[
R(q) = \{ \alpha \in PS(q) : g(\alpha) = q \}. \tag{1.3}
\]

This set consists, in fact, of all regular elements of \( PS(q) \), as shown in [3, Theorem 4]. Recently, in [4], the authors studied some properties of Mitsch’s natural partial order defined on a semigroup (see [5, Theorem 3]) and some other partial orders defined on \( PS(q) \). In particular, they described compatibility and the existence of maximal and minimal elements.

For any nonempty subset \( A \) of \( X \) such that \( |X \setminus A| \geq q \), let

\[
M_A = \{ \alpha \in BL(q) : A \notin X\alpha \text{ or } (A\alpha \subseteq A \text{ or } |X\alpha \setminus A| < q) \}. \tag{1.4}
\]

In other words, given \( \alpha \in BL(q) \), we have \( \alpha \in M_A \) if and only if \( X\alpha \) does not contain \( A \), or \( X\alpha \) contains \( A \) and either \( A\alpha \subseteq A \) or \( |X\alpha \setminus A| < q \). In [6], Levi and Wood showed that \( M_A \) is a maximal subsemigroup of \( BL(q) \). Later, Hotzel [7] showed that there are many other maximal subsemigroups of \( BL(q) \).

In this paper, we study maximal subsemigroups of \( PS(q) \). In particular, in Section 3 we describe all maximal subsemigroups of \( PS(q) \) when \( p > q \). We also determine some maximal subsemigroups of a subsemigroup \( S_r \) of \( PS(q) \) defined by

\[
S_r = \{ \alpha \in PS(q) : g(\alpha) \leq r \}, \tag{1.5}
\]

where \( q \leq r \leq p \). Moreover, we extend \( M_A \) to determine maximal subsemigroups of \( PS(q) \). In Section 4, we determine some maximal subsemigroups of \( PS(q) \) when \( p = q \).

2. Preliminaries

In this paper, \( Y = A \cup B \) means \( Y \) is a disjoint union of sets \( A \) and \( B \). As usual, \( \emptyset \) denotes the empty (one-to-one) mapping which acts as a zero for \( P(X) \). For each nonempty \( A \subseteq X \), we write \( id_A \) for the identity transformation on \( A \); these mappings constitute all the idempotents in \( I(X) \) and belong to \( PS(q) \) precisely when \( |X \setminus A| = q \).

We modify the convention introduced in [1, 2, vol 2, p 241]: namely, if \( \alpha \in I(X) \) is non-zero, then we write

\[
\alpha = \left( \begin{array}{c} a_i \\ x_i \end{array} \right) \tag{2.1}
\]
and take as understood that the subscript $i$ belongs to some (unmentioned) index set $I$, that
the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $X \alpha = \text{ran } \alpha = \{x_i\}$, $\alpha \alpha = x_i$ for each
$i$ and
$\text{dom } \alpha = \{a_i\}$. To simplify notation, if $A \subseteq X$, we sometimes write $A \alpha$ in place of $(A \cap \text{dom } \alpha) \alpha$.

Let $S$ be a semigroup and $\emptyset \neq A \subseteq S$. Then $\langle A \rangle$ denotes the subsemigroup of $S$ generated
by $A$. Recall that a proper subsemigroup $M$ of $S$ is maximal in $S$ if, whenever $M \subseteq N \subseteq S$ and $N$ is a subsemigroup of $S$, then $M = N$. Note that this is equivalent to each
one of the following:

(a) $\langle M \cup \{a\} \rangle = S$ for all $a \in S \setminus M$;

(b) for any $a,b \in S \setminus M$, $a$ can be written as a finite product of elements of $M \cup \{b\}$
(note that $a$ is not expressible as a product of elements of $M$ since $a \notin M$).

Throughout this paper, we will use this fact to show the maximality of subsemigroups
of $PS(q)$.

3. Maximal Subsemigroups of $PS(q)$ When $p > q$

The characterisation of maximal subsemigroups of a given semigroup is a natural topic to
consider when studying its structure. Sometimes, it is difficult to describe all of them (see
$[6,7]$, e.g.), but for a semigroup with some special properties, we can easily describe some of
its maximal subsemigroups.

**Lemma 3.1.** Let $S$ be a semigroup and suppose that $S$ is a disjoint union of a subsemigroup $T$ and an
ideal $I$ of $S$. Then,

(a) for any maximal subsemigroup $M$ of $T$, $M \cup I$ is a maximal subsemigroup of $S$;

(b) for any maximal subsemigroup $N$ of $S$ such that $T \setminus N \neq \emptyset$ and $T \cap N \neq \emptyset$, the set $T \cap N$ is
a maximal subsemigroup of $T$.

**Proof.** To see that (a) holds, let $M$ be a maximal subsemigroup of $T$. Since $I$ is an ideal, we
have $M \cup I$ is a subsemigroup of $S$. Clearly, $M \cup I \subseteq T \cup I = S$. If $a \in S \setminus (M \cup I)$, then
$a \in T \setminus M$ and thus $T = \langle M \cup \{a\} \rangle \subseteq \langle M \cup I \cup \{a\} \rangle$. Since $\langle M \cup I \cup \{a\} \rangle$ contains $I$, we have
$S = T \cup I = \langle M \cup I \cup \{a\} \rangle$ and so $M \cup I$ is maximal in $S$ as required.

