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Research Article

Radially Symmetric Solutions of a Nonlinear Elliptic Equation

Edward P. Krisner[1] and William C. Troy[2]

[1] *Department of Mathematics, University of Pittsburgh at Greensburg, Greensburg, PA 15601, USA*
[2] *Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA*

Correspondence should be addressed to Edward P. Krisner, epk15@pitt.edu

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We investigate the existence and asymptotic behavior of positive, radially symmetric singular solutions of $w'' + ((N-1)/r)w' - |w|^{p-1}w = 0$, $r > 0$. We focus on the parameter regime $N > 2$ and $1 < p < N/(N-2)$ where the equation has the closed form, positive singular solution $w_1 = (4 - 2(N-2)(p-1)/(p-1)^2)^{1/(p-1)} r^{-2/(p-1)}$, $r > 0$. Our advance is to develop a technique to efficiently classify the behavior of solutions which are positive on a maximal positive interval $(r_{\min}, r_{\max})$. Our approach is to transform the nonautonomous w equation into an autonomous ODE. This reduces the problem to analyzing the behavior of solutions in the phase plane of the autonomous equation. We then show how specific solutions of the autonomous equation give rise to the existence of several new families of singular solutions of the w equation. Specifically, we prove the existence of a family of singular solutions which exist on the entire interval $(0, \infty)$, and which satisfy $0 < w(r) < w_1(r)$ for all $r > 0$. An important open problem for the nonautonomous equation is presented. Its solution would lead to the existence of a new family of "super singular" solutions which lie entirely above $w_1(r)$.

1. Introduction

We investigate the behavior of solutions of

$$\Delta w - |w|^{p-1} w = 0, \tag{1.1}$$

where $w = w(x_1, \ldots, x_N)$, $N > 1$ and $p > 1$. Solutions of (1.1) are time-independent solutions of the nonlinear heat equation

$$\frac{\partial w}{\partial t} = \Delta w - |w|^{p-1} w. \tag{1.2}$$

In the mid 1980's, Brezis et al. [1], and Kamin and Peletier [2], investigated the existence and asymptotic behavior of positive, time-dependent singular solutions of (1.2). This led to the classical 1989 study by Kamin et al. [3], whose goal was to completely classify all positive, time-dependent solutions of (1.2). A natural extension of their study is to classify positive, time-*independent* solutions. Such solutions play an important role in analyzing the large time behavior of solutions of the time-dependent equation (1.2) (e.g., see the discussion following (1.5) below). Thus, in this paper, our goal is to extend the results in [1–3], and develop a method to efficiently classify the behavior of positive, time-independent solutions of (1.1). Our focus is on radially symmetric solutions, which have the form $w = w(r)$, where $r = (x_1^2 + \cdots + x_N^2)^{1/2}$, and satisfy

$$w'' + \frac{N-1}{r} w' - |w|^{p-1} w = 0, \qquad r > 0. \tag{1.3}$$

Equation (1.3) has the closed form, positive singular solution (see Figure 2)

$$w_1(r) = \left(\frac{4 - 2(N-2)(p-1)}{(p-1)^2} \right)^{1/(p-1)} r^{2/(1-p)}, \quad N > 2,\ 1 < p < \frac{N}{N-2}. \tag{1.4}$$

A Related Equation

A second, widely studied nonlinear heat equation is

$$\frac{\partial v}{\partial t} = \Delta v + |v|^{p-1} v. \tag{1.5}$$

Equation (1.5) has the closed form, stationary, positive singular solution

$$v_1(r) = \left(\frac{2(N-2)(p-1) - 4}{(p-1)^2} \right)^{1/(p-1)} r^{2/(1-p)}, \quad N > 2,\ \frac{N}{N-2} < p < \frac{N+2}{N-2}. \tag{1.6}$$

This well-known singular solution plays an important role in the analysis of blowup of solutions of (1.5). For example, when $v(x_1, \ldots, x_N, 0)$ is appropriately chosen, similarity solution methods developed by Haraux and Weissler [4], and Souplet and Weissler [5], show how $v(x_1, \ldots, x_N, t) \to c v_1(r)$ as $t \to \infty$, where $c > 0$ is a constant [4, 5]. In 1999, Chen and Derrick [6] developed comparison methods to determine the large time behavior of solutions of the general equation

$$\frac{\partial w}{\partial t} = \Delta w + f(w), \tag{1.7}$$

where $f(w)$ is super linear, as in (1.7) and (1.5). Their approach is to let positive, time independent solutions act as upper and/or lower bounds for initial values of solutions of (1.7). Their comparison technique allows them to prove either global existence or finite time blowup of solutions. It is hoped that the methods described above, combined with the new

singular solutions found in this paper, will lead to future analytical insights into the behavior of solutions of the time-dependent equation (1.2).

Specific Aims

We have three specific aims. The first two are listed below. The third is given later in this section. We assume throughout that $N > 2$ and $1 < p < N/(N-2)$, the parameter regime where $w_1(r)$ exists. In order to study properties of positive solutions of (1.3), our approach is to let $r_0 > 0$ be arbitrarily chosen and analyze solutions with initial values

$$w(r_0) = \alpha > 0, \qquad w'(r_0) = \beta \in R. \tag{1.8}$$

Let $(r_{\min}, r_{\max})$ denote the largest interval containing r_0 over which the solution of (1.3)–(1.8) is positive.

Specific Aim 1. For each solution of (1.3)–(1.8), prove whether $r_{\min} = 0$ or $r_{\min} > 0$, and determine $\lim_{r \to r_{\min}^+} (w(r), w'(r))$.

Specific Aim 2. For each solution of (1.3)–(1.8), prove whether $r_{\max} < \infty$ or $r_{\max} = \infty$, and determine $\lim_{r \to r_{\max}^-} (w(r), w'(r))$.

Analytical Methods

To address the issues raised in Specific Aims 1 and 2, we need to determine the behavior of each solution of (1.3)–(1.8) over the entire interval $(r_{\min}, r_{\max})$, where

$$\begin{aligned} r_{\min} &= \inf \{\widehat{r} \in (0, r_0) \mid w(r) > 0 \ \forall r \in (\widehat{r}, r_0]\}, \\ r_{\max} &= \sup \{\widehat{r} > r_0 \mid w(r) > 0 \ \forall r \in [r_0, \widehat{r})\}. \end{aligned} \tag{1.9}$$

Numerical Experiments

In Figure 1, we set $(N, p, r_0, \alpha) = (3, 2, 2, 2)$ and illustrate solutions of (1.3)–(1.8) for various β values. For example, when $\beta \le 0$, panels (a)–(d) show that both $r_{\min} = 0$ and $r_{\min} > 0$ are possible, and that

$$w'(r) < 0 \quad \forall r \in (r_{\min}, r_0), \qquad \lim_{r \to r_{\min}^+} \left(w(r), w'(r)\right) = (\infty, -\infty). \tag{1.10}$$

Panels (a)–(f) and also Figure 2 show that solutions can satisfy either $r_{\max} < \infty$ or $r_{\max} = \infty$.

Remark 1.1. It must be emphasized that it is illegitimate to claim that numerical results are rigorous proofs. Complete analytical proofs are needed to determine properties such as (1.10).

We now give a brief discussion of (1.10) which demonstrates the difficulties that arise in studying only the w equation to resolve Specific Aims 1 and 2. The proof of the first property in (1.10) follows from (1.3), which implies that $w''(r) > 0$ at any $r \in (r_{\min}, r_0]$, where

