Research Article

Strong Convergence Theorems of the General Iterative Methods for Nonexpansive Semigroups in Banach Spaces

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Let E be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping from E to E^* . Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on E such that $\text{Fix}(\mathcal{S}) := \bigcap_{t \geq 0} \text{Fix}(T(t)) \neq \emptyset$, and f is a contraction on E with coefficient $0 < \alpha < 1$. Let F be δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$ and γ a positive real number such that $\gamma < 1/\alpha(1 - \sqrt{1 - \delta/\lambda})$. When the sequences of real numbers $\{\alpha_n\}$ and $\{t_n\}$ satisfy some appropriate conditions, the three iterative processes given as follows: $x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n$, $n \geq 0$, $y_{n+1} = \alpha_n \gamma f(T(t_n)y_n) + (I - \alpha_n F)T(t_n)y_n$, $n \geq 0$, and $z_{n+1} = T(t_n)(\alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n)$, $n \geq 0$ converge strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\text{Fix}(\mathcal{S})$ of the variational inequality $\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x}) \rangle \geq 0$, $x \in \text{Fix}(\mathcal{S})$. Our results extend and improve corresponding ones of Li et al. (2009) Chen and He (2007), and many others.

1. Introduction

Let E be a real Banach space. A mapping T of E into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in E$. We denote by $\text{Fix}(T)$ the set of fixed points of T . A mapping $f : E \rightarrow E$ is called α -contraction if there exists a constant $0 < \alpha < 1$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in E$. A family $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ of mappings of E into itself is called a *nonexpansive semigroup* on E if it satisfies the following conditions:

- (i) $T(0)x = x$ for all $x \in E$;
- (ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

- (iii) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in E$ and $t \geq 0$;
- (iv) for all $x \in E$, the mapping $t \mapsto T(t)x$ is continuous.

We denote by $\text{Fix}(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is,

$$\text{Fix}(\mathcal{S}) := \{x \in E : T(t)x = x, 0 \leq t < \infty\} = \bigcap_{t \geq 0} \text{Fix}(T(t)). \quad (1.1)$$

In [1], Shioji and Takahashi introduced the following implicit iteration in a Hilbert space

$$x_n = \alpha_n x + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds, \quad \forall n \in \mathbb{N}, \quad (1.2)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{t_n\}$ is a sequence of positive real numbers which diverges to ∞ . Under certain restrictions on the sequence $\{\alpha_n\}$, Shioji and Takahashi [1] proved strong convergence of the sequence $\{x_n\}$ to a member of $F(\mathcal{S})$. In [2], Shimizu and Takahashi studied the strong convergence of the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds, \quad \forall n \in \mathbb{N} \quad (1.3)$$

in a real Hilbert space where $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on a closed convex subset C of a Banach space E and $\lim_{n \rightarrow \infty} t_n = \infty$. Using viscosity method, Chen and Song [3] studied the strong convergence of the following iterative method for a nonexpansive semigroup $\{T(t) : t \geq 0\}$ with $\text{Fix}(\mathcal{S}) \neq \emptyset$ in a Banach space:

$$x_{n+1} = \alpha_n f(x) + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds, \quad \forall n \in \mathbb{N}, \quad (1.4)$$

where f is a contraction. Note however that their iterate x_n at step n is constructed through the average of the semigroup over the interval $(0, t)$. Suzuki [4] was the first to introduce again in a Hilbert space the following implicit iteration process:

$$x_n = \alpha_n u + (1 - \alpha_n) T(t_n)x_n, \quad \forall n \in \mathbb{N}, \quad (1.5)$$

for the nonexpansive semigroup case. In 2002, Benavides et al. [5], in a uniformly smooth Banach space, showed that if $\mathcal{S}$ satisfies an asymptotic regularity condition and $\{\alpha_n\}$ fulfills the control conditions $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim_{n \rightarrow \infty} \alpha_n / \alpha_{n+1} = 0$, then both the implicit iteration process (1.5) and the explicit iteration process (1.6),

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T(t_n)x_n, \quad \forall n \in \mathbb{N}, \quad (1.6)$$

converge to a same point of $F(\mathcal{S})$. In 2005, Xu [6] studied the strong convergence of the implicit iteration process (1.2) and (1.5) in a uniformly convex Banach space which admits a

weakly sequentially continuous duality mapping. Recently, Chen and He [7] introduced the viscosity approximation process:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \beta_n)T(t_n)x_n, \quad \forall n \in \mathbb{N}, \quad (1.7)$$

where f is a contraction and $\{\alpha_n\}$ is a sequence in $(0, 1)$ and a nonexpansive semigroup $\{T(t) : t \geq 0\}$. The strong convergence theorem of $\{x_n\}$ is proved in a reflexive Banach space which admits a weakly sequentially continuous duality mapping. In [8], Chen et al. introduced and studied modified Mann iteration for nonexpansive mapping in a uniformly convex Banach space.

