Research Article

Statistical Convergence and Ideal Convergence of Sequences of Functions in 2-Normed Spaces

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We present various kinds of statistical convergence and $\mathcal{I}$ -convergence for sequences of functions with values in 2-normed spaces and obtain a criterion for $\mathcal{I}$ -convergence of sequences of functions in 2-normed spaces. We also define the notion of $\mathcal{I}$ -equistatistically convergence and study $\mathcal{I}$ -equistatistically convergence of sequences of functions.

1. Introduction

The concept of ideal convergence was introduced first by Kostyrko et al. [1] as an interesting generalization of statistical convergence [2–5].

Throughout this paper $\mathbb{N}$ will denote the set of positive integers. Let $(X, \|\cdot\|)$ be a normed space. Let K be a subset of positive integers $\mathbb{N}$ and $j \in \mathbb{N}$. The quotient $d_j(K) = \text{card}(K \cap \{1, \dots, j\})/j$ is called the j 'th *partial density* of K and d_j is a probability measure on $\mathcal{P}(\mathbb{N})$, with support $\{1, \dots, j\}$ [2, 3].

The limit $d(K) = \lim_{j \rightarrow \infty} d_j(K)$ (if exists) is called the *natural density* of K . Clearly, finite subsets have natural density zero and $d(K^c) = 1 - d(K)$ where $K^c = \mathbb{N} \setminus K$, that is, the complement of K . If $K_1 \subseteq K_2$ and K_1, K_2 have natural densities then $d(K_1) \leq d(K_2)$. Furthermore, if $d(K_1) = d(K_2) = 1$, then $d(K_1 \cap K_2) = 1$ [6].

Recall that a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X is called to be statistically convergent to $x \in X$ if the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\}$ has natural density zero for each $\epsilon > 0$. In this case we write $st\text{-}\lim_{n \rightarrow \infty} x_n = x$ [2–4].

A family $\mathcal{I} \subseteq \mathcal{P}(Y)$ of subsets a nonempty set Y is said to be an ideal in Y if

- (i) $\emptyset \in \mathcal{I}$,
- (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
- (iii) $A \in \mathcal{I}$, $B \subseteq A$ implies $B \in \mathcal{I}$,

while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ [7, 8]. Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to be $\mathcal{I}$ -convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\}$ belongs to $\mathcal{I}$ [1, 9].

2. Preliminaries

The notion of linear 2-normed spaces has been investigated by Gähler in the 60's [10, 11] and this has been developed extensively in different subjects by others [12–14]. Let X be a real linear space of dimension greater than 1, and $\|\cdot, \cdot\|$ be a nonnegative real-valued function on $X \times X$ satisfying the following conditions:

- (G1) $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors;
- (G2) $\|x, y\| = \|y, x\|$ for all x, y in X ;
- (G3) $\|\alpha x, y\| = |\alpha| \|x, y\|$ where α is real,
- (G4) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all x, y, z in X

$\|\cdot, \cdot\|$ is called a 2-norm on X and the pair $(X, \|\cdot, \cdot\|)$ is called a linear 2-normed space. In addition, for all scalars α and all x, y, z in X , we have the following properties:

- (1) $\|\cdot, \cdot\|$ is nonnegative;
- (2) $\|x, y\| = \|x, y + \alpha x\|$;
- (3) $\|x - y, y - z\| = \|x - y, x - z\|$.

Some of the basic properties of 2-norm are introduced in [14]. Given a 2-normed space $(X, \|\cdot, \cdot\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to be convergent to x in X if $\lim_{n \rightarrow \infty} \|x_n - x, z\| = 0$ for every $z \in X$. This can be written by the formula

$$(\forall z \in Y) (\forall \epsilon > 0) (\exists n_0 \in \mathbb{N}) (\forall n \geq n_0) \quad \|x_n - x, z\| < \epsilon. \quad (2.1)$$