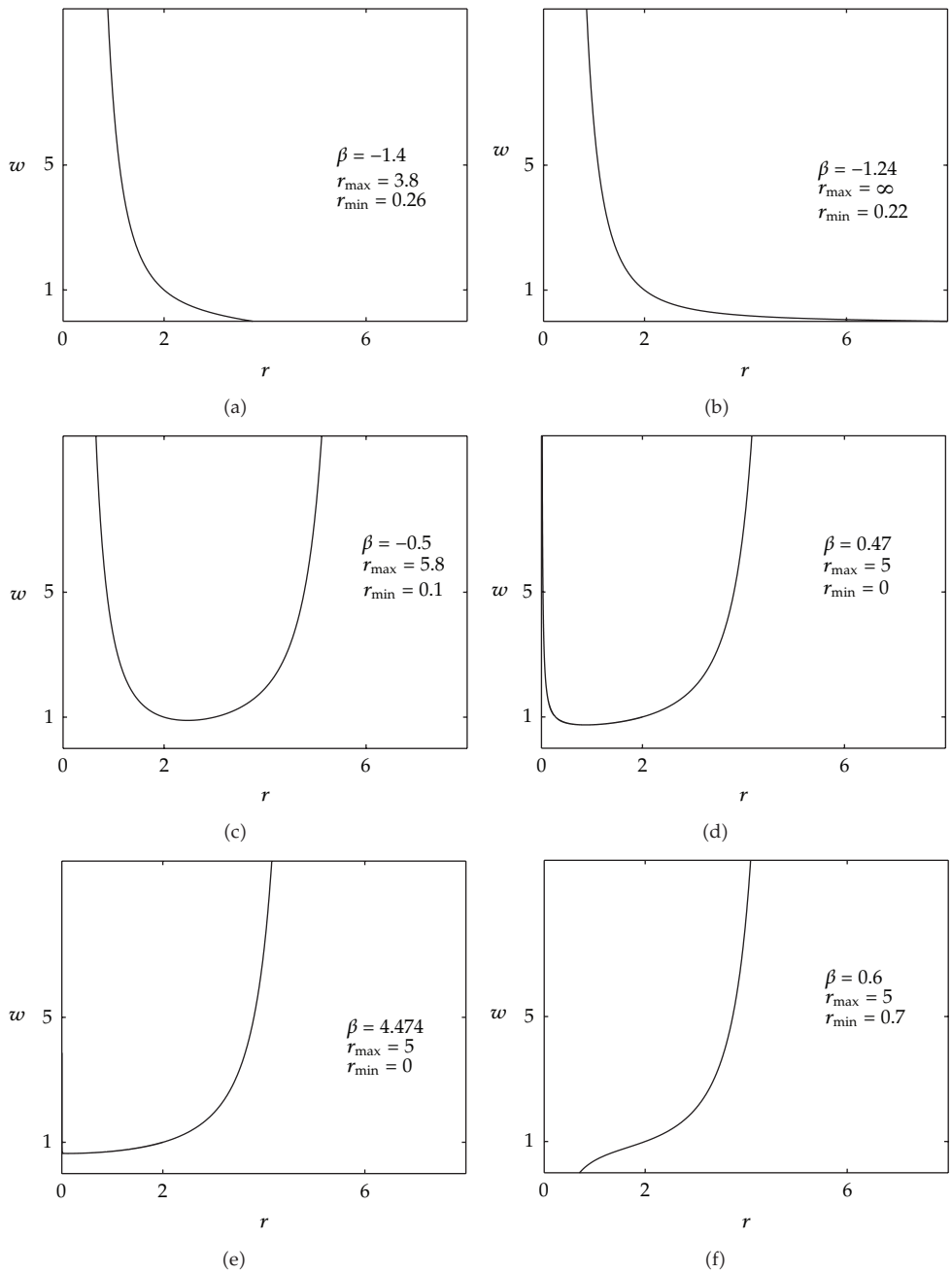

Figure 1: Solutions of (1.3)–(1.8) for various β values when $(N, p, r_0, \alpha) = (3, 2, 2, 2)$.

$w'(r) = 0$ and $w(r) > 0$. However, the fact that $w'(r) < 0$ for all $(r_{\min}, r_0)$ is *not* sufficient by itself to prove whether $r_{\min} = 0$ or $r_{\min} > 0$. Nor does it prove the second part of (1.10), that $\lim_{r \to r_{\min}^+} (w(r), w'(r)) = (\infty, -\infty)$. In fact, our study suggests that there is a unique $\beta_{\text{crit}} < 0$ where $r_{\min} = 0$ (see Figure 1(d)), and that $r_{\min} > 0$ at all other negative β values. The proof of these claims requires the development of further estimates. Such estimates might be obtained

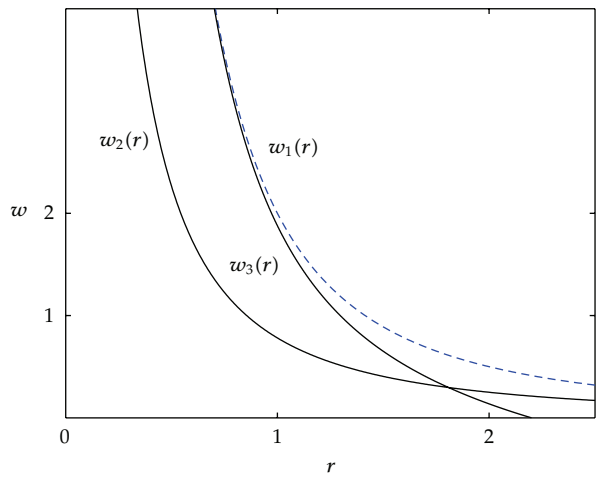

Figure 2: Solutions of (1.3)–(1.8) such that $r_{\min} = 0$ when $(N, p) = (3, 2)$; $w_1(r)$ is defined in (1.4); $w_2(r)$ and $w_3(r)$ satisfy parts (ii) and (iii) of Specific Aim 3. Also, $w_2(r)$ satisfies part (i) of Theorem 2.2, with $r_{\min} = 0$ and $r_{\max} = \infty$. $w_3(r)$ is a solution satisfying part (iv) of Theorem 2.2, with $r_{\min} = 0$ and $r_{\max} = 2.2$.

using Pohozaev-type identities [7] or topological shooting techniques [8]. Once the location of $r_{\min}$ and $\lim_{r \to r_{\min}^+} (w(r), w'(r))$ have been determined, we need to turn our attention to the interval $r > r_0$. As Figure 1 shows, there are several different types of behavior when $r > r_0$. For example, consider the solutions in panels (a), (b), and (c) in Figure 1. In each case,

$$r_{\min} > 0, \quad \lim_{r \to r_{\min}^+} w(r) = \infty. \tag{1.11}$$

When $r > r_0$, panels (a), (b), and (c) show three different behaviors of solutions, namely,

$$\begin{aligned} r_{\max} < \infty, &\quad \lim_{r \to r_{\max}^-} \left(w(r), w'(r)\right) = (0, -.25), \\ r_{\max} = \infty, &\quad \lim_{r \to r_{\max}^-} \left(w(r), w'(r)\right) = (0, 0), \\ r_{\max} < \infty, &\quad \lim_{r \to r_{\max}^-} \left(w(r), w'(r)\right) = (\infty, \infty). \end{aligned} \tag{1.12}$$

These results lead to the following analytical challenge: given *only* the fact that a solution satisfies property (1.11) when $r < r_0$, how can we prove which of the possibilities (1.12) occurs when $r > r_0$? It is not at all clear how to answer this question using standard methods such as Pohozaev identities or topological shooting.

Solutions with $r_{\min} = 0$

It is particularly important to understand the global behavior of solutions for which $r_{\min} = 0$ since such solutions may play an important role in analyzing the asymptotic behavior of blowup of solutions of the time-dependent equation (1.2). Figure 1(d) shows one such solution for which $r_{\min} = 0$. This solution lies entirely *above* $w_1(r)$, that is, $w(r) > w_1(r)$

for all $r \in (0, r_{\max})$. Figure 2 shows two other solutions, labeled $w_2(r)$ and $w_3(r)$, for which $r_{\min} = 0$. These solutions lie entirely *below* $w_1(r)$ on $(0, r_{\max})$. Our computations indicate that $w_2(r)$ satisfies $r_{\max} = \infty$, and that $r_{\max} < \infty$ for $w_3(r)$. These numerical experiments lead to

Specific Aim 3. Let $N > 2$ and $1 < p < N/(N-2)$. Prove that there are at least three families of solutions, other than $w_1(r)$, with $r_{\min} = 0$. The solutions in these families have the following properties:

(i) (see Figure 1(d)). For each $\alpha_0 > w_1(r_0)$ there exists $\beta_0 < 0$ such that if $w_0(r)$ is the solution of (1.3) with $(w_0(r_0), w_0'(r_0)) = (\alpha_0, \beta_0)$, then $r_{\min} = 0$, $r_{\max} < \infty$,

$$\begin{gathered} \lim_{r \to 0^+} \left(w(r), w'(r)\right) = (\infty, -\infty), \\ \lim_{r \to r_{\max}^{+-}} \left(w(r), w'(r)\right) = (\infty, \infty). \end{gathered} \tag{1.13}$$

(ii) (See Figure 2). For each $\alpha_2 \in (0, w_1(r_0))$ there exists $\beta_2 < 0$ such that if $w_2(r)$ is the solution of (1.3) with $(w_2(r_0), w_2'(r_0)) = (\alpha_2, \beta_2)$, then $r_{\min} = 0$, $r_{\max} = \infty$,

$$\begin{gathered} 0 < w_2(r) < w_1(r) \quad \forall r > 0, \\ \lim_{r \to 0^+} \left(w_2(r), w_2'(r)\right) = (\infty, -\infty), \\ \left(w_2(r), w_2'(r)\right) \sim \left(w_1(r), w_1'(r)\right) \quad \text{as } r \longrightarrow \infty. \end{gathered} \tag{1.14}$$