To prove (b), let $N$ be a maximal subsemigroup of $S$, where $T \setminus N \neq \emptyset$ and $T \cap N \neq \emptyset$,
and let $a \in T \setminus N$. Since $N$ is maximal in $S$, we have $\langle N \cup \{a\} \rangle = S$. Thus, for each $b \in T \setminus N$,
b $= c_1c_2\cdots c_n$ for some natural $n$ and some $c_i \in N \cup \{a\}$ for all $i = 1,2,\ldots,n$. Since $b \notin N$, we
have $c_i = a$ for some $i$. Moreover, since $b \notin I$, we have $c_j \in T \cap N$ for all $j \neq i$. It follows that
$T \setminus N \subseteq \langle (T \cap N) \cup \{a\} \rangle$, therefore

$$T = (T \setminus N) \cup (T \cap N) \subseteq \langle (T \cap N) \cup \{a\} \rangle,$$

that is, $T = \langle (T \cap N) \cup \{a\} \rangle$ and thus $T \cap N$ is maximal in $T$. \qed

Let $u$ be a cardinal number. The *successor* of $u$, denoted by $u'$, is defined as

$$u' = \min\{v : v > u\}. \tag{3.2}$$

Note that $u'$ always exists since the cardinals are wellordered, and when $u$ is finite we have
$u' = u + 1$. 

Lemma 3.4. Suppose that \( q < s \) then

\[
S_k = \{ \alpha \in PS(q) : g(\alpha) \leq k \}
\]

is a subsemigroup of \( PS(q) \). Also, when \( p > q \), the proper ideals of \( PS(q) \) are precisely the sets:

\[
T_s = \{ \alpha \in PS(q) : g(\alpha) \geq s \},
\]

where \( q < s \leq p \) (see [3, Theorem 13]). Thus, for any \( q \leq r < p \), it is clear that

\[
PS(q) = S_r \cup T_r,
\]

that is, \( PS(q) \) can be written as a disjoint union of the semigroup \( S_r \) and the ideal \( T_r \). Hence, the next result follows directly from Lemma 3.1(a).

Corollary 3.2. Suppose that \( p > r > q \geq \aleph_0 \). If \( M \) is a maximal subsemigroup of \( S_r \), then \( M \cup T_r \) is a maximal subsemigroup of \( PS(q) \).

Lemma 3.3. Let \( p > q \geq \aleph_0 \) and suppose that \( M \) is a maximal subsemigroup of \( PS(q) \). Then,

(a) \( S_r \cap M \neq \emptyset \) for all \( q \leq r < p \);

(b) if there exists \( \alpha \not\in M \) with \( g(\alpha) = k < p \), then \( S_k \setminus M \neq \emptyset \) for some \( q \leq k < p \).

Proof. To show that (a) holds, we first note that \( S_q \) is contained in \( S_r \) for all \( q \leq r < p \). If \( S_q \cap M = \emptyset \), then \( M \subseteq T_q \subseteq PS(q) \) and thus \( M = T_q \) by the maximality of \( M \). But \( T_q \not\subseteq T_q \cup BL(q) \subseteq PS(q) \) where \( T_q \cup BL(q) \) is a subsemigroup of \( PS(q) \) (since \( T_q \) is an ideal), so we get a contradiction. Therefore, \( \emptyset \neq S_q \cap M \subseteq S_r \cap M \) for all \( q \leq r < p \).

To show that (b) holds, suppose there is \( \alpha \not\in M \) with \( g(\alpha) = k < p \). If \( k < q \), then \( \alpha \in S_r \setminus M \) for all \( q \leq r < p \). Otherwise, if \( q \leq k \), then \( \alpha \in S_k \setminus M \). Hence (b) holds.

For what follows, for any cardinal \( r \leq p \), we let

\[
G_r = \{ \alpha \in PS(q) : g(\alpha) = r \}.
\]

Then \( G_0 = BL(q) \) and \( G_q = R(q) \). Moreover, if \( p > q \) and \( r > q \), then \( G_r = S_r \cap T_r \), and so \( G_r \) is a subsemigroup of \( S_r \) (since it is the intersection of two semigroups). Also, \( G_r \) is bisimple and idempotent-free, when \( p > q \) and \( r > q \) (see [3, Corollary 3]).

From [3, Theorem 5], if \( p \geq q \), then \( S_q = \alpha \cdot R(q) \) for each \( \alpha \in BL(q) \), and by [3, Theorem 6], \( S_q = BL(q) \cdot \mu \cdot BL(q) \) for each \( \mu \in R(q) \) when \( p \neq q \).

This motivates the following result.

Lemma 3.4. Suppose that \( p \geq r > q \geq \aleph_0 \). Then \( G_r = BL(q) \cdot \alpha \cdot BL(q) \) for each \( \alpha \in G_r \).

Proof. Let \( \alpha \in G_r \) and \( \beta, \gamma \in BL(q) \). Since

\[
X \setminus \text{dom} \alpha = [X\beta \cap (X \setminus \text{dom} \alpha)] \cup [(X \setminus X\beta) \cap (X \setminus \text{dom} \alpha)],
\]

then...
where \( g(\alpha) = |X \setminus \text{dom } \alpha| = r > q \) and the second intersection on the right has cardinal at most \( q \) (since \( |X \setminus X\beta| = q \)), we have \( |X\beta \cap (X \setminus \text{dom } \alpha)| = r \). This means that

\[
    r = |(X\beta \setminus \text{dom } \alpha)^{-1}| = |(X\beta \setminus \text{dom } \alpha)\beta^{-1}| = |\text{dom } \beta \setminus \text{dom } (\beta\alpha)| \tag{3.8}
\]

Since \( \text{dom } \gamma = X \), we have \( \text{dom } (\beta\gamma) = \text{dom } (\beta\alpha) \), and so \( g(\beta\gamma) = g(\beta\alpha) = r \). Hence \( \beta\gamma \in G_r \) and therefore \( BL(q) \cdot \alpha \cdot BL(q) \subseteq G_r \).