We write it as

$$x_n \xrightarrow{\|\cdot, \cdot\|_X} x. \quad (2.2)$$

Lemma 2.1 (see [13]). *Let $v = \{v_1, \dots, v_k\}$ be a basis of X . A sequence $(x_n)_{n \in \mathbb{N}}$ in X is convergent to x in X if and only if $\lim_{n \rightarrow \infty} \|x_n - x, v_i\| = 0$ for every $i = 1, \dots, k$. We can define the norm $\|\cdot\|_\infty$ on X by*

$$\|x\|_\infty := \max\{\|x, v_i\| : i = 1, \dots, d = k\}. \quad (2.3)$$

Lemma 2.2 (see [13]). *A sequence $(x_n)_{n \in \mathbb{N}}$ in X is convergent to x in X if and only if $\lim_{n \rightarrow \infty} \|x_n - x\|_\infty = 0$.*

Example 2.3. Let $X = \mathbb{R}^2$ be equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram spanned by the vectors x and y , which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \quad x = (x_1, x_2), \quad y = (y_1, y_2). \quad (2.4)$$

Take the standard basis $\{i, j\}$ for $\mathbb{R}^2$.

Then, $\|x, i\| = |x_2|$ and $\|x, j\| = |x_1|$, and so the derived norm $\|\cdot\|_\infty$ with respect to $\{i, j\}$ is

$$\|x\|_\infty = \max\{|x_1|, |x_2|\}, \quad x = (x_1, x_2). \quad (2.5)$$

Thus, here the derived norm $\|\cdot\|_\infty$ is exactly the same as the uniform norm on $\mathbb{R}^2$. Since the derived norm is a norm, it is equivalent to the Euclidean norm on $\mathbb{R}^2$.

Definition 2.4. Let $\mathcal{O} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n \in \mathbb{N}}$ of X is said to be $\mathcal{O}$ -convergent to x , if for each $\epsilon > 0$ and nonzero z in X the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x, z\| \geq \epsilon\}$ belongs to $\mathcal{O}$ [9].

If $(x_n)_{n \in \mathbb{N}}$ is $\mathcal{O}$ -convergent to x then we write it as

$$\mathcal{O} - \lim_{n \rightarrow \infty} \|x_n - x, z\| = 0 \quad \text{or} \quad \mathcal{O} - \lim_{n \rightarrow \infty} \|x_n, z\| = \|x, z\|. \quad (2.6)$$

The element x is $\mathcal{O}$ -limit of the sequence $(x_n)_{n \in \mathbb{N}}$.

Remark 2.5. If $(x_n)_{n \in \mathbb{N}}$ is any sequence in X and x is any element of X , then the set

$$\{n \in \mathbb{N} : \|x_n - x, z\| \geq \epsilon, \forall z \in X\} = \emptyset \quad (2.7)$$

since if $z = 0$, $\|x_n - x, z\| = 0 < \epsilon$ so the above set is empty.

Further we will give some examples of ideals and corresponding $\mathcal{O}$ -convergences.

Now we give an example of $\mathcal{O}$ -convergence in 2-normed spaces.

Example 2.6. (i) Let $\mathcal{O}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{O}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{O}_f$ -convergence coincides with usual convergence [11].

(ii) Put $\mathcal{O}_d = \{A \subset \mathbb{N} : d(A) = 0\}$. Then $\mathcal{O}_d$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{O}_d$ -convergence coincides with the statistical convergence [15].

Example 2.7. Let $\mathcal{O} = \mathcal{O}_d$. Define the $(x_n)_{n \in \mathbb{N}}$ in 2-normed space $(X, \|\cdot, \cdot\|)$ by

$$x_n = \begin{cases} (0, n), & n = k^2, \quad k \in \mathbb{N}, \\ (0, 0), & \text{otherwise} \end{cases} \quad (2.8)$$

and let $x = (0, 0)$ and $z = (z_1, z_2)$. Then for every $\epsilon > 0$ and $z \in X$

$$\{n \in \mathbb{N} : \|x_n - x, z\| > \epsilon\} \subseteq \{1, 4, 9, 16, \dots, n^2, \dots\} \quad (2.9)$$

we have that

$$d(\{n \in \mathbb{N} : \|x_n - x, z\| > \epsilon\}) = 0 \quad \text{for every } \epsilon > 0 \text{ and nonzero } z \in X. \quad (2.10)$$

This implies that $\mathcal{O}_d - \lim_{n \rightarrow \infty} \|x_n, z\| = \|x, z\|$. But, the sequence $(x_n)_{n \in \mathbb{N}}$ is not convergent to x .