(iii) (See Figure 2). For each $\alpha_3 \in (0, w_1(r_0))$ there exists $\beta_3 < 0$ such that if $w_3(r)$ is the solution of (1.3) with $(w_3(r_0), w_3'(r_0)) = (\alpha_3, \beta_3)$, then $r_{\min} = 0$, $r_{\max} < \infty$,

$$\begin{gathered} \lim_{r \to 0^+} \left(w_3(r), w_3'(r)\right) \sim \left(w_1(r), w_1'(r)\right) \quad \text{as } r \longrightarrow 0^+, \\ w(r_{\max}) = 0, \qquad w\left(r_{\max}'\right) < 0. \end{gathered} \tag{1.15}$$

Our Analytical Approach

Our goal is to develop techniques to efficiently prove the existence of solutions of the w equation (1.3) satisfying the properties described in Specific Aims 1, 2, and 3. Our experience shows that the analysis of (1.3) is especially complicated since useful estimates must include the independent variable r. Our advance is to significantly simplify the analysis by transforming (1.3) into an equation which is *autonomous*, that is, independent or r. For this, let $w(r)$ denote any solution of (1.3), and define

$$h(\tau) = \frac{w\left(\exp(\tau)\right)}{w_1\left(\exp(\tau)\right)}, \quad -\infty < \tau < \infty. \tag{1.16}$$

Then $h(\tau)$ solves

$$h'' + \frac{N-2}{p-1}\left(p - \frac{N+2}{N-2}\right)h' + \frac{2(N-2)}{(p-1)^2}\left(p - \frac{N}{N-2}\right)\left(|h|^{p-1} - 1\right)h = 0. \tag{1.17}$$

Remark 1.2. The effect of transformation (1.16) is to change (1.3) into (1.17). Transformation (1.16) is similar to the classical Emden-Fowler transformation $y = w/t$, $x = 1/t$, which changes the Emden-Fowler equation

$$y'' = Ax^n y^m \tag{1.18}$$

to the new equation

$$w'' = At^{-n-m-3}w^m. \tag{1.19}$$

Because (1.17) is autonomous, we can apply phase plane techniques to prove the behavior of its solutions. We then use the "inverse" formula

$$w(r) = h(\ln(r))w_1(r), \quad 0 < r < \infty \tag{1.20}$$

to determine the global behavior of corresponding solutions of the w equation (1.3). In Section 2, we demonstrate the utility of this two step procedure. First, in Theorem 2.1, we analyze the h equation (1.17), and prove the existence and global behavior of four new classes of solutions. Secondly, in Theorem 2.2, we demonstrate how these families generate four new families of singular solutions of the w equation (1.3). In parts (i), (iii), and (iv) of Theorem 2.2 we show how the formula $w(r) = h(\ln(r))w_1(r)$ can be efficiently used to prove the precise asymptotic behavior of each solution as $r \to r_{\min}^+$, and as $r \to r_{\max}^-$. These three solutions satisfy parts (i), (ii), and (iii) of Specific Aim 3. The final family of solutions in Theorem 2.2 (see part (ii)), is a family of "super singular solutions," which satisfy $r_{\max} = \infty$,

$$w(r) > w_1(r) \ \forall r \in (r_{\min}, \infty), \quad \lim_{r \to r_{\min}^+} \frac{w(r)}{w_1(r)} = \infty. \tag{1.21}$$

However, it remains a challenging open problem (see Open Problems 1 and 2 in Section 2) to prove whether $r_{\min} = 0$ or $r_{\min} > 0$. If the first possibility holds, then we have a fourth family of singular solutions, other than $w_1(r)$, which satisfy $r_{\min} = 0$.

2. The Main Result

In this section, we show how to make use of the autonomous h equation (1.17) to address the issues raised in Specific Aims 1, 2, and 3 for solutions of the nonautonomous w equation (1.3). In particular, our technique shows how the analysis of a solution of (1.17) can be used to completely determine the behavior of the corresponding solution of the w equation (1.3) on the maximal interval $(r_{\min}, r_{\max})$, where w is positive. To demonstrate the utility of our

method, we restrict our focus to four specific branches of solutions of the h equation (1.17). Our approach consists of two steps.

First, in Theorem 2.1, we classify the behavior of solutions of (1.17) whose trajectories lie on the stable and unstable manifolds leading to and from the constant solution $(h, h') = (1, 0)$ in the (h, h') plane. The stable manifold has two components, B_1 nd C_1, and the unstable manifold has two components, D_1 and E_1. Solutions on B_1, C_1, D_1, and E_1 are illustrated in Figure 4(a).

Secondly, in Theorem 2.2, we make use of the link

$$w(r) = h(\ln(r))w_1(r), \tag{2.1}$$

to show how solutions with initial values on B_1, C_1, D_1, and E_1 translate into four new continuous families of singular solutions of the w equation (1.3). For three of the four cases, we completely prove the behavior of solutions of the w equation on the maximal interval $(r_{\min}, r_{\max})$, where they are positive. For the fourth case, it remains a challenging open problem (see Open Problems 1 and 2 below) to prove the asymptotic behavior of the solution at the left end point $r = r_{\min}$. The important consequences of resolving these open problems is described in Section 3.

Theorem 2.1. *Let $N > 2$ and $1 < p < N/(N-2)$. Then*

(i) *There is a one-dimensional stable manifold Γ of solutions of* (1.17) *leading to* $(1, 0)$ *in the (h, h') phase plane. One component, B_1, of Γ points into the region $h < 1$, $h' > 0$. If $(h(0), h'(0)) \in B_1$, then*

$$0 < h(\tau) < 1, \quad 0 < h'(\tau) < \frac{N-2}{p-1}\left(\frac{N}{N-2} - p\right)h(\tau) \quad \forall \tau \in \mathbb{R}, \tag{2.2}$$

$$\lim_{\tau \to -\infty} \left(h(\tau), h'(\tau)\right) = (0, 0), \qquad \lim_{\tau \to -\infty} \frac{h'(\tau)}{h(\tau)} = \frac{N-2}{p-1}\left(\frac{N}{N-2} - p\right), \tag{2.3}$$

$$\lim_{\tau \to \infty} \left(h(\tau), h'(\tau)\right) = (1, 0). \tag{2.4}$$

(ii) *The second component, C_1, of Γ points into the region $h > 1$, $h' < 0$ of the (h, h') plane. If a solution satisfies $(h(0), h'(0)) \in C_1$, and $(\tau_{\min}, \infty)$ is its interval of existence, then*

$$h(\tau) > 1, \quad h'(\tau) < 0, \quad h''(\tau) > 0 \quad \forall \tau \in (\tau_{\min}, \infty), \tag{2.5}$$

$$\lim_{\tau \to \infty} \left(h(\tau), h'(\tau)\right) = (1, 0) \qquad \lim_{\tau \to \tau_{min^+}} h(\tau) = \infty. \tag{2.6}$$

(iii) *There is a one-dimensional unstable manifold Ω of solutions of* (1.17) *leading from* $(1, 0)$ *into the (h, h') plane. One component, D_1, of Ω points into the region $h > 1$, $h' > 0$. If*

a solution satisfies $(h(0), h'(0)) \in D_1$, *and* $(-\infty, \tau_{\max})$ *is its interval of existence, then* $\tau_{\max} < \infty$,

$$h(\tau) > 1, \quad h'(\tau) > 0, \quad h''(\tau) > 0 \quad \forall \tau \in (-\infty, \tau_{\max}), \tag{2.7}$$

$$\lim_{\tau \to -\infty} \left(h(\tau), h'(\tau)\right) = (1, 0), \quad \lim_{\tau \to \tau_{\max^-}} \left(h(\tau), h'(\tau)\right) = (\infty, \infty). \tag{2.8}$$

(iv) *The second component,* E_1, *of* Ω *points into the region* $h < 1$, $h' < 0$ *of the* (h, h') *plane. If* $(h(0), h'(0)) \in E_1$, *with* $0 < h(0) < 1$ *and* $h'(0) < 0$, *then there exists a value* $\tau^* > 0$ *such that*

$$\begin{gathered} 0 < h(\tau) < 1, \quad h'(\tau) < 0 \quad \forall \tau \in (-\infty, \tau^*), \\ \lim_{\tau \to -\infty} \left(h(\tau), h'(\tau)\right) = (1, 0), \quad h(\tau^*) = 0, \quad h'(\tau^*) < 0. \end{gathered} \tag{2.9}$$

Proof of (i). We need to prove properties (2.2)–(2.4). The first step is to linearize (1.17) around the constant solution $(h, h') = (0, 0)$. This gives

$$h'' + \frac{N-2}{p-1}\left(p - \frac{N+2}{N-2}\right)h' - \frac{2(N-2)}{(p-1)^2}\left(p - \frac{N}{N-2}\right)h = 0. \tag{2.10}$$