For the converse, if \( \alpha, \beta \in G_r \), then \( |X \setminus \text{dom } \alpha| = r = |X \setminus \text{dom } \beta| \). Since \( p > q \), every element in \( PS(q) \) has rank \( p \), so we write

\[
    \alpha = \left( \begin{array}{c} a_i \\ x_i \end{array} \right), \quad \beta = \left( \begin{array}{c} b_i \\ y_i \end{array} \right) \quad \text{where } |I| = p. \tag{3.9}
\]

Now write \( X \setminus \{ y_i \} = A \cup B \) and \( X \setminus \{ a_i \} = C \cup D \) where \( |A| = |B| = |C| = q \) and \( |D| = r \) (note that this is possible since \( d(\beta) = q \geq \aleph_0 \) and \( g(\alpha) = r > q \geq \aleph_0 \)). Define

\[
    \delta = \left( \begin{array}{c} b_i \\ X \setminus \{ b_i \} \\ a_i \\ D \end{array} \right), \quad e = \left( \begin{array}{c} x_i \\ X \setminus \{ x_i \} \\ y_i \\ A \end{array} \right), \tag{3.10}
\]

where \( \delta(X \setminus \{ b_i \}) \) and \( e(X \setminus \{ x_i \}) \) are bijections. Then \( \delta, e \in BL(q) \) and \( \beta = \delta\alpha\epsilon \), that is, \( G_r \subseteq BL(q) \cdot \alpha \cdot BL(q) \) and equality follows.

Now we can describe all maximal subsemigroups of \( PS(q) \) when \( p > q \).

**Theorem 3.5.** Suppose that \( p > q \geq \aleph_0 \). Then \( M \) is a maximal subsemigroup of \( PS(q) \) if and only if \( M \) equals one of the following sets:

(a) \( PS(q) \setminus G_p = \{ \alpha \in PS(q) : g(\alpha) < p \} \);

(b) \( N \cup T_r \), where \( q \leq r < p \) and \( N \) is a maximal subsemigroup of \( S_r \).

**Proof.** Let \( \alpha, \beta \in PS(q) \) be such that \( g(\alpha) < p \) and \( g(\beta) < p \). Clearly \( |X\alpha \setminus \text{dom } \beta| \leq |X \setminus \text{dom } \beta| = g(\beta) < p \). Then

\[
    |\text{dom } \alpha \setminus \text{dom } (\alpha\beta)| = \left| (X\alpha \setminus (X\alpha \cap \text{dom } \beta)) \alpha^{-1} \right| \tag{3.11}
\]

\[
    = \left| (X\alpha \setminus \text{dom } \beta) \alpha^{-1} \right| \tag{3.11}
\]

\[
    = |X\alpha \setminus \text{dom } \beta| < p.
\]

Hence,

\[
    |X \setminus \text{dom } (\alpha\beta)| = |X \setminus \text{dom } \alpha| + |\text{dom } \alpha \setminus \text{dom } (\alpha\beta)| < p, \tag{3.12}
\]
and this shows that $PS(q) \setminus G_p$ is a subsemigroup of $PS(q)$. To show that $PS(q) \setminus G_p$ is maximal in $PS(q)$, we let $\alpha, \beta \in PS(q) \setminus (PS(q) \setminus G_p) = G_p$. By Lemma 3.4, $\alpha = \lambda \beta \mu$ for some $\lambda, \mu \in BL(q) \subseteq PS(q) \setminus G_p$. Thus, $\alpha$ can be written as a finite product of elements of $(PS(q) \setminus G_p) \cup \{\beta\}$, and hence $PS(q) \setminus G_p$ is maximal in $PS(q)$. Also, if $q \leq r < p$ and $N$ is a maximal subsemigroup of $S_r$, then $N \cup T_r$ is maximal in $PS(q)$ by Corollary 3.2.

We now suppose that $M$ is a maximal subsemigroup of $PS(q)$ such that $M \neq PS(q) \setminus G_p$. Then there exists $\alpha \notin M$ with $g(\alpha) < p$. Thus, Lemma 3.3 implies that $S_k \setminus M \neq \emptyset$ and $S_k \cap M \neq \emptyset$ for some $k$, where $q \leq k < p$. Since $PS(q) = S_k \cup T_k$, Lemma 3.1(b) implies that $S_k \cap M$ is maximal in $S_k$. We also see that

$$M = (S_k \cap M) \cup (T_k \cap M) \subseteq (S_k \cap M) \cup T_k,$$

where $(S_k \cap M) \cup T_k$ is maximal in $PS(q)$ by Corollary 3.2. This means that $M = (S_k \cap M) \cup T_k$ by the maximality of $M$.

By the previous theorem, when $p > q$, most of the maximal subsemigroups of $PS(q)$ are induced by maximal subsemigroups of $S_r$ where $q \leq r < p$. Hence we now determine some maximal subsemigroups of $S_r$.

As mentioned in Section 1, for every nonempty subset $A$ of $X$ with $|X \setminus A| \geq q$, $M_A$ is a maximal subsemigroup of $BL(q)$. Here we extend the definition of $M_A$ and consider the set $\overline{M_A}$ defined as

$$\overline{M_A} = \{ \alpha \in PS(q) : A \nsubseteq X\alpha \text{ or } (A\alpha \subseteq A \subseteq \text{dom}\, \alpha \text{ or } |X\alpha \setminus A| < q) \},$$

that is, $\alpha$ in $PS(q)$ belongs to $\overline{M_A}$ if and only if

(a) $A \nsubseteq X\alpha$, or

(b) $A \subseteq X\alpha$ and either $A\alpha \subseteq A \subseteq \text{dom}\, \alpha$, or $|X\alpha \setminus A| < q$.

The next result gives more detail on $\overline{M_A}$.