3. Convergence for Sequences of Functions in 2-Normed Spaces

We discuss various kinds of convergence and $\mathcal{O}$ -convergence for sequences of functions with values in 2-normed spaces.

Let X, Y be 2-normed spaces and assume that functions

$$f : X \longrightarrow Y, \quad f_n : X \longrightarrow Y, \quad n \in \mathbb{N} \quad (3.1)$$

are given.

Definition 3.1. The sequence $(f_n)_{n \in \mathbb{N}}$ is said to be positive convergent to f (on X) if

$$f_n(x) \xrightarrow{\|\cdot, \cdot\|_Y} f(x) \quad \text{for each } x \in X. \quad (3.2)$$

We write

$$f_n \xrightarrow{\|\cdot, \cdot\|_Y} f. \quad (3.3)$$

This can be expressed by the formula

$$(\forall y \in Y) (\forall x \in X) (\forall \epsilon > 0) (\exists n_0 \in \mathbb{N}) (\forall n > n_0) \quad \|f_n(x) - f(x), y\|_Y < \epsilon. \quad (3.4)$$

Remark 3.2. If functions f, f_n are given as in Definition 3.1 and $\dim Y < \infty$ then (f_n) is pointwise convergent to f (on X) if and only if

$$(\forall x \in X) (\forall \epsilon > 0) (\exists n_0 \in \mathbb{N}) (\forall n > n_0) \quad \|f_n(x) - f(x), y\|_\infty < \epsilon. \quad (3.5)$$

We introduce uniform convergent of $(f_n)_{n \in \mathbb{N}}$ to f by the formula

$$(\forall y \in Y) (\forall \epsilon > 0) (\exists n_0 \in \mathbb{N}) (\forall n > n_0) (\forall x \in X) \quad \|f_n(x) - f(x), y\|_Y < \epsilon \quad (3.6)$$

and we write it as

$$f_n \xrightarrow[\text{uniform}]{\|\cdot, \cdot\|_Y} f. \quad (3.7)$$

Example 3.3. If $X = Y = \mathbb{R}^2$ is introduced in Lemma 2.1 then define

$$f(x_1, x_2) = \begin{cases} (0, 0) & \text{if } |x_2| < 1, \\ \left(0, \frac{1}{2}\right) & \text{if } |x_2| = 1, \\ (0, 1) & \text{if } |x_2| > 1 \end{cases}, \quad f_n(x) = \left(0, \frac{x_2^{2n}}{1 + x_2^{2n}}\right), \quad (3.8)$$

then

$$f_n \xrightarrow[\text{uniform}]{\|\cdot\|_Y} f, \quad f_n \xrightarrow{\|\cdot\|_Y} f. \quad (3.9)$$

Example 3.4. Let $X = Y = [0, 1] \times (0, 1) \subseteq \mathbb{R}^2$ and define

$$f_n(x_1, x_2) = \left(0, \frac{1}{nx_2 + 1}\right), \quad f(x_1, x_2) = (0, 0). \quad (3.10)$$

Then obviously $f_n \xrightarrow{\|\cdot\|_Y} f$. But we show that f_n does not converge uniformly to f in Y . Fix $\varepsilon = 1/2$ and for all $n_0 \in \mathbb{N}$ put $n_0 = n + 1$, $x_n = (0, 1/2n)$ then

$$\|f_n(x_1, x_2) - 0\|_\infty = \left|\frac{1}{nx_2 + 1}\right| = \frac{2}{3} > \varepsilon. \quad (3.11)$$