The eigenvalues associated with (2.10) satisfy

$$\mu_1 = \frac{N-2}{p-1}\left(\frac{N}{N-2} - p\right) > 0, \qquad \mu_2 = \frac{2}{p-1} > 0. \tag{2.11}$$

We will make use of the observation that (1.17) can be written as

$$h'' - (\mu_1 + \mu_2)h' + \mu_1\mu_2 h = \mu_1\mu_2 |h|^{p-1} h. \tag{2.12}$$

Next, a linearization of (1.17) about the constant solution $(h, h') = (1, 0)$ gives

$$h'' + \frac{N-2}{p-1}\left(p - \frac{N+2}{N-2}\right)h' + \frac{2(N-2)}{p-1}\left(p - \frac{N}{N-2}\right)(h-1) = 0. \tag{2.13}$$

Define $k = -2/(p-1)$. Then (2.13) becomes

$$h'' + \gamma h' + 2(\gamma - k)(h-1) = 0, \tag{2.14}$$

where

$$\gamma = \frac{N-2}{p-1}\left(p - \frac{N+2}{N-2}\right) < 0, \qquad \gamma - k = \frac{N-2}{p-1}\left(p - \frac{N}{N-2}\right) < 0. \tag{2.15}$$

Thus, the eigenvalues associated with (2.13) and (2.14) satisfy

$$\lambda_1 = \frac{-\gamma - \sqrt{\gamma^2 - 8(\gamma - k)}}{2} < 0, \qquad \lambda_2 = \frac{-\gamma + \sqrt{\gamma^2 - 8(\gamma - k)}}{2} > 0. \tag{2.16}$$

It follows from (2.16) and the Stable Manifold Theorem that there is a one-dimensional stable manifold Γ of solutions leading to $(1,0)$ in the (h,h') phase plane. Additionally,

$$\lim_{\tau\to\infty} \frac{h'(\tau)}{h(\tau)-1} = \lambda_1 \tag{2.17}$$

for $(h(\tau),h'(\tau)) \in \Gamma$. Thus, for sufficiently large τ, solutions on Γ satisfy $h(\tau) > 1$ if $h'(\tau) < 0$ and $h(\tau) < 1$ if $h'(\tau) > 0$. Let B_1 denote the component of Γ pointing into the region $h < 1$, $h' > 0$ of the (h,h') plane. Assume that $(h(0),h'(0)) \in B_1$. Then (2.4) holds. It remains to prove (2.2)-(2.3). Because of (2.11) and (2.17), and the translation invariance of (2.12), we can choose $1 - h(0) > 0$ and $h'(0) > 0$ small enough so that

$$0 < h(\tau) < 1, \quad 0 < h'(\tau) < \mu_1 h(\tau) \quad \forall \tau \in [0,\infty). \tag{2.18}$$

The definition of B_1, together with (2.18), imply that the maximal interval of existence is of the form $(\tau_{\min},\infty)$, where $\tau_{\min} < 0$.

Next, we show that $B_1 \subset U^o$, where U^o is the bounded open triangular region

$$U^o = \{(h_1,h_2) \mid 0 < h_1 < 1,\ 0 < h_2 < \mu_1 h_1\}. \tag{2.19}$$

Figure 4(b) shows U^o when $(N,p) = (3,2)$. Because of (2.18), it suffices to show that $(h(\tau),h'(\tau)) \in U^o$ for all $\tau \in (\tau_{\min},0]$. For contradiction, assume that $(h(\tau),h'(\tau))$ leaves U^o at some point in $(\tau_{\min},0)$. Define

$$H = \frac{dh}{d\tau} - \mu_1 h. \tag{2.20}$$

It follows from (2.12) that H satisfies

$$H' - \mu_2 H = \mu_1\mu_2 |h|^{p-1} h. \tag{2.21}$$

Suppose that $(h(\tau),h'(\tau))$ leaves U^o across the line $H = 0$. That is, (see Figure 3(a)) suppose that there exists $\tau_0 \in (\tau_{\min},0)$ such that

$$H(\tau) < 0, \quad 0 < h(\tau) < 1 \quad \text{on } (\tau_0,0),\ H(\tau_0) = 0. \tag{2.22}$$

If $h(\tau_0) = 0$, then (2.20) implies that $h'(\tau_0) = 0$, contradicting uniqueness of the constant solution $(h,h') = (0,0)$. Thus, $h(\tau_0) > 0$. Also, (2.22) implies that

$$H'(\tau_0) \le 0. \tag{2.23}$$

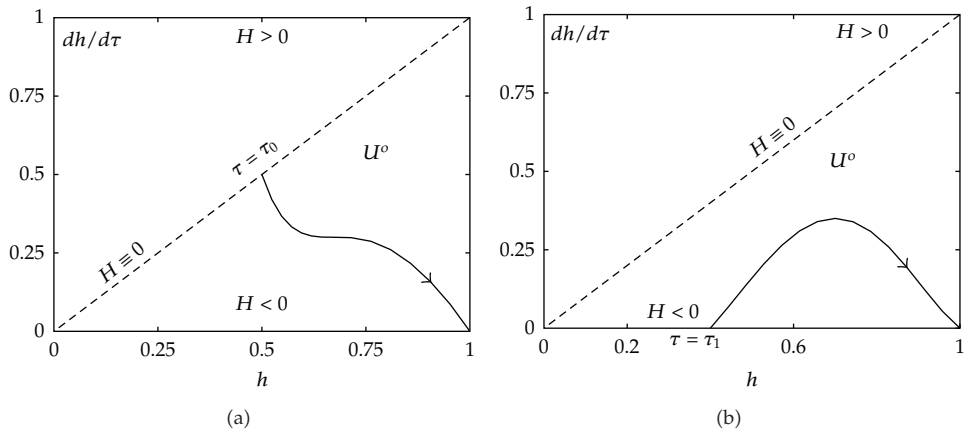

Figure 3: (a) solid curve illustrates properties (2.22). (b) solid curve illustrates properties (2.25) and (2.26).

The fact that $h(\tau_0) > 0$, combined with (2.21), results in

$$H'(\tau_0) = \mu_1\mu_2(h(\tau_0))^p > 0, \tag{2.24}$$

contradicting (2.23). Thus, $(h(\tau), h'(\tau))$ can only leave U^o across the line segment $0 < h < 1$, $h' = 0$. If so, there is a $\tau_1 \in (\tau_{\min}, 0)$ such that

$$H(\tau) < 0, \quad 0 < h(\tau) < 1, \quad h'(\tau) > 0 \quad \forall \tau \in (\tau_1, 0), \tag{2.25}$$

$$0 < h(\tau_1) < 1, \qquad h'(\tau_1) = 0, \tag{2.26}$$

as depicted in the right panel of Figure 3. Hence,

$$h''(\tau_1) \geq 0. \tag{2.27}$$

It follows from (2.12) and (2.26) that

$$h''(\tau_1) = \mu_1\mu_2\Big((h(\tau_1))^{p-1} - 1\Big)h(\tau_1) < 0, \tag{2.28}$$

contradicting (2.27). We conclude that $(h(\tau), h'(\tau))$ cannot leave U^o on $(\tau_{\min}, \infty)$, hence $B_1 \subset U^o$ as claimed. Moreover, since $(h(\tau), h'(\tau))$ is bounded, then $\tau_{\min} = -\infty$ follows from standard ODE theory. Thus, $(h(\tau), h'(\tau)) \in U^o$ for all $\tau \in \mathbb{R}$, and, therefore, $h'(\tau) > 0$ for all $\tau \in \mathbb{R}$.