On the other hand, iterative approximation methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [9–11] and the references therein. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let A be a strongly positive bounded linear operator on H ; that is, there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \bar{\gamma}\|x\|^2 \quad \forall x \in H. \quad (1.8)$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H :

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \quad (1.9)$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H . In 2003, Xu [10] proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \quad n \geq 0, \quad (1.10)$$

converges strongly to the unique solution of the minimization problem (1.9) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Using the viscosity approximation method, Moudafi [12] introduced the following iterative process for nonexpansive mappings (see [13] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H . Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \quad n \geq 0, \quad (1.11)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. It is proved [12, 13] that, under certain appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.11) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad x \in H. \quad (1.12)$$

Recently, Marino and Xu [14] mixed the iterative method (1.10) and the viscosity approximation method (1.11) and considered the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \quad (1.13)$$

where A is a strongly positive bounded linear operator on H . They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies the certain conditions, then the sequence $\{x_n\}$ generated by (1.13) converges strongly to the unique solution x^* in H of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in H \quad (1.14)$$

which is the optimality condition for the minimization problem, $\min_{x \in C} (1/2)\langle Ax, x \rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Very recently, Li et al. [15] introduced the following iterative procedures for the approximation of common fixed points of a one-parameter nonexpansive semigroup on a Hilbert space H :

$$x_0 = x \in H, \quad x_{n+1} = (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds + \alpha_n \gamma f(x_n), \quad n \geq 0, \quad (1.15)$$

where A is a strongly positive bounded linear operator on H .

Let δ and λ be two positive real numbers such that $\delta, \lambda < 1$. Recall that a mapping F with domain $D(F)$ and range $R(F)$ in E is called δ -strongly accretive if, for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y) \rangle \geq \delta \|x - y\|^2, \quad (1.16)$$

where J is the normalized duality mapping from E into the dual space E^* . Recall also that a mapping F is called λ -strictly pseudocontractive if, for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|(x - y) - (Fx - Fy)\|^2. \quad (1.17)$$

It is easy to see that (1.17) can be rewritten as

$$\langle (I - F)x - (I - F)y, j(x - y) \rangle \geq \lambda \|(I - F)x - (I - F)y\|^2, \quad (1.18)$$

see [16].

In this paper, motivated by the above results, we introduce and study the strong convergence theorems of the general iterative scheme $\{x_n\}$ defined by (1.19) in the framework of a reflexive Banach space E which admits a weakly sequentially continuous duality mapping:

$$x_0 = x \in E, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n, \quad n \geq 0, \quad (1.19)$$

where F is δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$, f is a contraction on E with coefficient $0 < \alpha < 1$, γ is a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$, and $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ is a nonexpansive semigroup on E . The strong convergence theorems are proved under some appropriate control conditions on parameters $\{\alpha_n\}$ and $\{t_n\}$. Furthermore, by using these results, we obtain strong convergence theorems of the following new general iterative schemes $\{y_n\}$ and $\{z_n\}$ defined by

$$y_0 = y \in E, \quad y_{n+1} = \alpha_n \gamma f(T(t_n)y_n) + (I - \alpha_n F)T(t_n)y_n, \quad n \geq 0, \quad (1.20)$$

$$z_0 = z \in E, \quad z_{n+1} = T(t_n)(\alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n), \quad n \geq 0. \quad (1.21)$$

The results presented in this paper extend and improve the main results in Li et al. [15], Chen and He [7], and many others.

