Research Article

On a Gauss-Kuzmin-Type Problem For a Generalized Gauss-Kuzmin Operator

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Received 20 April 2011; Accepted 15 June 2011

Academic Editor: Naseer Shahzad

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A generalized limit probability measure associated with a random system with complete connections for a generalized Gauss-Kuzmin operator, only for a special case, is defined, and its behaviour is investigated. As a consequence a specific version of Gauss-Kuzmin-type problem for the above generalized operator is obtained.

1. Introduction

Let \( Y = C([0,1]) \) be the Banach space of complex-valued continuous functions on \([0,1]\) under the supremum norm, and let \( N^* = \{1,2,\ldots\}, N = \{0,1,2,\ldots\} \). Then for every \( f \in Y \) and for every \( \alpha \geq 1 \) the function \( G_\alpha f \) introduced by Fluch [1] and defined by

\[
(G_\alpha f)(w) = \sum_{x \in N^*} \frac{\alpha^2}{(ax + w) \cdot (ax + \alpha - 1 + w)} \cdot f\left(\frac{\alpha}{ax + \alpha - 1 + w}\right),
\]

for all \( w \in [0,1] \), is called a generalized Gauss-Kuzmin operator.

The present paper arises as an attempt to determine a generalized limit probability measure, only for a special case, associated with a random system with complete connections for the above generalized Gauss-Kuzmin operator obtained in Ganatsiou [2], for every \( \alpha > 2 \). This will give us the possibility to obtain a specific variant of Gauss-Kuzmin-type problem for the above operator.

Our approach is given in the context of the theory of dependence with complete connections (see Iosifescu and Grigorescu [3]). For a more detailed study of the theory and
applications of dependence with complete connections to the metrical problems and other interesting aspects of number theory we refer the reader to [4–9] and others.

The paper is organized as follows. In Section 2, we present all the necessary results regarding the ergodic behaviour of a random system with complete connections associated with the generalized Gauss-Kuzmin operator $G_\alpha$ obtained in [2], in order to make more comprehensible the presentation of the paper. In Section 3, we introduce the determination of a limit probability measure associated with the above random system with complete connections, only for a special case, for every $\alpha > 2$, which will give us the possibility to study in Section 4 a specific version of the associated Gauss-Kuzmin type problem.

2. Auxiliary Results

For every $\alpha \geq 1$, we consider the function $\rho_\alpha$ defined by

$$\rho_\alpha(w) = \frac{\alpha}{\alpha + w}, \quad w \in [0,1],$$

and set

$$g_n = \frac{G^n f}{\rho_\alpha}, \quad n \in \mathbb{N},$$

where $G^{n+1} f = G_\alpha(G^n f)$, for every $n \in \mathbb{N}$ and for every $f \in Y$.

Then we obtain the following statement which gives a relation deriving from an analogous of the Gauss-Kuzmin type equation.

**Proposition 2.1.** The function $g_n$ satisfies

$$g_{n+1}(w) = \sum_{x \in \mathbb{N}^*} \frac{\alpha \cdot (\alpha + w)}{(ax + w) \cdot (ax + \alpha + w)} \cdot g_n\left(\frac{\alpha}{ax + \alpha - 1 + w}\right),$$

for any $n \in \mathbb{N}$ and $w \in [0,1]$.

Furthermore we obtain the following.

**Proposition 2.2.** For every $\alpha \geq 1$, the function

$$P_\alpha(w,x) = \frac{\alpha \cdot (\alpha + w)}{(ax + w) \cdot (ax + \alpha + w)}, \quad w \in [0,1], \quad x \in \mathbb{N}^*,$$

defines a transition probability function from $([0,1], B_{[0,1]})$ to $(X, P(X))$, where $X = \mathbb{N}^*$ and $P(X)$ the power set of $X$.

Equation (2.3) and Proposition 2.2 lead to the consideration of a family of random systems with complete connections (RSCCs)

$$\{(W,W)(X,X), u_\alpha, P_\alpha\}, \quad \alpha \geq 1,$$
In the next, we consider the transition probability function $Q_\alpha$. This means that

$$W = [0,1], \quad W = B_{[0,1]}, \quad X = N^*, \quad X = P(X),$$

$$u_\alpha(w,x) = \frac{a}{ax + a - 1 + w}, \quad P_\alpha(w,x) = \frac{a \cdot (a + w)}{(ax + w) \cdot (ax + a + w)}, \quad w \in W, \ x \in X. \quad (2.6)$$

In the next, we consider the transition probability function $Q_\alpha$, $\alpha \geq 1$, of the Markov chain associated with the family of the RSCCs (2.5) and the corresponding Markov operator $U_\alpha$, $\alpha \geq 1$, defined by

$$U_\alpha f(w) = \sum_{n \in N^*} \frac{a \cdot (a + w)}{(ax + w) \cdot (ax + a + w)} \cdot f\left(\frac{a}{ax + a - 1 + w}\right), \quad (2.7)$$

for all complex-valued measurable bounded functions $f$ on $[0,1]$.