**Lemma 3.6.** Suppose that $p \geq q \geq \aleph_0$, and let $A$ be a nonempty subset of $X$ such that $|X \setminus A| \geq q$. Then,

(a) for any cardinal $k$ such that $0 \leq k \leq p$, there exist $\alpha, \beta \in PS(q)$ such that $g(\alpha) = k = g(\beta)$ and $\alpha \in \overline{M_A}$, $\beta \notin \overline{M_A}$;

(b) for each $\gamma \notin \overline{M_A}$, $|\text{dom}\, \gamma \setminus A^{\gamma^{-1}}| = |X \setminus A| = |X\gamma \setminus A|$ and $|A^{\gamma^{-1}}| = |A|$.

**Proof.** To show that (a) holds, let $|X \setminus A| = r \geq q$, and let $k$ be a cardinal such that $0 \leq k \leq p$. We write $X \setminus A = R \cup Q$ where $|R| = r$ and $|Q| = q$. If $r = p$, then $|A \cup R| = r = p$; if not, then $|X \setminus A| < p$, and this implies $|A| = p$, and so $|A \cup R| = p$. Fix $a \in A$ and let $B = (A \setminus \{a\}) \cup R$. Then, $|B| = p$ and $|X \setminus B| = |Q \cup \{a\}| = q$. We write $X = K \cup L$ where $|K| = k$ and $|L| = p$. Then there exists a bijection $\alpha : L \to B$ and so $g(\alpha) = k$, $d(\alpha) = q$. Also, since $A \nsubseteq B = X\alpha$, we have $\alpha \in \overline{M_A}$. 
To find $\beta \in PS(q) \setminus \overline{M_A}$ with $g(\beta) = k$, we consider two cases. First, if $r = p$, we write $X \setminus A = P \cup Q \cup K$ where $|P| = p$, $|Q| = q$, $|K| = k$. Fix $a \in A$ and define

$$\beta = \left( \frac{P \cup Q \cup \{a\}}{P \cup K \cup \{a\}} \right) (X \setminus A \setminus \{a\})$$

(3.15)

where $\beta|_{(P \cup Q \cup \{a\})}$ and $\beta|_{(A \setminus \{a\})}$ are bijections and $a\beta \neq a$. On the other hand, if $r < p$, then $|A| = p$. In this case we write $A = A' \cup K'$ and $X \setminus A = R \cup Q$ where $|A'| = p$, $|K'| = k$, $|R| = r$ and $|Q| = q$. Fix $a \in A'$ and redefine

$$\beta = \left( \frac{(X \setminus A \setminus \{a\}) \setminus \{a\}}{R \cup \{a\}} \right)$$

(3.16)

where $\beta|_{(X \setminus A \setminus \{a\})}$ and $\beta|_{(A' \setminus \{a\})}$ are bijections and $a\beta \neq a$. In both cases, we have $d(\beta) = q$, $g(\beta) = k$, $A \subseteq X\beta$, $A\beta \subseteq A$, and $|X\beta \setminus A| \geq q$, that is $\beta \in PS(q) \setminus \overline{M_A}$.

To see that (b) holds, suppose that there is $\gamma \not\in \overline{M_A}$, then $A \subseteq X\gamma$ and $|X\gamma \setminus A| \geq q$. So $|A\gamma^{-1}| = |A|$ since $\gamma$ is injective. Also,

$$X \setminus A = (X \setminus X\gamma) \cup (X\gamma \setminus A),$$

(3.17)

where $|X \setminus X\gamma| = q$. Since $|X \setminus A| \geq q$ and by our assumption $|X\gamma \setminus A| \geq q$, we have $|X \setminus A| = |X\gamma \setminus A| = |X\gamma \setminus A\gamma^{-1}| = |\text{dom } \gamma \setminus A\gamma^{-1}|$ as required. \qed

In [6, Theorem 1], the authors proved that $M_A$ is a maximal subsemigroup of $BL(q)$ for every nonempty subset $A$ of $X$ such that $|X \setminus A| \geq q$. Using a similar argument, we show that $M_A$ is a subsemigroup of $PS(q)$.

**Lemma 3.7.** Suppose that $p \geq q \geq \aleph_0$, and let $A$ be a nonempty subset of $X$ such that $|X \setminus A| \geq q$. Then $M_A$ is a proper subsemigroup of $PS(q)$.

**Proof.** Let $\alpha, \beta \in M_A$. If $A \not\subseteq X\alpha\beta$, then $a\beta \in M_A$. Now we suppose that $A \subseteq X\alpha\beta$. Then, $A \subseteq X\beta$ and since $\beta \in M_A$, we either have $A\beta \subseteq A \subseteq \text{dom } \beta$, or $|X\beta \setminus A| < q$. If $|X\beta \setminus A| < q$, then

$$|X\alpha\beta \setminus A| \leq |X\beta \setminus A| < q$$

(3.18)

and so $a\beta \in M_A$. Otherwise, we have $A\beta \subseteq A \subseteq X\alpha\beta$ and hence $A \subseteq X\alpha$ since $\beta$ is injective. Since $a \in M_A$, we either have $A\alpha \subseteq A \subseteq \text{dom } a$, or $|X\alpha \setminus A| < q$. If the latter occurs, then

$$|X\alpha\beta \setminus A| \leq |X\alpha \beta \setminus A\beta| = |(X\alpha \setminus A)\beta| \leq |X\alpha \setminus A| < q,$$

(3.19)

therefore $a\beta \in M_A$. On the other hand, if $A\alpha \subseteq A \subseteq \text{dom } a$, we have $A\alpha\beta \subseteq A\beta \subseteq A$. Moreover, $A\alpha \subseteq X\alpha \cap \text{dom } \beta$, that is, $A \subseteq (X\alpha \cap \text{dom } \beta)^{-1} = \text{dom}(a\beta)$. Therefore $a\beta \in M_A$, and hence
$\overline{M}_A$ is a subsemigroup of $PS(q)$. Finally, this subsemigroup is properly contained in $PS(q)$ by Lemma 3.6(a).