2. Preliminaries

Throughout this paper, it is assumed that E is a real Banach space with norm $\|\cdot\|$ and let J denote the normalized duality mapping from E into E^* given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\} \quad (2.1)$$

for each $x \in E$, where E^* denotes the dual space of E , $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing, and $\mathbb{N}$ denotes the set of all positive integers. In the sequel, we will denote the single-valued duality mapping by j , and consider $F(T) = \{x \in C : Tx = x\}$. When $\{x_n\}$ is a sequence in E , then $x_n \rightarrow x$ (resp., $x_n \rightharpoonup x$, $x_n \xrightarrow{*} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x . In a Banach space E , the following result (*the subdifferential inequality*) is well known [17, Theorem 4.2.1]: for all $x, y \in E$, for all $j(x+y) \in J(x+y)$, for all $j(x) \in J(x)$,

$$\|x\|^2 + 2\langle y, j(x) \rangle \leq \|x+y\|^2 \leq \|x\|^2 + \langle y, j(x+y) \rangle. \quad (2.2)$$

A real Banach space E is said to be *strictly convex* if $\|x+y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if, for all $\epsilon \in [0, 2]$, there exists $\delta_\epsilon > 0$ such that

$$\|x\| = \|y\| = 1 \quad \text{with} \quad \|x-y\| \geq \epsilon \quad \text{implies} \quad \frac{\|x+y\|}{2} < 1 - \delta_\epsilon. \quad (2.3)$$

The following results are well known and can be founded in [17]:

- (i) a uniformly convex Banach space E is reflexive and strictly convex [17, Theorems 4.2.1 and 4.1.6],
- (ii) if E is a strictly convex Banach space and $T : E \rightarrow E$ is a nonexpansive mapping, then fixed point set $F(T)$ of T is a closed convex subset of E [17, Theorem 4.5.3].

If a Banach space E admits a sequentially continuous duality mapping J from weak topology to weak star topology, then from Lemma 1 of [18], it follows that the duality mapping J is single-valued and also E is smooth. In this case, duality mapping J is also said to be *weakly sequentially continuous*, that is, for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, then $J(x_n) \xrightarrow{*} J(x)$ (see [18, 19]).

In the sequel, we will denote the single-valued duality mapping by j . A Banach space E is said to satisfy *Opial's condition* if, for any sequence $\{x_n\}$ in E , $x_n \rightharpoonup x$ as $n \rightarrow \infty$ implies

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\| \quad \forall y \in E \text{ with } x \neq y. \quad (2.4)$$

By Theorem 1 of [18], we know that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition and E is smooth; for the details, see [18].

Now, we present the concept of uniformly asymptotically regular semigroup (also see [20, 21]). Let C be a nonempty closed convex subset of a Banach space E , $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ a continuous operator semigroup on C . Then, $\mathcal{S}$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) on C if, for all $h \geq 0$ and any bounded subset D of C ,

$$\lim_{t \rightarrow \infty} \sup_{x \in D} \|T(h)(T(t)x) - T(t)x\| = 0. \quad (2.5)$$

The nonexpansive semigroup $\{\sigma_t : t > 0\}$ defined by the following lemma is an example of u.a.r. operator semigroup. Other examples of u.a.r. operator semigroup can be found in [20, Examples 17 and 18].

Lemma 2.1 (see [3, Lemma 2.7]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E , D a bounded closed convex subset of C , and $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. For each $h > 0$, set $\sigma_t(x) = (1/t) \int_0^t T(s)x ds$, then*

$$\lim_{t \rightarrow \infty} \sup_{x \in D} \|\sigma_t(x) - T(h)\sigma_t(x)\| = 0. \quad (2.6)$$

Example 2.2. The set $\{\sigma_t : t > 0\}$ defined by Lemma 2.1 is u.a.r. nonexpansive semigroup. In fact, it is obvious that $\{\sigma_t : t > 0\}$ is a nonexpansive semigroup. For each $h > 0$, we have

$$\begin{aligned} \|\sigma_t(x) - \sigma_h\sigma_t(x)\| &= \left\| \sigma_t(x) - \frac{1}{h} \int_0^h T(s)\sigma_t(x) ds \right\| \\ &= \left\| \frac{1}{h} \int_0^h (\sigma_t(x) - T(s)\sigma_t(x)) ds \right\| \\ &\leq \frac{1}{h} \int_0^h \|\sigma_t(x) - T(s)\sigma_t(x)\| ds. \end{aligned} \quad (2.7)$$

Applying Lemma 2.1, we have

$$\lim_{t \rightarrow \infty} \sup_{x \in D} \|\sigma_t(x) - \sigma_h \sigma_t(x)\| \leq \frac{1}{h} \int_0^h \lim_{t \rightarrow \infty} \sup_{x \in D} \|\sigma_t(x) - T(s)\sigma_t(x)\| ds = 0. \quad (2.8)$$

Let C be a nonempty closed and convex subset of a Banach space E and D a nonempty subset of C . A mapping $Q : C \rightarrow D$ is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx, \quad (2.9)$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q : C \rightarrow D$ is called a retraction if $Qx = x$ for all $x \in D$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D . The following lemma concerns the sunny nonexpansive retraction.