This gives us the possibility of obtain the following.

**Proposition 2.3.** The family of RSCCs (2.5) is with contraction. Moreover, its associated Markov operator $U_\alpha$ given by (2.7) is regular with respect to $L([0,1])$, the Banach space of all real-valued bounded Lipschitz functions on $[0,1]$.

On the contrary the RSCC associated with a concrete piecewise fractional linear map (see Ganatsiou [10]) is not an RSCC with contraction since $r_1 = 1$ and its associated Markov chain is not compact and regular with respect to the set $L([0,1]),$ even though there exists a point $y^* \in (0,1)$, such that

$$\lim_{n \to \infty} \left| \sum_{n} (y) - y^* \right| = 0, \quad (2.8)$$

for all $y \in (0,1)$. This corrects the escape of [10] gives an RSCC associated with a concrete piecewise fractional linear map which is not uniformly ergodic (a special case of [4]).

By virtue of Proposition 2.3, it follows from [3, Theorem 3.4.5] that the family of RSCCs (2.5) is uniformly ergodic. Furthermore, Theorem 3.1.24 in [3] implies that, for every $\alpha \geq 1$, there exists a unique probability measure $\gamma_\alpha$ on $B_{[0,1]}$, which is stationary for the kernel $Q_\alpha$, such that

$$\lim_{n \to \infty} U_\alpha^n f = \int_{0}^{1} f \, d\gamma_\alpha, \quad f \in L([0,1]). \quad (2.9)$$

This means that

$$\gamma_\alpha(B) = \int_{0}^{1} Q_\alpha(w,B) \gamma_\alpha(dw), \quad (2.10)$$

where

$$Q_\alpha(w,B) = \sum_{x \in B_{w}} P_\alpha(w,x), \quad (2.11)$$
with
\[ B_w = \{ x \in N^* \mid u_x(w, x) \in B \} \quad \text{for every } B \in W, w \in [0, 1]. \tag{2.12} \]

Moreover, for some \( c > 0 \) and \( 0 < \theta < 1 \), we have
\[ \left\| U^n f - \int_0^1 f d\gamma_n \right\| \leq c \cdot \theta^n \cdot \| f \|_L, \tag{2.13} \]
for all \( n \in N^* \) and \( f \in L([0, 1]) \), where \( \| \cdot \|_L \) denotes the usual norm in \( L([0, 1]) \), where
\[ U^n f = \int_0^1 f(w) \gamma_n(dw). \tag{2.14} \]

In general the form of the limit probability measure associated with the family of random systems with complete connections (2.5) cannot be determined but this is possible only for a special case as we prove in the following section.

For the proofs of the above results we refer the reader to Ganatsiou [2].

### 3. A Limit Probability Measure Associated with the Family of RSCCs

Now, we are able to determine a limit probability measure associated with the family of RSCCs (2.5) as is shown in the following.

**Proposition 3.1.** The probability measure \( \gamma_a \) has the density
\[ \rho_a(w) = \frac{a}{a + w}, \quad \text{for every } w \in [0, 1], \tag{3.1} \]
with constant \( 1/a \cdot \log(1 + a^{-1}) \) only for the special case \( a \cdot u^{-1} + 1 - a[u^{-1} + a^{-1}] < 1 \), for every \( a > 2, 0 < u \leq 1 \).

**Proof.** By virtue of uniqueness of \( \gamma_a \) we have to show that it satisfies relation (2.10). Since the intervals \([0, u]\), \( 0 < u \leq 1 \) generate \( B_{[0,1]} \) it is sufficient to verify (2.10) only for \( B = [0, u] \), \( 0 < u \leq 1 \).

Suppose that \( B = [0, u] \). Then, for every \( w \in [0, 1] \), we have
\[ B_w = \{ x \in N^* \mid u_x(w, x) \in [0, u] \} = \left\{ x \in N^* \mid \frac{a}{ax + a - 1 + w} < u \right\} \]
\[ = \left\{ x \in N^* \mid x \geq \left[ u^{-1} - w \cdot a^{-1} + a^{-1} \right] \right\}. \tag{3.2} \]