Definition 3.5. Let X and Y be 2-normed spaces with $\dim Y < \infty$ and let $f : X \rightarrow Y$ be a function. The function f is said to be sequentially continuous at $x_0 \in X$ if for any sequence $(x_n)_{n \in \mathbb{N}}$ of X converging to x_0 one has

$$f(x_n) \xrightarrow{\|\cdot\|_Y} f(x_0). \quad (3.12)$$

Definition 3.6. Let X and Y be two 2-normed spaces, and $\dim Y < \infty$. If $f_n : X \rightarrow Y$ is a sequence of functions, we say $(f_n)_{n \in \mathbb{N}}$ is equi-continuous (on X) if

$$(\forall z \in X) (\forall \varepsilon > 0) (\exists \delta > 0) (\forall x, x_0 \in X) \quad \|x - x_0, z\|_X < \delta \implies \|f_n(x) - f_n(x_0)\|_\infty < \varepsilon. \quad (3.13)$$

Corollary 3.7. Let X and Y be two 2-normed spaces, $x_0 \in X$ with $\dim Y < \infty$.

If $f : X \rightarrow Y$ is a function such that satisfying the following formula

$$(\forall z \in X) (\forall \varepsilon > 0) (\exists \delta > 0) (\forall x \in X) \quad \|x - x_0, z\|_X < \delta \implies \|f(x) - f(x_0)\|_\infty < \varepsilon \quad (3.14)$$

then f is sequentially continuous at x_0 .

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X such that $x_n \xrightarrow{\|\cdot\|_X} x_0$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $\|f(x) - f(x_0)\|_\infty < \varepsilon$ for every $x \in X$ where $\|x - x_0, z\|_X \leq \delta$ for each $z \in X$. On the other hand $x_n \xrightarrow{\|\cdot\|_X} x_0$ hence for all $z \in X$ there exist n_0 such that $\|x_n - x_0, z\|_X < \delta$ for all $n \geq n_0$. Therefore $f(x_n) \xrightarrow{\|\cdot\|_Y} f(x_0)$ and f is sequentially continuous at x_0 . $\square$

4. $\mathcal{O}$ -Convergence of Functions in 2-Normed Spaces

Let X, Y be 2-normed spaces. Fix an admissible ideal $\mathcal{O} \subseteq \mathcal{P}(\mathbb{N})$ and assume that functions $f : X \rightarrow Y, f_n : X \rightarrow Y, n \in \mathbb{N}$ are given.

Definition 4.1. A sequence $(f_n)_{n \in \mathbb{N}}$ of functions is said to be $\mathcal{O}$ -pointwise convergent to f (on X) if $\mathcal{O}\text{-}\lim_{n \rightarrow \infty} \|f_n(x) - f(x), z\|_Y = 0$ (in $(Y, \|\cdot, \cdot\|_Y)$) for each $x \in X$. We Write

$$f_n \xrightarrow{\|\cdot\|_Y}_{\mathcal{O}} f. \quad (4.1)$$

This can be expressed by the formula

$$(\forall z \in Y) (\forall x \in X) (\forall \varepsilon > 0) (\exists M \in \mathcal{O}) (\forall n \in \mathbb{N} \setminus M) \quad \|f_n(x) - f(x), z\|_Y < \varepsilon. \quad (4.2)$$

Definition 4.2. A sequence, $(f_n)_{n \in \mathbb{N}}$ is said to be $\mathcal{O}$ -uniformly convergent to f (on X) if and only if

$$(\forall z \in Y) (\forall \varepsilon > 0) (\exists M \in \mathcal{O}) (\forall n \in \mathbb{N} \setminus M) (\forall x \in X) \quad \|f_n(x) - f(x), z\|_Y \leq \varepsilon. \quad (4.3)$$

We write $f_n \xrightarrow[\text{uniform}]{\|\cdot\|_Y}_{\mathcal{O}} f$.