Proof of the first part (2.3). First, we prove that $h \to 0^+$ as $\tau \to -\infty$. Since $h'(\tau) > 0$ and $0 < h(\tau) < 1$ on $\mathbb{R}$, then $0 \leq \overline{h} < 1$ where, $\overline{h} = \lim_{\tau\to-\infty} h$. To obtain a contradiction suppose that $\overline{h} > 0$. Then $0 < \overline{h} < 1$ and (2.12) yield

$$\frac{d^2h}{d\tau^2} - (\mu_1 + \mu_2)\frac{dh}{d\tau} \longrightarrow \mu_1\mu_2\left(\overline{h}^{p-1} - 1\right)\overline{h} < 0 \quad \text{as } \tau \longrightarrow -\infty. \tag{2.29}$$

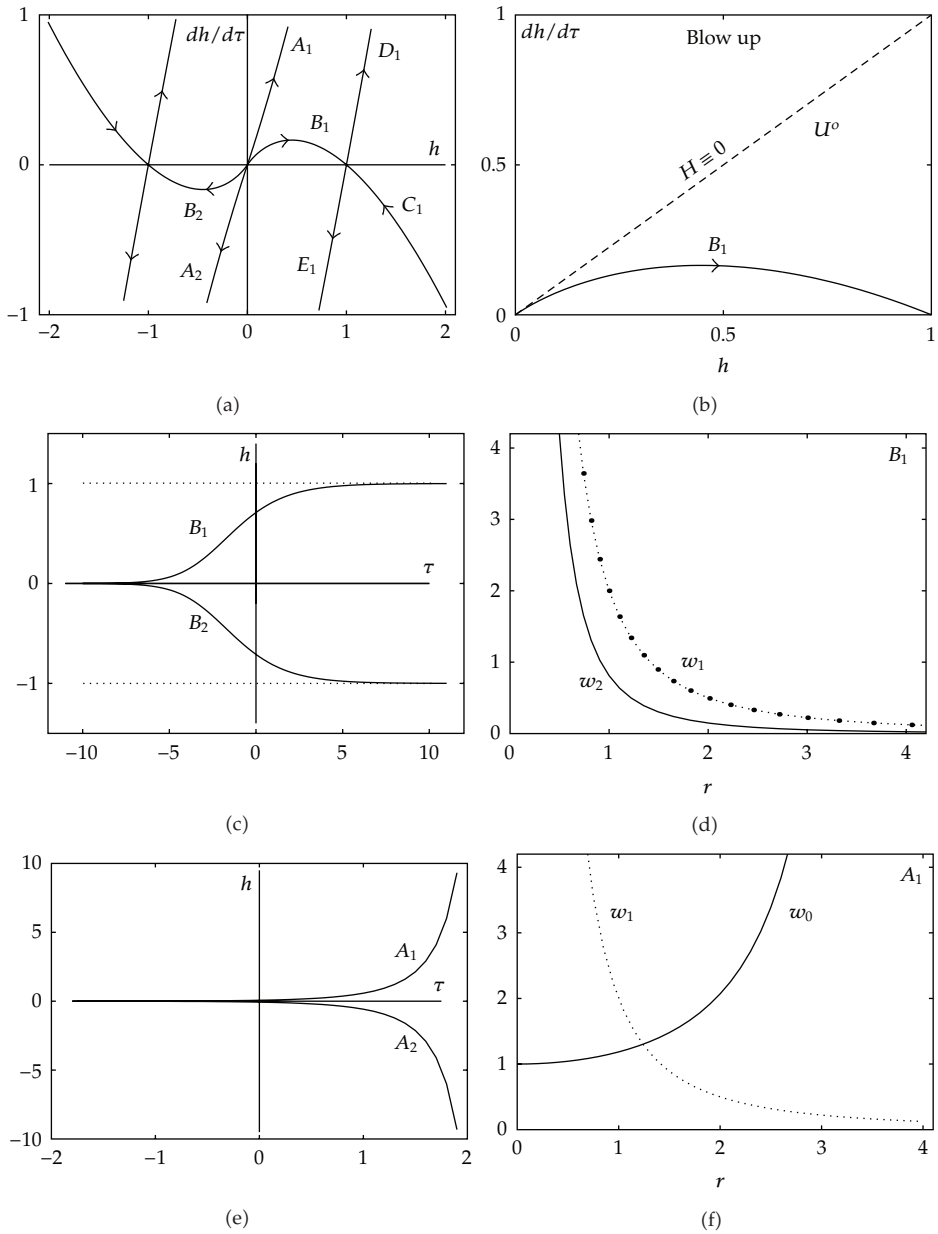

Figure 4: $N = 3$, $p = 2$. Row 1: Solutions on the stable and unstable manifolds associated with $(h, h') \equiv (\pm 1, 0)$ and $(h, h') \equiv (0, 0)$. Rows 2 and 3: h components on A_1, A_2, B_1, B_2 and w components along A_1 and B_1; $w_0(r)$ is bounded at $r = 0$, $w_1(r) = 2r^{-2}$ is the known singular solution; $w_2(r)$ is the new, positive singular solution corresponding to heteroclinic orbit B_1.

It follows from (2.29) that $h'(\tau) - (\mu_1 + \mu_2)h(\tau) \to \infty$ as $\tau \to -\infty$ which contradicts the fact that U^o is bounded and $(h(\tau), h'(\tau)) \in U^o$ for all $\tau \in \mathbb{R}$. Thus, $h(\tau) \to 0^+$ as $\tau \to -\infty$. Next, we show that $h'(\tau) \to 0^+$ as $\tau \to -\infty$. Note that $0 < h'(\tau) < \mu_1 h(\tau)$ on $(-\infty, 0]$ is an immediate consequence of $H(\tau) < 0$ and $h'(\tau) > 0$ on $(-\infty, 0]$. Therefore, $h'(\tau) \to 0^+$ as $\tau \to -\infty$ follows from the fact that $h(\tau) \to 0^+$ as $\tau \to -\infty$. □

Proof of second part of (2.3). Finally, we need to prove that $\rho = (h'/h) \to \mu_1$ as $\tau \to -\infty$. The definition of ρ together with (2.12) gives

$$\rho' + \rho^2 - (\mu_1 + \mu_2)\rho = \mu_1\mu_2\left(h^{p-1} - 1\right). \tag{2.30}$$

We now show that $\rho \to \mu_1$ monotonically as $\tau \to -\infty$. Differentiating (2.30) yields

$$\rho'' + (2\rho - \mu_1 - \mu_2)\rho' = \mu_1\mu_2(p-1)h^{p-2}h'. \tag{2.31}$$

Hence, if $\rho'(\tau_*) = 0$ for some $\tau_* \in \mathbb{R}$, then

$$\rho''(\tau_*) = \mu_1\mu_2(p-1)h^{p-2}(\tau_*)h'(\tau_*) > 0. \tag{2.32}$$

This implies that ρ' has at most one zero on $\mathbb{R}$. Furthermore,

$$0 < \rho(\tau) = \frac{h'(\tau)}{h(\tau)} < \mu_1 \quad \forall \tau \in \mathbb{R} \tag{2.33}$$

since

$$\left(h(\tau), h'(\tau)\right) \in U^o \quad \forall \tau \in \mathbb{R}. \tag{2.34}$$

Thus, $\overline{\rho} = \lim_{\tau\to-\infty}\rho$ exists and $0 \le \overline{\rho} \le \mu_1$. Moreover, the fact that $\overline{\rho}$ is finite ensures the existence of an unbounded decreasing sequence $\{\tau_n\}$ such that $\lim_{\tau_n\to-\infty}\rho'(\tau_n) = 0$. Substituting

$$\lim_{\tau_n\to-\infty}\rho'(\tau_n) = 0 = \lim_{\tau_n\to-\infty}h(\tau_n), \overline{\rho} = \lim_{\tau_n\to-\infty}\rho \tag{2.35}$$

into (2.30) results in

$$\overline{\rho}^2 - (\mu_1 + \mu_2)\overline{\rho} + \mu_1\mu_2 = 0. \tag{2.36}$$

The bound $0 \le \overline{\rho} \le \mu_1$ and (2.36) imply that $\overline{\rho} = \mu_1$. Thus, $\rho \to \mu_1$ as $\tau \to -\infty$ as claimed. □

□

Proof of (ii). It follows from the Stable Manifold Theorem and (2.17) that there is a second component, C_1, of Γ which points into the region $h > 1, h' < 0$ of the (h, h') plane (Figure 4(a)). Thus, if $(h(0), h'(0)) \in C_1$, and $h(0) - 1 > 0$ is sufficiently small, then

$$h(\tau) > 1 \quad h'(\tau) < 0 \quad \forall \tau \in [0, \infty), \tag{2.37}$$

$$\lim_{\tau\to\infty}\left(h(\tau), h'(\tau)\right) = (1, 0). \tag{2.38}$$

Let $(\tau_{\min}, \infty)$ denote the interval of existence of this solution. It remains to prove (2.5) and the second part of (2.6), that is, that

$$h(\tau) > 1, \quad h'(\tau) < 0, \quad h''(\tau) > 0 \quad \forall \tau \in (\tau_{\min}, \infty), \tag{2.39}$$

$$\lim_{\tau \to \tau_{\min}^{+}} h(\tau) = \infty. \tag{2.40}$$

Let (τ_*, ∞) denote the maximal subinterval of $(\tau_{\min}, \infty)$ such that $h'(\tau) < 0$ for all $\tau \in (\tau_*, \infty)$. From the definition of τ_* and (2.37), it follows that $h(\tau) > 1$ for all $\tau > \tau_*$. Next, we prove that $\tau_* = \tau_{\min}$. Suppose, for contradiction, that $\tau_* > \tau_{\min}$. Then

$$h(\tau_*) > 1 \quad h'(\tau_*) = 0, \quad h''(\tau_*) \leq 0. \tag{2.41}$$

From (1.17), and the fact that $h(\tau_*) > 1$ and $h'(\tau_*) = 0$, it follows that

$$h''(\tau_*) = \frac{2(N-2)}{(p-1)^2}\left(p - \frac{N}{N-2}\right)\left(|h(\tau_*)|^{p-1} - 1\right)h(\tau_*) > 0, \tag{2.42}$$

which contradicts (2.41). We conclude that $\tau_* = \tau_{\min}$, hence $h(\tau) > 1$ and $h'(\tau) < 0$ for all $\tau \in (\tau_{\min}, \infty)$. Finally, suppose that $h''(\hat{\tau}) = 0$ at some $\hat{\tau} \in (\tau_{\min}, \infty)$. A differentiation (1.17) gives

$$h'''(\hat{\tau}) = -\frac{2(N-2)}{(p-1)^2}\left(p - \frac{N}{N-2}\right)\left(p|h(\hat{\tau})|^{p-1} - 1\right)h'(\hat{\tau}) < 0. \tag{2.43}$$