Lemma 2.3 (see [22, 23]). *Let C be a closed convex subset of a smooth Banach space E . Let D be a nonempty subset of C and $Q : C \rightarrow D$ be a retraction. Then, Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu) \rangle \leq 0 \quad (2.10)$$

for all $u \in C$ and $y \in D$.

Lemma 2.4 (see [24, Lemma 2.3]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - t_n)a_n + t_n c_n + b_n, \quad \forall n \geq 0, \quad (2.11)$$

where $\{t_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy the restrictions

- (i) $\sum_{n=1}^{\infty} t_n = \infty$;
- (ii) $\sum_{n=1}^{\infty} b_n < \infty$;
- (iii) $\limsup_{n \rightarrow \infty} c_n \leq 0$.

Then, $\lim_{n \rightarrow \infty} a_n = 0$.

The following lemma will be frequently used throughout the paper and can be found in [25].

Lemma 2.5 (see [25, Lemma 2.7]). *Let E be a real smooth Banach space and $F : E \rightarrow E$ a mapping.*

- (i) *If F is δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$, then $I - F$ is contractive with constant $\sqrt{(1 - \delta)/\lambda}$.*
- (i) *If F is δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$, then, for any fixed number $\tau \in (0, 1)$, $I - \tau F$ is contractive with constant $1 - \tau(1 - \sqrt{(1 - \delta)/\lambda})$.*

3. Main Results

Now, we are in a position to state and prove our main results.

Theorem 3.1. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J . Let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that $\text{Fix}(\mathcal{S}) \neq \emptyset$. Suppose that the real sequences $\{\alpha_n\} \subset [0, 1]$, $\{t_n\} \subset (0, \infty)$ satisfy the conditions*

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n \rightarrow \infty} t_n = \infty. \quad (3.1)$$

Let F be δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \rightarrow E$ a contraction mapping with coefficient $\alpha \in (0, 1)$, and γ a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{x_n\}$ defined by (1.19) converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\text{Fix}(\mathcal{S})$ of the variational inequality

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x}) \rangle \geq 0, \quad x \in \text{Fix}(\mathcal{S}) \quad (3.2)$$

or equivalently $\tilde{x} = Q_{\text{Fix}(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{\text{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\text{Fix}(\mathcal{S})$.

Proof. Note that $\text{Fix}(\mathcal{S})$ is a nonempty closed convex set. We first show that $\{x_n\}$ is bounded. Let $q \in \text{Fix}(\mathcal{S})$. Thus, by Lemma 2.5, we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n - (I - \alpha_n F)q - \alpha_n Fq\| \\ &\leq \alpha_n \|\gamma f(x_n) - Fq\| + \|I - \alpha_n F\| \|T(t_n)x_n - q\| \\ &\leq \alpha_n \gamma \|f(x_n) - f(q)\| + \alpha_n \|\gamma f(q) - Fq\| + \|I - \alpha_n F\| \|x_n - q\| \\ &\leq \alpha_n \alpha \gamma \|x_n - q\| + \alpha_n \|\gamma f(q) - Fq\| \\ &\quad + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|x_n - q\| \\ &= \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \alpha \gamma\right)\right) \|x_n - q\| \\ &\quad + \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \alpha \gamma\right) \frac{\|\gamma f(q) - Fq\|}{1 - \sqrt{(1-\delta)/\lambda} - \alpha \gamma} \\ &\leq \max \left\{ \|x_n - q\|, \frac{1}{1 - \sqrt{(1-\delta)/\lambda} - \alpha \gamma} \|\gamma f(q) - Fq\| \right\}, \quad \forall n \geq 0. \end{aligned} \quad (3.3)$$

By induction, we get

$$\|x_n - q\| \leq \max \left\{ \|x_0 - q\|, \frac{1}{1 - \sqrt{(1-\delta)/\lambda} - \alpha\gamma} \|\gamma f(q) - Fq\| \right\}, \quad n \geq 0. \quad (3.4)$$