Remark 4.3. If $\mathcal{O} = \mathcal{O}_d$ then $\xrightarrow{\|\cdot\|_Y}_{\mathcal{O}_d}$ and $\xrightarrow[\text{uniform}]{\|\cdot\|_Y}_{\mathcal{O}_d}$ will be read (respectively) as $\mathcal{O}$ -pointwise and $\mathcal{O}$ -uniform statistically convergence. If $f_n \xrightarrow{\|\cdot\|_Y}_{\mathcal{O}_d} f$, then for all $x \in X$ $f_n(x) \xrightarrow{\|\cdot\|_Y}_{\mathcal{O}_d} f(x)$ which may be given by the formula

$$(\forall x \in X) (\forall \varepsilon > 0) \quad \{n \in \mathbb{N} : \|f_n(x) - f(x)\|_\infty \geq \varepsilon\} \in \mathcal{O}_d \quad (4.4)$$

we have by [15]

$$(\forall x \in X) (\forall \varepsilon, \delta > 0) (\exists n_0 \in \mathbb{N}) (\forall n \geq n_0) \quad d_j(\{n \in \mathbb{N} : \|f_n(x) - f(x)\|_\infty \geq \varepsilon\}) < \delta. \quad (4.5)$$

Remark 4.4. We obviously have

$$\begin{aligned} f_n \xrightarrow[\text{uniform}]{\|\cdot\|_Y}_{\mathcal{O}} f &\implies f_n \xrightarrow{\|\cdot\|_Y}_{\mathcal{O}} f, \\ f_n \xrightarrow[\text{uniform}]{\|\cdot\|_Y}_{\mathcal{O}} f &\iff \sup_{x \in X} \|f_n(x) - f(x), z\|_Y \xrightarrow{\|\cdot\|_Y}_{\mathcal{O}} 0 \quad \forall z \in Y. \end{aligned} \quad (4.6)$$

Remark 4.5. Let $\mathcal{O}$ be such that $\mathcal{O}$ -convergence of sequences of points in $(Y, \|\cdot, \cdot\|_Y)$ is strictly more general than the usual convergence. Then there is a sequence $(y_n)_{n \in \mathbb{N}} \subseteq Y$, such that

$$y_n \xrightarrow{\|\cdot, \cdot\|_Y} \mathcal{O} y \quad \text{but} \quad \lim_{n \rightarrow \infty} \|y_n - y, z\|_Y \neq 0 \quad \text{for each } z \in Y. \quad (4.7)$$

Putting $f_n(x) = y_n$ and $f(x) = y$ for $x \in X$ and $n \in \mathbb{N}$, we have

$$f_n \xrightarrow[\text{uniform}]{\|\cdot, \cdot\|_Y} \mathcal{O} f \quad \text{but} \quad \neg f_n \xrightarrow{\text{uniform}} f. \quad (4.8)$$

Thus, in this situation, $\mathcal{O}$ -uniform convergence of sequences of functions is strictly more general than the usual uniform convergence.

Theorem 4.6. Let $\mathcal{O} \subseteq \mathcal{P}(\mathbb{N})$ be an admissible ideal and X, Y be two 2-normed spaces with $\dim Y < \infty$. Assume that $f_n \xrightarrow{\|\cdot, \cdot\|_Y} \mathcal{O} f$ (on X) where functions $f_n : X \rightarrow Y, n \in \mathbb{N}$ are equi-continuous (on X) and $f : X \rightarrow Y$. Then f is sequentially continuous (on X).

Proof. Let $z, x_0 \in X$ and $\varepsilon > 0$. By equi-continuity of f_n 's, there exist $\delta > 0$ such that $\|f_n(x_0) - f_n(x)\|_\infty \leq \varepsilon$ for every $n \in \mathbb{N}$ whenever $\|x - x_0, z\| < \delta$.

Fix $x \in X$ such that $\|x - x_0, z\| < \delta$. Since $f_n \xrightarrow{\|\cdot, \cdot\|_Y} \mathcal{O} f$, the set

$$\left\{ n \in \mathbb{N} : \|f_n(x_0) - f(x_0)\|_\infty \geq \frac{\varepsilon}{3} \right\} \cup \left\{ n \in \mathbb{N} : \|f_n(x) - f(x)\|_\infty \geq \frac{\varepsilon}{3} \right\} \quad (4.9)$$

is in $\mathcal{O}$ and different from $\mathbb{N}$. Hence there exists $n_0 \in \mathbb{N}$ such that

$$\|f_{n_0}(x_0) - f(x_0)\|_\infty < \frac{\varepsilon}{3}, \quad \|f_{n_0}(x) - f(x)\|_\infty < \frac{\varepsilon}{3}. \quad (4.10)$$