Remark 3.8. For any cardinal $r$ such that $q \leq r \leq p$, $S_r \cap \overline{M}_A$ is a proper subsemigroup of $S_r$ but it is not maximal when $q < r$. To see this, suppose $S_r \cap \overline{M}_A$ is maximal and choose $\alpha, \beta \notin \overline{M}_A$ such that $g(\alpha) = r$ and $g(\beta) = 0$ (possible by Lemma 3.6(a)). Then $\alpha, \beta \in S_r \setminus \overline{M}_A$ where $\text{dom } \beta = X$. Moreover $\langle (S_r \cap \overline{M}_A) \cup \{\alpha\} \rangle = S_r$, and so

$$\beta = \gamma_1 \gamma_2 \cdots \gamma_n \alpha \lambda_1 \lambda_2 \cdots \lambda_m$$

for some $n, m \in \mathbb{N}_0$ and $\gamma_i, \lambda_j \in (S_r \cap \overline{M}_A) \cup \{\alpha\}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. If $n = 0$ or $\gamma_1 = \alpha$, then $\text{dom } \beta \subseteq \text{dom } \alpha$ and so $g(\alpha) = 0$, a contradiction. Thus, $n \neq 0$ and $\gamma_1 \neq \alpha$. Since $X = \text{dom } \beta \subseteq \text{dom } (\gamma_1 \gamma_2 \cdots \gamma_n)$, it follows that $\gamma = \gamma_1 \gamma_2 \cdots \gamma_n \in BL(q)$. Moreover, $X \gamma \subseteq \text{dom } \alpha$, and this implies,

$$q \leq r = |X \setminus \text{dom } \alpha| \leq |X \setminus X \gamma| = q,$$

and hence $r = q$.

Since $M_A$ is maximal in $BL(q)$, a subsemigroup of $PS(q)$, it is natural to think that $\overline{M}_A$ is maximal in $PS(q)$. But when $p > q$, by taking $r = p$, the above observation shows that this claim is false since $S_p = PS(q)$. Thus, $\overline{M}_A$ is not always a maximal subsemigroup of $PS(q)$.

The proof of the next result follows some ideas from [6, Theorem 1].

**Theorem 3.9.** Suppose that $p \geq r \geq q \geq \aleph_0$, and let $A$ be a nonempty subset of $X$ such that $|X \setminus A| \geq q$. Then $S_r \cap \overline{M}_A$ is a maximal subsemigroup of $S_r$, precisely when $r = q$.

**Proof.** In Remark 3.8, we have shown that $S_r \cap \overline{M}_A$ is not maximal in $S_r$ when $r > q$. It remains to show $S_q \cap \overline{M}_A$ is maximal in $S_q$. Let $\alpha, \beta \in S_q \setminus \overline{M}_A$. Then $g(\alpha), g(\beta) \leq q$ and Lemma 3.6(b) implies that

$$\left| AA^{-1} \right| = |A| = \left| A\beta^{-1} \right|,$$

$$\left| \text{dom } \alpha \setminus AA^{-1} \right| = \left| \text{dom } \beta \setminus A\beta^{-1} \right| = |X \beta \setminus A| = |X \alpha \setminus A| = |X \setminus A| = s \quad \text{(say)} \geq q.$$

We also have $A\beta \notin A$ or $A \notin \text{dom } \beta$. In the case that $A\beta \notin A$, we have $A\beta \cap (X \setminus A) \neq \emptyset$. Thus, there exists $y \in A \cap (X \setminus A)\beta^{-1}$, so $y \notin A\beta^{-1}$. Since $|\text{dom } \beta \setminus (A\beta^{-1} \cup \{y\})| = s$, we can write

$$\text{dom } \beta \setminus (A\beta^{-1} \cup \{y\}) = \{c_i\} \cup \{d_k\},$$
where \(|J| = s\) and \(|K| = q\). Also, since \(\alpha, \beta \not\in \overline{M}_A\), we have \(A \subseteq X\alpha\) and \(A \subseteq X\beta\). Thus, for convenience, write \(A = \{a_i\}\), let \(y_i, z_i \in X\) be such that \(y_i \alpha = a_i = z_i \beta\) for each \(i\), and let

\[
\beta = \begin{pmatrix}
  z_i & c_j & d_k & y \\
  a_i & c_j \beta & d_k \beta & y \beta
\end{pmatrix}.
\]

(3.24)

Now define \(\gamma \in P(X)\) by

\[
\gamma = \begin{pmatrix}
  y_i & b_j \\
  z_i & c_j
\end{pmatrix}.
\]

(3.25)

Then \(d(\gamma) = |\{d_k\} \cup \{y\}| + g(\beta) = q\), that is, \(\gamma \in \text{PS}(q)\). Also, since \(\text{dom } \gamma = \text{dom } \alpha\), we have \(g(\gamma) = g(\alpha) \leq q\) and so \(\gamma \in S_q\). Moreover, since \(y \in A\) and \(y \not\in X\gamma\), we have \(A \not\subseteq X\gamma\), that is, \(\gamma \not\in \overline{M}_A\). Also, since \(d(\alpha) = q\), we can write \(X \setminus X\alpha = \{m_k\} \cup \{n_k\} \cup \{z\}\) and define \(\mu \in P(X)\) by

\[
\mu = \begin{pmatrix}
  a_i & c_j \beta & d_k \beta & y \beta \\
  a_i & b_j \alpha & m_k & z
\end{pmatrix}.
\]

(3.26)

Then \(d(\mu) = |\{n_k\}| = q = d(\beta) = g(\mu)\), that is, \(\mu \in S_q\). Moreover, \(\mu \in \overline{M}_A\) since \(A\mu = A \subseteq \text{dom } \mu\). Finally, we can see that \(\alpha = \gamma \beta \mu\) where \(\gamma, \mu \in S_q \cap \overline{M}_A\).