This implies that $\{x_n\}$ is bounded and, hence, so are $\{f(x_n)\}$ and $\{FT(t_n)x_n\}$. This implies that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T(t_n)x_n\| = \lim_{n \rightarrow \infty} \alpha_n \|\gamma f(x_n) - FT(t_n)x_n\| = 0. \quad (3.5)$$

Since $\{T(t)\}$ is a u.a.r. nonexpansive semigroup and $\lim_{n \rightarrow \infty} t_n = \infty$, we have, for all $h > 0$,

$$\lim_{n \rightarrow \infty} \|T(h)(T(t_n)x_n) - T(t_n)x_n\| \leq \lim_{n \rightarrow \infty} \sup_{x \in \{x_n\}} \|T(h)(T(t_n)x) - T(t_n)x\| = 0. \quad (3.6)$$

Hence, for all $h > 0$,

$$\begin{aligned} \|x_{n+1} - T(h)x_{n+1}\| &\leq \|x_{n+1} - T(t_n)x_n\| + \|T(t_n)x_n - T(h)T(t_n)x_n\| + \|T(h)T(t_n)x_n - T(h)x_{n+1}\| \\ &\leq 2\|x_{n+1} - T(t_n)x_n\| + \|T(t_n)x_n - T(h)T(t_n)x_n\| \longrightarrow 0. \end{aligned} \quad (3.7)$$

That is, for all $h > 0$,

$$\lim_{n \rightarrow \infty} \|x_n - T(h)x_n\| = 0. \quad (3.8)$$

Let $\Phi = Q_{\text{Fix}(S)}$. Then, $\Phi(I - F - \gamma f)$ is a contraction on E . In fact, from Lemma 2.5(i), we have

$$\begin{aligned} \|\Phi(I - F - \gamma f)x - \Phi(I - F - \gamma f)y\| &\leq \|(I - F - \gamma f)x - (I - F - \gamma f)y\| \\ &\leq \|(I - F)x - (I - F)y\| + \gamma\|f(x) - f(y)\| \\ &\leq \sqrt{\frac{1-\delta}{\lambda}}\|x - y\| + \alpha\gamma\|x - y\| \\ &= \left(\sqrt{\frac{1-\delta}{\lambda}} + \alpha\gamma \right) \|x - y\|, \quad \forall x, y \in E. \end{aligned} \quad (3.9)$$

Therefore, $\Phi(I - F - \gamma f)$ is a contraction on E due to $(\sqrt{(1-\delta)/\lambda} + \alpha\gamma) \in (0, 1)$. Thus, by Banach contraction principle, $Q_{\text{Fix}(S)}(I - F - \gamma f)$ has a unique fixed point $\tilde{x}$. Then, using Lemma 2.3, $\tilde{x}$ is the unique solution in $\text{Fix}(S)$ of the variational inequality (3.2). Next, we show that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x}) \rangle \leq 0. \quad (3.10)$$

Indeed, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x}) \rangle = \lim_{k \rightarrow \infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n_k} - \tilde{x}) \rangle. \quad (3.11)$$

We may assume that $x_{n_k} \rightharpoonup p \in E$ as $k \rightarrow \infty$, since a Banach space E has a weakly sequentially continuous duality mapping J satisfying Opial's condition [13]. We will prove that $p \in \text{Fix}(\mathcal{S})$. Suppose the contrary, $p \notin \text{Fix}(\mathcal{S})$, that is, $T(h_0)p \neq p$ for some $h_0 > 0$. It follows from (3.8) and Opial's condition that

$$\begin{aligned} \liminf_{k \rightarrow \infty} \|x_{n_k} - p\| &< \liminf_{k \rightarrow \infty} \|x_{n_k} - T(h_0)p\| \\ &\leq \liminf_{k \rightarrow \infty} \{ \|x_{n_k} - T(h_0)x_{n_k}\| + \|T(h_0)x_{n_k} - T(h_0)p\| \} \\ &\leq \liminf_{k \rightarrow \infty} \{ \|x_{n_k} - T(h_0)x_{n_k}\| + \|x_{n_k} - p\| \} \\ &= \liminf_{k \rightarrow \infty} \|x_{n_k} - p\|. \end{aligned} \quad (3.12)$$