We have

$$\begin{aligned} \|f(x_0) - f(x)\|_\infty &\leq \|f(x_0) - f_{n_0}(x_0)\|_\infty + \|f_{n_0}(x_0) - f_{n_0}(x)\|_\infty + \|f_{n_0}(x) - f(x)\|_\infty \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned} \quad (4.11)$$

and by (Corollary 3.7) f is sequentially continuous at $x_0 \in X$. □

5. $\mathcal{O}$ -Equistatistically Convergent

Let X, Y be two 2-normed spaces with $\dim Y < \infty$ and $\mathcal{O} = \mathcal{O}_d \subseteq 2^X$ be admissible ideal on X .

Definition 5.1. A $(f_n)_{n \in \mathbb{N}}$ is called $\mathcal{D}$ -equi-statistically convergent to f (we write it as $f_n \overset{\|\cdot\|_Y}{\rightsquigarrow}_{\mathcal{D}} f$) if for every $\varepsilon > 0$ the sequence $(g_{j,\varepsilon})_{j \in \mathbb{N}}$ of functions $g_{j,\varepsilon} : X \rightarrow \mathbb{R}$ given by

$$g_{j,\varepsilon}(x) = d_j(\{n \in \mathbb{N} : \|f_n(x) - f(x)\|_\infty \geq \varepsilon\}), \quad x \in X \quad (5.1)$$

is uniformly convergent to the zero function (on X). Hence $f_n \overset{\|\cdot\|_Y}{\rightsquigarrow}_{\mathcal{D}} f$ if and only if the following formula holds:

$$(\forall \varepsilon, \delta > 0) (\exists n_0 \in \mathbb{N}) (\forall j \geq n_0) (\forall x \in X), \quad d_j(\{n \in \mathbb{N} : \|f_n(x) - f(x)\|_\infty \geq \varepsilon\}) < \delta. \quad (5.2)$$

Corollary 5.2. *The following properties hold:*

- (i) $f_n \overset{\|\cdot\|_Y}{\rightsquigarrow}_{\mathcal{D}} f$ implies $f_n \xrightarrow{\|\cdot\|_Y}_{\mathcal{D}} f$,
- (ii) $f_n \xrightarrow[\text{uniform}]{\|\cdot\|_Y}_{\mathcal{D}} f$ implies $f_n \overset{\|\cdot\|_Y}{\rightsquigarrow}_{\mathcal{D}} f$.

Proof.

- (i) If $f_n \overset{\|\cdot\|_Y}{\rightsquigarrow}_{\mathcal{D}} f$ by the monotonicity of operator d_j , we take $\varepsilon = \delta$ in Definition 4.2. Thus it is obvious.
- (ii) Assume $f_n \xrightarrow[\text{uniform}]{\|\cdot\|_Y}_{\mathcal{D}} f$ and $\varepsilon > 0$. By Definition 4.2 there exist a set $M \in \mathcal{D}$ such that $\|f_n(x) - f(x)\|_\infty < \varepsilon$ for all $n \in \mathcal{D} \setminus M$ and $x \in X$. Since $M \in \mathcal{D}$. We can pick $n_0 \in \mathbb{N}$ such that $d_j(M) < \varepsilon$ for all $j \geq n_0$. Let $x \in X$ and $n \in \mathbb{N}$. Thus $\|f_n(x) - f(x)\|_\infty \geq \varepsilon$ implies $n \in M$. Hence for each $j \geq n_0$, we have

$$d_j(\{n \in \mathbb{N} : \|f_n(x) - f(x)\|_\infty \geq \varepsilon\}) \leq d_j(M) < \varepsilon \quad (5.3)$$

by Definition 4.2 witnesses that $f_n \overset{\|\cdot\|_Y}{\rightsquigarrow}_{\mathcal{D}} f$. □