On the other hand, if \(A \not\subseteq \text{dom } \beta\), then there exists \(w \in A \cap (X \setminus \text{dom } \beta)\). In this case, we rewrite \(\text{dom } \beta \setminus A\beta^{-1} = \{c_j\} \cup \{d_k\}\) and \(X \setminus X\alpha = \{m_k\} \cup \{n_k\}\) where \(|J| = s, |K| = q\). Like before, we write \(A = \{a_i\}\) and \(\text{dom } \alpha = \{y_i\} \cup \{b_j\}\) where \(\{b_j\} = \text{dom } \alpha \setminus A\alpha^{-1}\), then

\[
\beta = \begin{pmatrix}
  z_i & c_j & d_k \\
  a_i & c_j \beta & d_k \beta
\end{pmatrix}.
\]

(3.27)

Define \(\gamma, \mu \in P(X)\) by

\[
\gamma = \begin{pmatrix}
  y_i & b_j \\
  z_i & c_j
\end{pmatrix}, \quad \mu = \begin{pmatrix}
  a_i & c_j \beta & d_k \beta \\
  a_i & b_j \alpha & m_k
\end{pmatrix}.
\]

(3.28)

Then, \(d(\gamma) = |\{d_k\}| + g(\beta) = q, g(\gamma) = g(\alpha) \leq q, d(\mu) = |\{n_k\}| = q = d(\beta) = g(\mu)\), and so \(\gamma, \mu \in S_q\). Also, \(\gamma, \mu \in \overline{M}_A\) since \(A \not\subseteq X\gamma\) (note that \(w \in A \setminus \text{dom } \beta \subseteq A \setminus X\gamma\)) and \(A\mu = A \subseteq \text{dom } \mu\). Moreover, \(\alpha = \gamma \beta \mu\). In other words, we have shown that for every \(\alpha, \beta \in S_q \cap \overline{M}_A\), \(\alpha\) can be written as a finite product of elements of \((S_q \cap \overline{M}_A) \cup \{\beta\}\). Therefore, \(S_q \cap \overline{M}_A\) is maximal in \(S_q\).

We now determine some other classes of maximal subsemigroups of \(S_r\).
Lemma 3.10. Suppose that $p \geq r \geq q \geq \aleph_0$. Let $k$ be a cardinal such that $k = 0$ or $q \leq k \leq r$. Then

$$S_r \setminus G_k = \{ \alpha \in PS(q) : k \neq g(\alpha) \leq r \}$$

is a proper subsemigroup of $S_r$.

Proof. Since $k \leq r$, we have $S_r \setminus G_k \subseteq S_r$. If $k = 0$, then $S_r \setminus G_0 = S_r \setminus BL(q)$, and this is a subsemigroup of $S_r$ since, for $\alpha, \beta \in S_r \setminus BL(q)$, $\text{dom}(\alpha \beta) \subseteq \text{dom} \alpha \subseteq X$, and this implies $\alpha \beta \in S_r \setminus BL(q)$. Now suppose $q \leq k \leq r$ and let $\alpha, \beta \in S_r$ be such that $g(\alpha \beta) = k$. We claim that $g(\alpha) = k$ or $g(\beta) = k$. To see this, assume that $g(\alpha) \neq k$. Since

$$k = |X \setminus \text{dom}(\alpha \beta)| = |X \setminus \text{dom} \alpha| + |\text{dom} \alpha \setminus \text{dom}(\alpha \beta)|,$$

we have $|X \setminus \text{dom} \alpha| < k$, thus

$$k = |\text{dom} \alpha \setminus \text{dom}(\alpha \beta)| = |(X \alpha \setminus (X \alpha \cap \text{dom} \beta)) \alpha^{-1}|$$

$$= |(X \alpha \setminus \text{dom} \beta) \alpha^{-1}| = |X \alpha \setminus \text{dom} \beta|.$$

Note that

$$X \setminus \text{dom} \beta = [(X \alpha \setminus \text{dom} \beta)] \cup [(X \setminus X \alpha) \cap (X \setminus \text{dom} \beta)],$$

where the intersection on the right has cardinal at most $q$. Hence, $g(\beta) = |X \setminus \text{dom} \beta| = k$ and we have shown that $S_r \setminus G_k$ is a subsemigroup of $S_r$. \hfill \Box

Remark 3.11. Observe that, if $0 < k < q$ then $S_r \setminus G_k$ is not a semigroup for all $q \leq r \leq p$. To see this, let $\alpha \in BL(q)$ and $\beta = \text{id}_{X \alpha, K}$ for some subset $K$ of $X \alpha$ such that $|K| = k$ (possible since $|X \alpha| = p > k$), then $\alpha, \beta \in PS(q)$ since $d(\beta) = d(\alpha) + k = q$. Moreover, since $g(\alpha) = 0$ and $g(\beta) = q \neq k$, we have $\alpha, \beta \in S_r \setminus G_k$. But

$$\text{dom}(\alpha \beta) = (X \alpha \cap \text{dom} \beta) \alpha^{-1} = (X \alpha \setminus K) \alpha^{-1} = X \setminus K \alpha^{-1},$$

thus $g(\alpha \beta) = |K \alpha^{-1}| = k$, that is, $\alpha \beta \in G_k$.