This is a contradiction, which shows that $p \in F(T(h))$ for all $h > 0$, that is, $p \in \text{Fix}(\mathcal{S})$. In view of the variational inequality (3.2) and the assumption that duality mapping J is weakly sequentially continuous, we conclude

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x}) \rangle &= \lim_{k \rightarrow \infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n_k} - \tilde{x}) \rangle \\ &\leq \langle \gamma f(\tilde{x}) - F\tilde{x}, j(p - \tilde{x}) \rangle \leq 0. \end{aligned} \quad (3.13)$$

Finally, we will show that $x_n \rightarrow \tilde{x}$. For each $n \geq 0$, we have

$$\begin{aligned} \|x_{n+1} - \tilde{x}\|^2 &= \|\alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n - (I - \alpha_n F)\tilde{x} - \alpha_n F\tilde{x}\|^2 \\ &\leq \|\alpha_n \gamma f(x_n) - \alpha_n F\tilde{x} + (I - \alpha_n F)T(t_n)x_n - (I - \alpha_n F)\tilde{x}\|^2 \\ &= \|(I - \alpha_n F)T(t_n)x_n - (I - \alpha_n F)\tilde{x}\|^2 + 2\alpha_n \langle \gamma f(x_n) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle \\ &\leq \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}} \right) \right)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n \langle \gamma f(x_n) - \gamma f(\tilde{x}), j(x_{n+1} - \tilde{x}) \rangle \\ &\quad + 2\alpha_n \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle. \end{aligned} \quad (3.14)$$

On the other hand,

$$\begin{aligned}
 & \langle \gamma f(x_n) - \gamma f(\tilde{x}), j(x_{n+1} - \tilde{x}) \rangle \\
 & \leq \gamma \alpha \|x_n - \tilde{x}\| \|x_{n+1} - \tilde{x}\| \\
 & \leq \gamma \alpha \|x_n - \tilde{x}\| \left[\sqrt{\left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n |\langle \gamma f(x_n) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle|} \right] \\
 & \leq \gamma \alpha \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|x_n - \tilde{x}\|^2 \\
 & \quad + \gamma \alpha \|x_n - \tilde{x}\| \sqrt{2 |\langle \gamma f(x_n) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle|} \sqrt{\alpha_n} \\
 & \leq \gamma \alpha \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|x_n - \tilde{x}\|^2 + \sqrt{\alpha_n} M_0,
 \end{aligned} \tag{3.15}$$

where M_0 is a constant satisfying $M_0 \geq \gamma \alpha \|x_n - \tilde{x}\| \sqrt{2 |\langle \gamma f(x_n) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle|}$. Substituting (3.15) in (3.14), we obtain

$$\begin{aligned}
 \|x_{n+1} - \tilde{x}\|^2 & \leq \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n \gamma \alpha \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \\
 & \quad \times \|x_n - \tilde{x}\|^2 + 2\alpha_n \sqrt{\alpha_n} M_0 + 2\alpha_n \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle \\
 & = \left(1 - 2\alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) + \alpha_n^2 \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^2\right) \|x_n - \tilde{x}\|^2 \\
 & \quad + 2\alpha_n \gamma \alpha \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|x_n - \tilde{x}\|^2 \\
 & \quad + 2\alpha_n \sqrt{\alpha_n} M_0 + 2\alpha_n \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle \\
 & = \left(1 - 2\alpha_n \left[\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) - \alpha \gamma + \alpha_n \gamma \alpha \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) \right]\right) \|x_n - \tilde{x}\|^2 \\
 & \quad + \alpha_n \left[\alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^2 \|x_n - \tilde{x}\|^2 + 2M_0 \sqrt{\alpha_n} + 2 \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle \right] \\
 & = (1 - \alpha_n \gamma_n) \|x_n - \tilde{x}\|^2 + \alpha_n \gamma_n \frac{\beta_n}{\gamma_n},
 \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} \gamma_n &= 2 \left[\left(1 - \sqrt{\frac{1-\delta}{\lambda}} \right) - \alpha\gamma + \alpha_n\gamma\alpha \left(1 - \sqrt{\frac{1-\delta}{\lambda}} \right) \right], \\ \beta_n &= \left[\alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}} \right)^2 \|x_n - \tilde{x}\|^2 + 2M_0\sqrt{\alpha_n} + 2\langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle \right]. \end{aligned} \quad (3.17)$$

It is easily seen that $\sum_{n=1}^{\infty} \alpha_n \gamma_n = \infty$. Since $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \alpha_n = 0$, by (3.46), we obtain $\limsup_{n \rightarrow \infty} \beta_n / \gamma_n \leq 0$, applying Lemma 2.4 to (3.16) to conclude $x_n \rightarrow \tilde{x}$ as $n \rightarrow \infty$. This completes the proof. $\square$

Using Theorem 3.1, we obtain the following two strong convergence theorems of new iterative approximation methods for a nonexpansive semigroup $\{T(t) : 0 \leq t < \infty\}$.