Thus, since $h''' < 0$ whenever $h'' = 0$, we conclude that $h''(\tau) < 0$ for all $\tau > \hat{\tau}$. This implies that $h'(\infty) < 0$, contradicting (2.38). Therefore, it must be the case that $h''(\tau) > 0$ for all $\tau \in (\tau_{\min}, \infty)$. This completes the proof of (2.39). It then follows from (2.39) and standard theory that $\lim_{\tau \to \tau_{\min}^{+}} h(\tau) = \infty$, and (2.40) is proved. □

Open Problem 1. The issue of whether $\tau_{\min} = -\infty$ or $\tau_{\min} > -\infty$ remains unresolved. Its resolution may lead to new classes of solutions of the w equation (1.3). Precise details of the implications for solutions of (1.3) are given below, both in the proof of Theorem 2.2, and in the discussion which follows its proof.

Proof of (iii). It follows from (2.16) and the Stable Manifold Theorem that there is a one-dimensional unstable manifold Ω of solutions leading from $(1, 0)$ into the (h, h') plane. Additionally, solutions on Ω satisfy

$$\lim_{\tau \to -\infty} \frac{h'(\tau)}{h(\tau) - 1} = \lambda_2 > 0. \tag{2.44}$$

Thus, for sufficiently large τ, solutions on Ω satisfy $h(\tau) > 1$ if $h'(\tau) > 0$, and $h(\tau) < 1$ if $h'(\tau) < 0$. Let D_1 denote the component of Ω pointing into the region $h > 1$, $h' > 0$ of the (h, h') plane (Figure 4(a)). Let $(h(0), h'(0)) \in D_1$. Then $\lim_{\tau \to -\infty}(h(\tau), h'(\tau)) = (1, 0)$, hence,

the first part of (2.8) is proved. Next, because of (2.44) and the translation invariance of (2.12), we can choose $h(0) - 1 > 0$ and $h'(0) > 0$ small enough so that

$$h(\tau) > 1, \quad h'(\tau) > 0, \quad \forall \tau \in (-\infty, 0]. \tag{2.45}$$

The interval of existence of this solution is of the form $(-\infty, \tau_{\max})$, where $\tau_{\max} > 0$. It remains to prove that finite time blowup occurs, that is, that $\tau_{\max} < \infty$, and

$$h(\tau) > 1, \quad h'(\tau) > 0, \quad h''(\tau) > 0 \quad \forall \tau \in (-\infty, \tau_{\max}), \tag{2.46}$$

$$\lim_{\tau \to \tau_{\max}^-} \left(h(\tau), h'(\tau)\right) = (\infty, \infty). \tag{2.47}$$

Let $(-\infty, \tau^*)$ denote the maximal subinterval of $(-\infty, \tau_{\max})$ such that $h'(\tau) > 0$ for all $\tau \in (-\infty, \tau^*)$. It follows from (2.45) and the definition of τ^* that $h(\tau) > 1$ for all $\tau \in (-\infty, \tau^*)$. We claim that $\tau^* = \tau_{\max}$. Suppose, for contradiction, that $\tau^* < \tau_{\max}$. Then

$$h(\tau^*) > 1, \quad h(\tau^*) = 0, \quad h''(\tau^*) \le 0. \tag{2.48}$$

However, (1.17) and the fact that $h(\tau^*) > 1$ and $h'(\tau^*) = 0$, imply that

$$h''(\tau^*) = \frac{2(N-2)}{(p-1)^2}\left(p - \frac{N}{N-2}\right)\left(|h(\tau^*)|^{p-1} - 1\right)h(\tau^*) > 0, \tag{2.49}$$

contradicting (2.48). Thus, $\tau^* = \tau_{\max}$, hence $h(\tau) > 1$ and $h'(\tau) > 0$ for all $\tau \in (-\infty, \tau_{\max})$. Also, it follows exactly as in the Proof of (ii) that $h''(\tau)$ does not change sign on $(-\infty, \tau_{\max})$, and that $h''(\tau) > 0$ for all $\tau \in (-\infty, \tau_{\max})$. This completes the proof of (2.46). Next, we prove that $\tau_{\max} < \infty$. Suppose, however, that $\tau_{\max} = \infty$. Then $h''(\tau) > 0$ for all $\tau \in (-\infty, \infty)$. This implies that

$$h'(\tau) \ge h'(0) > 0 \quad \forall \tau \ge 0, \quad \lim_{\tau \to \infty} h(\tau) = \infty. \tag{2.50}$$

To use (2.50) to contradict the assumption that $\tau_{\max} < \infty$, we analyze

$$S = \frac{(h')^2}{2} + \frac{2(N-2)}{(p-1)^2}\left(p - \frac{N}{N-2}\right)\left(\frac{h^{p+1}}{p+1} - \frac{h^2}{2}\right), \tag{2.51}$$

which satisfies

$$S' = \frac{N-2}{p-1}\left(\frac{N+2}{N-2} - p\right)(h')^2. \tag{2.52}$$

Since $h'(\tau) \geq h'(0) > 0$ for all $\tau \geq 0$, it follows from an integration of (2.52) that $S(\tau) \to \infty$ as $\tau \to \infty$. These, (2.51) and the fact that $h(\tau) \to \infty$ as $\tau \to \infty$, imply that there is a $\tau_1 \geq 0$ such that $S(\tau) \geq 0$ for all $\tau \geq \tau_1$, that is, that

$$(h')^2 \geq \frac{2(N-2)}{(p-1)^2}\left(\frac{N}{N-2} - p\right)\frac{h^{p+1}}{p+1} \quad \forall \tau \geq \tau_1. \tag{2.53}$$

An integration of (2.53) gives

$$(h(\tau))^{(1-p)/2} \leq (h(\tau_1))^{(1-p)/2} + a\frac{(1-p)}{2}(\tau - \tau_1), \quad \tau \geq \tau_1, \tag{2.54}$$

where $a = (2(N-2)/(p-1)(p-1)^2)(N/(N-2)-p))^{1/2} > 0$ since $1 < p < N/(N-2)$. The right side of (2.54) is negative when $\tau > \tau_2 = \tau_1 + (2(p-1)/a)(h(\tau_1))^{(1-p)/2}$. Thus, (2.54) reduces to $(h(\tau))^{(1-p)/2} < 0$ when $\tau > \tau_2$, a contradiction. We conclude that $\tau_{\max} < \infty$, as claimed. Since $\tau_{\max} < \infty$, it follows from (1.17), (2.7), and standard theory that $(h(\tau), h'(\tau)) \to (\infty, \infty)$ as $\tau \to \tau_{\max}^-$. This proves property (2.47). □

Proof of (iv). It follows from the Stable Manifold Theorem and (2.44) that there is a second component, E_1, of Ω which points into the region $0 < h < 1$, $h' < 0$ of the (h, h') plane. Thus, if $(h(0), h'(0)) \in E_1$, and $1 - h(0) > 0$ is sufficiently small, then

$$\begin{gathered} 0 < h(\tau) < 1, \quad h'(\tau) < 0 \quad \forall \tau \in (-\infty, 0], \\ \lim_{\tau \to -\infty}\left(h(\tau), h'(\tau)\right) = (1, 0). \end{gathered} \tag{2.55}$$

Define

$$\tau^* = \sup\left\{\hat{\tau} > 0 \mid 0 < h(\tau) < 1, h'(\tau) < 0 \ \forall \tau \in [0, \hat{\tau})\right\}. \tag{2.56}$$