Theorem 3.12. Suppose that $p \geq r \geq q \geq \aleph_0$. Then the following statements hold:

(a) $S_r \setminus G_0$ is a maximal subsemigroup of $S_r$;

(b) if $p > q$, then for each cardinal $k$ such that $q \leq k \leq r$, $S_r \setminus G_k$ is a maximal subsemigroup of $S_r$.

Proof. By Lemma 3.10, $S_r \setminus G_0$ is a subsemigroup of $S_r$. To see that it is maximal, let $\alpha, \beta \in G_0 = BL(q) \subseteq S_q$. By [3, Theorem 5], $S_q = \beta \cdot R(q)$, and this implies that $\alpha = \beta \gamma$ for some $\gamma \in R(q) \subseteq S_r \setminus G_0$. Hence $S_r \setminus G_0$ is maximal in $S_r$.

Now suppose that $p > q$ and let $q \leq k \leq r$. Let $\alpha, \beta \in G_k$. If $k = q$, then $G_k = R(q) \subseteq S_q$ and, by [3, Theorem 6], $S_q = BL(q) \cdot \beta \cdot BL(q)$. If $k > q$, then $G_k = BL(q) \cdot \beta \cdot BL(q)$ (by
Lemma 3.4). Therefore, $\alpha = \gamma \beta \mu$ for some $\gamma, \mu \in BL(q) \subseteq S_r \setminus G_k$, and so $S_r \setminus G_k$ is maximal in $S_r$. \hfill \square

**Corollary 3.13.** Suppose that $p > q \geq \aleph_0$ and let $A$ be a nonempty subset of $X$ such that $|X \setminus A| \geq q$. Then the following statements hold:

(a) $\overline{MA} \cup T_q$;
(b) $N_k = \{ \alpha \in PS(q) : g(\alpha) \neq k \}$ where $k = 0$ or $q \leq k \leq p$.

**Proof.** By Theorem 3.9, $S_q \cap \overline{MA}$ is maximal in $S_q$. Then Corollary 3.2 implies that $(S_q \cap \overline{MA}) \cup T_q$ is maximal in $PS(q)$. But

\[ (S_q \cap \overline{MA}) \cup T_q = (S_q \cup T_q) \cap (\overline{MA} \cup T_q) = PS(q) \cap (\overline{MA} \cup T_q) = \overline{MA} \cup T_q, \quad (3.34) \]

and so (a) holds. To show that (b) holds, let $r = p$ in Theorem 3.12. Then $S_p = PS(q)$ and thus $N_k = S_p \setminus G_k$ is maximal in $PS(q)$. \hfill \square

**Theorem 3.14.** Suppose that $p > q \geq \aleph_0$ and $k$ equals $0$ or $q$. Let $A$ be a nonempty subset of $X$ such that $|X \setminus A| \geq q$. Then the two classes of maximal subsemigroups $S_q \cap \overline{MA}$ and $S_q \setminus G_k$ of $S_q$ are always disjoint.

**Proof.** By Theorems 3.9 and 3.12, $S_q \cap \overline{MA}$ and $S_q \setminus G_k$ are maximal subsemigroups of $S_q$. By Lemma 3.6(a), there exists $\alpha \in \overline{MA}$ with $g(\alpha) = k$. Then $\alpha \in S_k \cap \overline{MA} \subseteq S_q \cap \overline{MA}$ but $\alpha \notin S_q \setminus G_k$, that is, $S_q \cap \overline{MA} \notin S_q \setminus G_k$. Also, $S_q \setminus G_k \notin S_q \cap \overline{MA}$ by the maximality of $S_q \cap \overline{MA}$ and $S_q \setminus G_k$. Therefore, $S_q \cap \overline{MA}$ is not equal to $S_q \setminus G_k$. \hfill \square

**4. Maximal Subsemigroups of $PS(q)$ When $p = q$**

We first recall that, when $p = q$, the empty transformation $\emptyset$ belongs to $PS(q)$ since $d(\emptyset) = p = q$. In this case, the ideals of $PS(q)$ are precisely the sets:

\[ J_r = \{ \alpha \in PS(q) : r(\alpha) < r \}, \quad (4.1) \]

where $1 \leq r \leq p'$ (see [3, Theorem 14]). Clearly, $J_p' = PS(q)$ and $J_p = \{ \alpha \in PS(q) : r(\alpha) < p \}$ is the largest proper ideal. In this case, the complement of each $J_r$ in $PS(q)$ is not a semigroup. To see this, write $X = A \cup B \cup C$ where $|A| = p$ and $|B| = r = |C|$. Then $id_B, id_C \in PS(q) \setminus J_r$, whereas $id_B, id_C = \emptyset \in J_r$. Hence, unlike what was done in Section 3, we cannot use Lemma 3.1 to find maximal subsemigroups of $PS(q)$ when $p = q$. In this section, we determine some maximal subsemigroups of $PS(q)$, for $p = q$, using a different approach. We first describe some properties of each maximal subsemigroup in this case.

**Lemma 4.1.** Suppose that $p = q \geq \aleph_0$ and $M$ is a maximal subsemigroup of $PS(q)$. Then the following statements hold:

(a) $M$ contains all $\alpha \in PS(q)$ with $r(\alpha) < p$,
(b) if $R(q) \subseteq M$, then $M \cap BL(q) = \emptyset$. 