Example 5.3. Define $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^2, f_n : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^2, n \in \mathbb{N}$

$$f_n(x_1, x_2) = \begin{cases} \left(0, \frac{1}{n}\right), & \text{if } x_2 = \frac{1}{n}, \\ (0, 0), & \text{otherwise,} \end{cases} \quad f(x_1, x_2) = (0, 0), \quad (5.4)$$

Then $f_n \overset{\|\cdot\|_Y}{\rightsquigarrow}_{\mathcal{D}} f$ but $\neg f_n \xrightarrow[\text{uniform}]{\|\cdot\|_Y}_{\mathcal{D}} f$. Indeed, let $\varepsilon > 0$ and $k \in \mathbb{N}$ such that $1/k < \varepsilon$. Then for all $j \geq k$ and $x = (x_1, x_2) \in [0, 1] \times [0, 1]$ we have

$$d_j(\{n \in \mathbb{N} : \|f_n(x) - f(x)\|_\infty > \varepsilon\}) \leq \frac{1}{j} \leq \frac{1}{k} \leq \varepsilon. \quad (5.5)$$

Hence $f_n \overset{\|\cdot\|_Y}{\rightsquigarrow}_{\mathcal{D}} f$.

Suppose that $f_n \xrightarrow[\text{uniform}]{\|\cdot\|_Y} \mathcal{O}_d f$. Thus there is the set $M \in \mathcal{O}_d$ such that for all $n \in \mathcal{O}_d \setminus M$ and $x \in [0, 1] \times [0, 1]$ we have $\|f_n(x) - f(x)\|_\infty < 1$.

Choose $k \in \mathcal{O}_d \setminus M$. Then f_k must be the zero function, a contradiction.

Theorem 5.4. Assume $f : X \rightarrow Y$ and $f_n : X \rightarrow Y$ for $n \in \mathbb{N}$ fix $x_0 \in X$. If $f_n \xrightarrow{\|\cdot\|_Y} \mathcal{O}_d f$ and all functions $f_n, n \in \mathbb{N}$, are sequentially continuous at x_0 then f is sequentially continuous at x_0 .

Proof. Let $\varepsilon > 0$. Since $f_n \xrightarrow{\|\cdot\|_Y} \mathcal{O}_d f$, we can find $n_0 \in \mathbb{N}$ such that

$$d_{n_0} \left(\left\{ n \in \mathbb{N} : \|f_n(x) - f(x)\|_\infty \geq \frac{\varepsilon}{3} \right\} \right) < \frac{1}{2} \quad \forall x \in X. \quad (5.6)$$

Put $E(x) = \{n \leq K : \|f_n(x) - f(x)\|_\infty < \varepsilon/3\}$, $x \in X$. In other word d_{n_0} is a probability measure on $\mathcal{P}(\mathbb{N})$ with the support $\{1, \dots, n_0\}$, it follows that $d_{n_0}(E(x)) > 1/2$ for all $x \in X$. By the sequentially continuity of $f_1, \dots, f_{n_0}$ at x_0 , there exist $\delta > 0$ such that $\|f_i(x) - f_i(x_0)\|_\infty < \varepsilon/3$ for all $1 \leq i \leq n_0$ and $x \in X, \|x - x_0, z\| < \delta$ for each $z \in X$. Fix x such that $x \in X, \|x - x_0, z\| < \delta$ for each $z \in X$.

Since $d_{n_0}(E(x)) > 1/2$ and $d_{n_0}(E(x_0)) > 1/2$, there exists $p \in E(x) \cap E(x_0)$ such that

$$\begin{aligned} \|f(x) - f(x_0)\|_\infty &\leq \|f(x) - f_p(x)\|_\infty + \|f_p(x) - f_p(x_0)\|_\infty + \|f_p(x_0) - f(x_0)\|_\infty \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned} \quad (5.7)$$

Thus we show that $\|f(x) - f(x_0)\|_\infty < \varepsilon$ for all $x \in X, \|x - x_0, z\| < \delta$ for each $z \in X$. $\square$

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