We need to prove that $\tau^* < \infty$, that $h(\tau^*)$ and $h'(\tau^*)$ are finite,

$$h(\tau^*) = 0, \qquad h'(\tau^*) < 0. \tag{2.57}$$

For this, integrate (1.17) and get

$$h'(\tau)e^{A\tau} = h'(0) + B\int_0^\tau e^{A\eta}\left(\left|h(\eta)\right|^{p-1} - 1\right)h(\eta)\, d\eta, \quad 0 \leq \tau < \tau^*, \tag{2.58}$$

where $A = ((N-2)/(p-1))(p-(N+2)/(N-2)) < 0$ and $B = (2(N-2)/(p-1)^2)(N/(N-2)-p) > 0$. Because $(|h|^{p-1} - 1)h > -1$ for all $h \in [0, 1]$, it follows that

$$\int_0^\tau e^{A\eta}\left(\left|h(\eta)\right|^{p-1} - 1\right)h(\eta)\, d\eta \geq -\int_0^\tau e^{A\eta}\, d\eta = \frac{-1}{A}\left(e^A\tau - 1\right) \quad \forall \tau \in [0, \tau^*). \tag{2.59}$$

Combining (2.58) and (2.59) gives

$$0 > h'(\tau)e^{A\tau} \geq h'(0) - \frac{B}{A}\left(e^{A\tau} - 1\right) \quad \forall \tau \in [0, \tau^*). \tag{2.60}$$

We conclude from (2.60) that if τ^* is finite, then $h(\tau)$ and $h'(\tau)$ are bounded on the closed interval $[0, \tau^*]$. This, (2.58), the definition of τ^*, and (2.60) imply that $h'(\tau^*) < 0$ and $h(\tau^*) = 0$ if τ^* is finite. Thus, (2.57) is proved if τ^* is shown to be finite. We assume, for contradiction, that $\tau^* = \infty$. Then

$$0 < h(\tau) < 1, \quad h'(\tau) < 0 \quad \forall \tau \geq 0. \tag{2.61}$$

Since the integral term in (2.58) is negative for all $\tau \geq 0$, then (2.58) reduces to $h'(\tau)e^{A\tau} \leq h'(0)$ for all $\tau \geq 0$. An integration gives

$$h(\tau) \leq h(0) - \frac{h'(0)}{A}\left(e^{-A\tau} - 1\right) \quad \forall \tau \in [0, \infty). \tag{2.62}$$

The right side of (2.62) is negative when $\tau > -(1/A)\ln(h(0)A/h'(0) + 1)$, contradicting (2.61). We conclude that $\tau^* < \infty$, as claimed. This completes the proof of Theorem 2.1. □

Solutions of the w equation

Below, in Theorem 2.2, we show how to combine parts (i)–(iv) of Theorem 2.1 together with the formula

$$w(r) = h(\ln(r))w_1(r), \tag{2.63}$$

to generate new families of solutions of the w equation (1.3). In each of the four cases (i)–(iv), we show how to use (2.63) to prove the existence of an entire continuum of new singular solutions of (1.3). In each case, our approach is to let $(h(0), h'(0))$ be an arbitrarily chosen element of one of the four continuous curves B_1, C_1, D_1 or E_1. Since $r = e^{\tau}$, the initial conditions for the corresponding solution of (1.3) are given at $r = e^0 = 1$, and satisfy

$$w(1) = h(0)w_1(1), \qquad w'(1) = h'(0)w(1) + h(0)w'(1). \tag{2.64}$$

Because the curves B_1, C_1, D_1, and E_1 are continuous, this technique generates four new continua of solutions of the w equation. In addition, for cases (i), (iii), and (iv), our analytical technique allows us to completely resolve the issues raised in Specific Aims 1, 2, and 3 in Section 1. That is, for each of the solutions described in (i), (iii), and (iv) we show how to efficiently prove the limiting behavior of the solution at both ends of the maximal interval $(r_{\min}, r_{\max})$, where it is positive. For part (ii), our analysis of the behavior of solutions at $r_{\min}$ is incomplete, and this leads to Open Problem 2 which is stated at the end of the proof of (ii). This problem is directly related to Open Problem 1 described above at the end of the proof of part (ii) of Theorem 2.1.

Open Problem 3 Prove the existence of other families of solutions of (1.3). For example, the existence and limiting behavior of the solutions labeled (c) and (f) in Figure 1 have not yet been proved. It is our hope that our analytical techniques can be extended to prove the existence and limiting behavior of these and many other new families of solutions.

Theorem 2.2. *Let $N > 2$ and $1 < p < N/(N-2)$, and let $w_1(r)$ denote the positive singular solution of* (1.3) *defined in* (1.4).

(1) A Continuum of Singular Solutions Generated by B_1

Let $h_2(\tau)$ denote a solution of (1.17) which satisfies $(h_2(0), h_2'(0)) \in B_1$ in part (i) of Theorem 2.1. The corresponding solution $w_2(r) = h_2(\ln(r))w_1(r)$ of (1.3) has initial values

$$w_2(1) = h_2(0)w_1(1), \qquad w_2'(1) = h_2'(0)w(1) + h_2(0)w'(1), \tag{2.65}$$

and satisfies

$$\begin{gathered} 0 < w_2(r) < w_1(r) \quad \forall r > 0, \qquad \frac{w_2(r)}{w_1(r)} \longrightarrow 1 \quad \text{as } r \longrightarrow \infty, \\ w_2(r) \sim \left(\frac{4 - 2(N-2)(p-1)}{(p-1)^2}\right)^{1/(p-1)} r^{-(N-2)} \quad \text{as } r \longrightarrow 0^+. \end{gathered} \tag{2.66}$$

Figures 2 and 4(d) show solutions of (1.3) with these properties.

(2) A Continuum of Singular Solutions Generated by C_1

Let $h_3(\tau)$ denote a solution of (1.17) which satisfies $(h_3(0), h_3'(0)) \in C_1$ in part (ii) of Theorem 2.1. The corresponding solution $w_3(r) = h_3(\ln(r))w_1(r)$ of (1.3) has initial values

$$w_3(1) = h_3(0)w_1(1), \qquad w_3'(1) = h_3'(0)w(1) + h_3(0)w'(1). \tag{2.67}$$

Let $(r_{\min}, r_{\max})$ be the maximal interval where $w_3(r) > 0$. Then $r_{\max} = \infty$,

$$w_3(r) > w_1(r) \quad \forall r > r_{\min}, \tag{2.68}$$

$$\lim_{r \to r_{\min}^+} \frac{w_3(r)}{w_1(r)} = \infty, \qquad \lim_{r \to \infty} \frac{w_3(r)}{w_1(r)} = 1. \tag{2.69}$$

Figure 1(b) shows a solution of (1.3) with these properties.

(3) A Continuum of Singular Solutions Generated by D_1

Let $h_4(\tau)$ denote a solution of (1.17) which satisfies $(h_4(0), h_4'(0)) \in D_1$ in part (iii) of Theorem 2.1. The corresponding solution $w_4(r) = h_4(\ln(r))w_1(r)$ of (1.3) has initial values

$$w_4(1) = h_4(0)w_1(1), \qquad w_4'(1) = h_4'(0)w(1) + h_4(0)w'(1). \tag{2.70}$$

Let $(r_{\min}, r_{\max})$ be the maximal interval where $w_4(r) > 0$. Then $r_{\min} = 0$ and $r_{\max} < \infty$,

$$w_4(r) > w_1(r) \quad \forall r \in (0, r_{\max}), \tag{2.71}$$

$$\lim_{r \to 0^+} \frac{w_4(r)}{w_1(r)} = 1, \qquad \lim_{r \to r_{\max}} w_4(r) = \infty. \tag{2.72}$$

Figure 1(d) shows a solution of (1.3) with these properties.

(4) A Continuum of Singular Solutions Generated by E_1

Let $h_5(\tau)$ denote a solution of (1.17) which satisfies $(h_5(0), h_5'(0)) \in E_1$ in part (iv) of Theorem 2.1. The corresponding solution $w_5(r) = h_5(\ln(r))w_1(r)$ of (1.3) has initial values

$$w_5(1) = h_5(0)w_1(1), \qquad w_5'(1) = h_5'(0)w(1) + h_5(0)w'(1). \tag{2.73}$$

Let $(r_{\min}, r_{\max})$ be the maximal interval where $w_5(r) > 0$. Then $r_{\min} = 0$ and $r_{\max} < \infty$,

$$0 < w_5(r) < w_1(r) \quad \forall r \in (0, r_{\max}), \tag{2.74}$$

$$\lim_{r \to 0^+} \frac{w_5(r)}{w_1(r)} = 1, \qquad \lim_{r \to r_{max}} w_4(r) = 0. \tag{2.75}$$

Figure 2 shows a solution of (1.3) with these properties.