Proof. Suppose that there exists $\alpha \notin M$ with $r(\alpha) = k < p$. Then $g(\alpha) = p$, and we write in the usual way

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}.\quad (4.2)$$

Also, write $X \setminus \{a_i\} = P \cup Q$ and $X \setminus \{x_i\} = R \cup S$ where $|P| = |Q| = p = |R| = |S|$, and define $\beta, \gamma$ in $P(X)$ by

$$\beta = \begin{pmatrix} a_i \\ P \end{pmatrix}, \quad \gamma = \begin{pmatrix} a_i \\ Q \end{pmatrix}.$$ \quad (4.3)

where $\beta|P$ and $\gamma|Q$ are bijections. Then $\beta, \gamma \in PS(q)$. Also,

$$\alpha = \beta \cdot \alpha \cdot \text{id}_{X,\alpha} \in PS(q) \cdot \alpha \cdot PS(q),$$ \quad (4.4)

thus $M \subseteq M \cup (PS(q) \cdot \alpha \cdot PS(q))$. But $M \cup (PS(q) \cdot \alpha \cdot PS(q))$ is a subsemigroup of $PS(q)$ and this means that $M \cup (PS(q) \cdot \alpha \cdot PS(q)) = PS(q)$ by the maximality of $M$. Since all mappings in $PS(q) \cdot \alpha \cdot PS(q)$ have rank at most $k$, it follows that $M$ contains all mappings with rank greater than $k$. Therefore $\beta, \gamma \in M$ and thus $\alpha = \beta \gamma \in M$, a contradiction.

To show that (b) holds, suppose that $R(q) \subseteq M$. If there exists $\alpha \in M \cap BL(q)$, then $[3, \text{Theorem 5}]$ implies that $PS(q) = \alpha \cdot R(q) \subseteq M$ (note that $S_q = PS(q)$ when $p = q$), so $M = PS(q)$, contrary to the maximality of $M$. Thus $M \cap BL(q) = \emptyset$. \qed

Remark 4.2. If $p > q$, then every $\alpha \in PS(q)$ has rank $p$. This contrasts with Lemma 4.1(a). Also, by Corollary 3.13, if $p > q$ and $q < k \leq p$, $N_k$ is a maximal subsemigroup of $PS(q)$ containing $R(q) \cup BL(q)$, this contrasts with Lemma 4.1(b).

As in Section 3, for any cardinal $k$, we let

$$N_k = \{ \alpha \in PS(q) : g(\alpha) \neq k \}.$$

By Lemma 3.10 and Remark 3.11, if $p = q$, then $N_k$ is a subsemigroup of $PS(q)$ exactly when $k = 0$ or $k = p$. From Corollary 3.13, when $p > q$, $N_k$ is a maximal subsemigroup of $PS(q)$. But when $p = q$, Lemma 4.1(b) implies that $N_p$ is not maximal since $\emptyset \notin N_p$. Moreover, Lemma 4.1(a) implies that every maximal subsemigroup of $PS(q)$ must contain the largest proper ideal

$$J_p = \{ \alpha \in PS(q) : r(\alpha) < p \}.$$ \quad (4.6)

Note that $J_p$ itself is a subsemigroup of $PS(q)$, but it is not maximal since $J_p \subseteq R(q)$ (in case $p = q$, $r(\alpha) < p$ implies $g(\alpha) = p$).
Theorem 4.3. Suppose that \( p = q \geq \aleph_0 \) and let \( A \) be a nonempty subset of \( X \) such that \( |X \setminus A| \geq q \). The following are maximal subsemigroups of \( PS(q) \):

(a) \( \overline{M}_A \);

(b) \( N_0 \);

(c) \( N_p \cup J_p \).

Proof. If \( p = q \), then \( S_q = PS(q) \), and so (a) holds by Theorem 3.9. Also, by taking \( r = p \) in Theorem 3.12(a), we see that (b) holds. To show that (c) holds, take \( r = p = k \) in Lemma 3.10, we have \( N_p = S_p \setminus G_p \) is a subsemigroup of \( PS(q) \). Moreover, \( N_p \cup J_p \) is also a subsemigroup of \( PS(q) \) since \( J_p \) is an ideal. To show the maximality of \( N_p \cup J_p \), let \( \alpha, \beta \in PS(q) \setminus (N_p \cup J_p) \). Then \( g(\alpha) = g(\beta) = p = r(\alpha) = r(\beta) \). Write in the usual way

\[
\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \quad \beta = \begin{pmatrix} b_i \\ y_i \end{pmatrix},
\]

where \( |I| = p \), and let

\[
X \setminus \{a_i\} = A \cup B, \quad X \setminus \{b_i\} = C \cup D,
\]

where \( |A| = |B| = |C| = |D| = p \). Then define \( \gamma, \mu \in P(X) \) by

\[
\gamma = \begin{pmatrix} b_i \\ X \setminus \{b_i\} \\ a_i \\ A \end{pmatrix}, \quad \mu = \begin{pmatrix} x_i \\ X \setminus \{x_i\} \\ y_i \\ C \end{pmatrix},
\]

where \( \gamma \mid (X \setminus \{b_i\}) \) and \( \mu \mid (X \setminus \{x_i\}) \) are bijections. Thus \( \gamma, \mu \in PS(q) \) since \( d(\gamma) = |B| = p = |D| = d(\mu) \). Moreover \( \gamma, \mu \in N_p \cup J_p \) since \( g(\gamma) = g(\mu) = 0 < p \). It is clear that \( \beta = \gamma \alpha \mu \) and therefore \( N_p \cup J_p \) is maximal in \( PS(q) \).

Remark 4.4. When \( p = q \), if \( M \) is a maximal subsemigroup containing \( R(q) \), then

\[
M \subseteq (PS(q) \setminus BL(q)) = N_0
\]

by Lemma 4.1(b). Thus, \( M = N_0 \) by the maximality of \( M \). So we conclude that \( N_0 \) is the only maximal subsemigroup of \( PS(q) \) containing \( R(q) \).

Remark 4.5. As we showed in Section 3, to see all maximal subsemigroups of \( PS(q) \) when \( p > q \), it is necessary to describe all maximal subsemigroups of \( S_r \) where \( q \leq r < p \). So we leave this as a direction for future research.

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