Hence by (2.11), we have that
\[ Q_a(w, [0, u]) = \frac{a + w}{a \left[ u^{-1} - w \cdot a^{-1} + a^{-1} \right] + w}. \tag{3.3} \]
where
\[
[u^{-1} - w \cdot \alpha^{-1} + \alpha^{-1}] = \begin{cases} 
[u^{-1} + \alpha^{-1}], & \text{if } 0 \leq w < \alpha \cdot u^{-1} + 1 - \alpha \cdot [u^{-1} + \alpha^{-1}], \\
[u^{-1} + \alpha^{-1}] - 1, & \text{if } \alpha \cdot u^{-1} + 1 - \alpha \cdot [u^{-1} + \alpha^{-1}] < w \leq 1.
\end{cases}
\] (3.4)

We consider the case \(\alpha u^{-1} + 1 - \alpha \cdot [u^{-1} + \alpha^{-1}] < 1\) or \(u^{-1} < [u^{-1} + \alpha^{-1}]\), for every \(\alpha > 2, 0 < u \leq 1\). Consequently, we obtain that
\[
\int_0^1 Q_a(w, [0, u)) \cdot \rho_a(w)dw = \frac{1}{\log(1 + \alpha^{-1})} \cdot \int_0^1 \frac{dw}{\alpha \cdot [u^{-1} - w \cdot \alpha^{-1} + \alpha^{-1}] + w}
\]
\[
= \frac{1}{\log(1 + \alpha^{-1})} \cdot \left[ \log(\alpha \cdot u^{-1} + 1) - \log(\alpha \cdot u^{-1} + 1 - \alpha) + \log(\alpha \cdot [u^{-1} + \alpha^{-1}] - \alpha + 1) - \log(\alpha \cdot [u^{-1} + \alpha^{-1}]) \right].
\] (3.5)

In the next we put
\[
I = \log(\alpha \cdot [u^{-1} + \alpha^{-1}] - \alpha + 1) - \log(\alpha \cdot [u^{-1} + \alpha^{-1}])
\]
\[
= \log\left(1 - \frac{1}{[u^{-1} + \alpha^{-1}]} + \frac{1}{\alpha \cdot [u^{-1} + \alpha^{-1}]}\right),
\] (3.6)
\[
II = \log(\alpha \cdot u^{-1} + 1) - \log(\alpha \cdot u^{-1} + 1 - \alpha)
\]
\[
= \log\left(1 + \frac{u}{\alpha}\right) - \log\left(1 + \frac{u}{\alpha - u}\right)
\]

By taking the limit of
\[
III = I - \log\left(1 + \frac{u}{\alpha - u}\right)
\]
\[
= \log\left(1 - \frac{1}{[u^{-1} + \alpha^{-1}]} + \frac{1}{\alpha \cdot [u^{-1} + \alpha^{-1}]}\right) - \log\left(1 + \frac{u}{\alpha - u}\right)
\] (3.7)

when \(u \to 1\) we have that
\[
\lim_{u \to 1} \log\left(1 + \frac{u}{\alpha - u}\right) = \log\left(\frac{1}{\alpha}\right),
\]
\[
\lim_{u \to 1} \log\left(1 - \frac{1}{[u^{-1} + \alpha^{-1}]} + \frac{1}{\alpha \cdot [u^{-1} + \alpha^{-1}]}\right) = \log\left(\frac{1}{\alpha}\right), \text{ for every } \alpha > 2.
\] (3.8)
So part III tends to 0 when \( u \to 1 \). This means that

\[
\lim_{u \to 1} \left[ \int_0^1 Q_\alpha(w, [0, u)) \cdot \rho_\alpha(w) \, dw \right] = \frac{1}{\log(1 + \alpha^{-1})} \cdot \lim_{u \to 1} \log \left( 1 + \frac{u}{\alpha} \right)
\]

which is equal to

\[
\lim_{u \to 1} \int_0^u \rho_\alpha(w) \, dw = \lim_{u \to 1} \int_0^u \frac{1}{\alpha} \cdot \log \left( 1 + \frac{\alpha + u}{\alpha} \right) \, dw \\
= \frac{1}{\log(1 + \alpha^{-1})} \lim_{u \to 1} \left[ \log(\alpha + u) - \log \alpha \right] \\
= \frac{1}{\log(1 + \alpha^{-1})} \lim_{u \to 1} \log \left( 1 + \frac{u}{\alpha} \right)
\]

and the proof is complete.

\[\square\]

4. A Version of the Gauss-Kuzmin-Type Problem

Let \( \mu \) be a nonatomic measure on the \( \sigma \)-algebra \( B_{[0,1]} \). Then we may define

\[ V_0(w) = \mu([0, w]), \]

\[ V_n(w) = V_n(w, \mu) = \int_0^w G_n^\alpha f(t) \, dt, \quad n \in \mathbb{N}^*, \ w \in [0, 1]. \]