Proof of (1). Let h_2 denote a solution of (1.3) which satisfies part (i) of Theorem 2.1. By (1.20), the solution of (1.3) corresponding to h_2 is

$$w_2(r) = h_2(\ln(r))w_1(r). \tag{2.76}$$

It follows from (2.2) in Theorem 2.1 that $0 < h_2(\ln(r)) < 1$ for all $r > 0$. This, as well as (2.76), implies that

$$0 < w_2(r) < w_1(r) \quad \forall r > 0 \tag{2.77}$$

(see Figure 4(d)). We claim that w_2 is singular at $r = 0$. The first step in proving this claim is to observe that (2.3) and (2.11) imply that $(h_2'(\tau)/h_2(\tau)) \to \mu_1$ as $\tau \to -\infty$. Thus, $\ln(h_2(\tau)) \sim \mu_1\tau$ as $\tau \to -\infty$. This and the fact that $\tau = \ln(r)$ lead to

$$h_2(\tau) = h_2(\ln(r)) \sim r^{\mu_1} \quad \text{as } r \to 0^+. \tag{2.78}$$

Substituting (1.4) and (2.78) into (2.76) gives

$$w_2(r) \sim \left(\frac{4 - 2(N-2)(p-1)}{(p-1)^2} \right)^{1/(p-1)} r^{\mu_1 - \mu_2} \quad \text{as } r \longrightarrow 0^+. \tag{2.79}$$

Our claim that w_2 is singular at $r = 0$ follows from (2.79) and the fact that $\mu_1 - \mu_2 = 2 - N < 0$. It remains to determine the asymptotic behavior of $w_2(r)$ as $r \to \infty$. Since $h_2(\ln(r)) \to 1^-$ as $r \to \infty$, then $(w_2(r)/w_1(r)) \to 1$ as $r \to \infty$. This completes the proof of properties (2.66). □

Proof of (2). Let h_3 denote a solution of (1.3) which satisfies part (ii) of Theorem 2.1. By (1.20), the solution of (1.3) corresponding to h_3 is

$$w_3(r) = h_3(\ln(r))w_1(r). \tag{2.80}$$

Initial conditions (2.67) follow exactly as in the proof of part (i). Let $(r_{\min}, r_{\max})$ denote the maximal interval over which $w_3(r) > 0$. It follows from (2.5) in Theorem 2.1 that $r_{\max} = \infty$ and $h_4(\ln(r)) > 1$ for all $r > r_{\min}$. This, together with (2.80), implies that

$$w_3(r) > w_1(r) \quad \forall r \in (r_{\min}, \infty). \tag{2.81}$$

This proves (2.68). Property (2.6) in Theorem 2.1, as well as (2.80), imply that

$$\lim_{r\to\infty} \frac{w_3(r)}{w_1(r)} = \lim_{r\to\infty} h_3(\ln(r)) = 1, \tag{2.82}$$

$$\lim_{r\to r_{\min}^+} \frac{w_3(r)}{w_1(r)} = \lim_{r\to r_{\min}^+} h_3(\ln(r)) = \infty. \tag{2.83}$$

□

Open Problem 2. Prove whether $r_{\min} = 0$ or $r_{\min} > 0$. This problem arises as a direct consequence of Open Problem 1 described above at the end of the proof of part (ii) of Theorem 2.1. If it can be proved that $r_{\min} = 0$, then

$$w_3(r) > w_1(r) \qquad \forall r \in (0, \infty), \qquad \lim_{r\to 0^+} \frac{w_3(r)}{w_1(r)} = \infty. \tag{2.84}$$

Because $w_3(r) \to \infty$ much faster than $w_1(r)$, we refer to any solution satisfying either (2.83) of (2.84) as a *Super Singular Solution*. This class of solutions has not previously been reported.

Proof of (3). Let h_4 denote a solution of (1.3) which satisfies part (iii) of Theorem 2.1. By (1.20), the solution of (1.3) corresponding to h_4 is

$$w_4(r) = h_4(\ln(r))w_1(r). \tag{2.85}$$

Initial conditions (2.70) follow exactly as in the proof of part (i). Let $(r_{\min}, r_{\max})$ denote the maximal interval over which $w_4(r) > 0$. It follows from (2.7) in Theorem 2.1 that $r_{\min} = 0$ and $r_{\max} < \infty$, and $h_4(\ln(r)) > 1$ for all $r \in (0, r_{\max})$. This, together with (2.85), implies that

$$w_4(r) > w_1(r) \quad \forall r \in (0, r_{\max}). \tag{2.86}$$

This proves (2.71). Property (2.8), in Theorem 2.1, as well as (2.85), implies that

$$\begin{gathered}\lim_{r\to r_{\max}^-} w_4(r) = \lim_{r\to r_{\max}^-} h_4(\ln(r))w_1(r) = \infty,\\ \lim_{r\to 0^+}\frac{w_4(r)}{w_1(r)} = \lim_{r\to 0^+} h_4(\ln(r)) = 1.\end{gathered} \tag{2.87}$$

This completes the proof of (2.72). □

Proof of (4). Let h_5 denote a solution of (1.3) which satisfies part (iv) of Theorem 2.1. By (1.20), the solution of (1.3) corresponding to h_5 is

$$w_5(r) = h_5(\ln(r))w_1(r). \tag{2.88}$$

Initial conditions (2.73) follow exactly as in the proof of part (i). Let $(r_{\min}, r_{\max})$ be the maximal interval over which $w_5(r) > 0$. It follows from (2.9) in Theorem 2.1 that $r_{\min} = 0$ and $r_{\max} < \infty$, and $h_5(\ln(r)) < 1$ for all $r \in (0, r_{\max})$. This, together with (2.88), implies that

$$0 < w_5(r) < w_1(r) \quad \forall r \in (0, r_{\max}). \tag{2.89}$$

This proves (2.74). Properties (2.9) in Theorem 2.1, and (2.88), imply that

$$\begin{gathered}\lim_{r\to 0^+}\frac{w_3(r)}{w_1(r)} = \lim_{r\to 0^+} h_3(\ln(r)) = 1,\\ \lim_{r\to r_{\max}^-} w_5(r) = \lim_{r\to r_{\max}^-} h_5(\ln(r))w_1(r) = 0.\end{gathered} \tag{2.90}$$

This completes the proof of (2.75). Therefore, Theorem 2.2 is proved. □

3. Conclusions

In this paper, our analytic advance is the development of methods to efficiently prove the existence and asymptotic behavior of families of positive singular solutions of (1.3). Our approach consists of the following three steps.

Step 1. Transform the nonautonomous w equation (1.3) into the autonomous h equation (1.17) by setting

$$h(\tau) = \frac{w(\exp(\tau))}{w_1(\exp(\tau))}, \quad -\infty < \tau < \infty. \tag{3.1}$$

Step 2. Analyze the existence and asymptotic behavior of solutions of (1.17) which are positive on a maximal interval $(\tau_{\min}, \tau_{\max})$.

Step 3. For each such solution of the h equation, make use of the inverse transformation

$$w(r) = h(\ln(r))w_1(r), \quad 0 < r < \infty \tag{3.2}$$

to prove the existence and asymptotic behavior of the associated solution (3.2) of the w equation on the maximal interval $(r_{\min}, r_{\max})$, where $w(r) > 0$.

In Section 2, we used this three-step procedure (see Theorems 2.1 and 2.2) to prove the existence and asymptotic behavior of three new families of solutions of (1.3). Open Problems 1 and 2 describe a fourth family of solutions whose existence is also proved in these theorems, and which satisfy

$$w(r) > w_1(r), \quad w'(r) < 0 \quad \forall r \in (r_{\min}, \infty), \ \lim_{r \to \infty} w(r) = 0. \tag{3.3}$$

The unresolved issue is to prove whether $r_{\min} = 0$ or $r_{\min} > 0$. If $r_{\min} = 0$, then solutions in this family satisfy the limiting property

$$\lim_{r \to 0^+} \frac{w(r)}{w_1(r)} = \infty. \tag{3.4}$$

Thus, as $r \to 0^+$, these "super singular" solutions approach ∞ much faster than the closed form solution $w_1(r)$.

Open Problem 3. Prove the existence and asymptotic behavior of families of positive solutions of (1.3) other than those found in Theorem 2.2. For example, the existence and limiting behavior of the solutions labelled (c) and (f) in Figure 1 have not yet been proved. It is our hope that our techniques can be extended to prove the existence and limiting behavior of these and many other new families of solutions.

Open Problem 4. Determine the role that the singular solutions proved in Theorem 2.2 play in the analysis of the full time-dependent PDE (1.2). Can the analytic techniques developed by Souplet and Weissler [5], and those of Chen and Derrick [6], be extended to apply to these new solutions?

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