Suppose that \( V'_0 \) exists and it is bounded (\( \mu \) has bounded density). Then by induction we have that \( V'_n \) exists and it is bounded for any \( n \in \mathbb{N}^* \) with

\[ V'_n(w) = G_n^\alpha f(w) = G_a \left( (G_a^{n-1} f)(w) \right), \quad f \in L([0,1]), \ n \in \mathbb{N}^*. \]

So

\[ \int_0^w V'_n(t) \, dt = \int_0^w G_n^\alpha f(t) \, dt, \quad V_n(w) = \int_0^w G_n^\alpha f(t) \, dt \]

while

\[ g_n(w) = \frac{G_n^\alpha f(w)}{p_\alpha(w)} = \frac{V'_n(w)}{p_\alpha(w)}, \quad n \in \mathbb{N}. \]

Now, we are able to determine the limit \( \lim_{n \to \infty} V_n(1/w) \) and to give the rate of this convergence, that is, a specific version of the associated Gauss-Kuzmin type problem.
Proposition 4.1. (i) If the density \( V'_0 \) of \( \mu \) is a Riemann integrable function, then

\[
\lim_{{n \to \infty}} V_n \left( \frac{1}{w} \right) = \frac{1}{\log(1 + a^{-1})} \cdot \log \left( \frac{aw + 1}{aw} \right), \quad \text{for all} \quad w \geq 1, \quad a > 0, \quad n \in N^*.
\]  

(ii) If the density \( V'_0 \) of \( \mu \) is an element of \( L([0, 1]) \), then there exist two positive constants \( c \) and \( \theta < 1 \) such that

\[
\lim_{{n \to \infty}} V_n \left( \frac{1}{w} \right) = (1 + q\theta^n) \cdot \frac{1}{\log(1 + a^{-1})} \cdot \log \left( \frac{aw + 1}{aw} \right),
\]  

for all \( w \geq 1, \quad a > 0, \quad n \in N^* \), where \( q = q(\mu, n, w) \) with \( |q| \leq c \).

Proof. Let \( V'_0 \in L([0, 1]) \). Then \( g_o \in L([0, 1]) \), and by using relation (2.14) we have

\[
U^\infty_{\alpha g_0} \equiv \lim_{{n \to \infty}} U^n_{\alpha g_0} = \int_0^1 g_0(w) \gamma_\alpha \left( dw \right) = \int_0^1 V'_0(w) \left( dw \right) = 1.
\]  

According to relation (2.13), there exist two positive constants \( c \) and \( \theta < 1 \) such that

\[
U^n_{\alpha g_0} = U^\infty_{\alpha g_0} + T^n_{\alpha g_0}, \quad \text{for every} \quad g_o \in C([0, 1])
\]  

If we consider the Banach space \( C([0, 1]) \) of all real continuous functions defined on \([0, 1]\) with the supremum norm, then since \( L([0, 1]) \) is a dense subset of \( C([0, 1]) \) we have

\[
\lim_{{n \to \infty}} |T^n_{\alpha g_0}| = 0, \quad \text{for every} \quad g_o \in C([0, 1]).
\]  

This means that it is valid for any measurable function \( g_o \), which is \( \gamma_\alpha \)-almost surely continuous, that is, for any Riemann integrable function \( g_o \). Consequently we obtain

\[
\lim_{{n \to \infty}} V_n \left( \frac{1}{w} \right) = \lim_{{n \to \infty}} \int_0^{1/w} U^n_{\alpha g_0}(t) \rho_\alpha(t) dt \\
= \int_0^{1/w} \rho_\alpha(t) dt = \int_0^{1/w} \frac{1}{\log(1 + a^{-1})} \cdot \frac{\alpha}{\alpha + t} dt \\
= \frac{1}{\log(1 + a^{-1})} \cdot \log \left( \frac{aw + 1}{aw} \right),
\]

that is the solution of the associated Gauss-Kuzmin type problem. \( \square \)

Remarks. (1) It is notable that for \( a = 1 \) the RSCC associated with the generalized Gauss-Kuzmin operator is identical to that associated with the ordinary continued fraction expansion (see Iosifescu and Grogorescu [3]). Moreover the corresponding limit probability measure associated with the family of RSCCs (2.5) for \( a = 1 \) is identical to the limit
probability measure associated with the above random system with complete connections for the ordinary continued fraction expansion, that is, identical to the Gauss’s measure \( \gamma \) on \( B_{[0,1]} \) defined by

\[
\gamma(A) = \frac{1}{\log 2} \int_A \frac{dt}{t+1}, \quad A \in B_{[0,1]},
\]

(4.11)

(2) It is an open problem the determination of an analogous limit probability measure for the case \( \alpha u^{-1} + 1 - \alpha : [u^{-1} + \alpha^{-1}] > 1 \).

References