Corollary 3.2. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J . Let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that $\text{Fix}(\mathcal{S}) \neq \emptyset$. Suppose that the real sequences $\{\alpha_n\} \subset [0, 1]$, $\{t_n\} \subset (0, \infty)$ satisfy the conditions*

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n \rightarrow \infty} t_n = \infty. \quad (3.18)$$

Let F be δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \rightarrow E$ a contraction mapping with coefficient $\alpha \in (0, 1)$, and γ a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{y_n\}$ defined by (1.20) converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\text{Fix}(\mathcal{S})$ of the variational inequality

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x}) \rangle \geq 0, \quad x \in \text{Fix}(\mathcal{S}) \quad (3.19)$$

or equivalently $\tilde{x} = Q_{\text{Fix}(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{\text{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\text{Fix}(\mathcal{S})$.

Proof. Let $\{x_n\}$ be the sequence given by $x_0 = y_0$ and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n, \quad \forall n \geq 0. \quad (3.20)$$

Form Theorem 3.1, $x_n \rightarrow \tilde{x}$. We claim that $y_n \rightarrow \tilde{x}$. Indeed, we estimate

$$\begin{aligned} &\|x_{n+1} - y_{n+1}\| \\ &\leq \alpha_n \gamma \|f(T(t_n)y_n) - f(x_n)\| + \|I - \alpha_n F\| \|T(t_n)x_n - T(t_n)y_n\| \\ &\leq \alpha_n \gamma \alpha \|T(t_n)y_n - x_n\| + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}} \right) \right) \|x_n - y_n\| \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \gamma \alpha \|T(t_n)y_n - T(t_n)\tilde{x}\| + \alpha_n \gamma \alpha \|T(t_n)\tilde{x} - x_n\| + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|x_n - y_n\| \\
&\leq \alpha_n \gamma \alpha \|y_n - \tilde{x}\| + \alpha_n \gamma \alpha \|\tilde{x} - x_n\| + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|x_n - y_n\| \\
&\leq \alpha_n \gamma \alpha \|y_n - x_n\| + \alpha_n \gamma \alpha \|x_n - \tilde{x}\| + \alpha_n \gamma \alpha \|\tilde{x} - x_n\| + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|x_n - y_n\| \\
&= \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \gamma \alpha\right)\right) \|x_n - y_n\| \\
&\quad + \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \gamma \alpha\right) \frac{2\alpha\gamma}{\left(1 - \sqrt{(1-\delta)/\lambda} - \gamma \alpha\right)} \|\tilde{x} - x_n\|.
\end{aligned} \tag{3.21}$$

It follows from $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n \rightarrow \infty} \|x_n - \tilde{x}\| = 0$, and Lemma 2.4 that $\|x_n - y_n\| \rightarrow 0$. Consequently, $y_n \rightarrow \tilde{x}$ as required. $\square$

Corollary 3.3. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J . Let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that $\text{Fix}(\mathcal{S}) \neq \emptyset$. Suppose that the real sequences $\{\alpha_n\} \subset [0, 1]$, $\{t_n\} \subset (0, \infty)$ satisfy the conditions*

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n \rightarrow \infty} t_n = \infty. \tag{3.22}$$

Let F be δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \rightarrow E$ a contraction mapping with coefficient $\alpha \in (0, 1)$, and γ a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{z_n\}$ defined by (1.21) converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\text{Fix}(\mathcal{S})$ of the variational inequality

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x}) \rangle \geq 0, \quad x \in \text{Fix}(\mathcal{S}) \tag{3.23}$$

or equivalently $\tilde{x} = Q_{\text{Fix}(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{\text{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\text{Fix}(\mathcal{S})$.

Proof. Define the sequences $\{y_n\}$ and $\{\beta_n\}$ by

$$y_n = \alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n, \quad \beta_n = \alpha_{n+1} \quad \forall n \in \mathbb{N}. \tag{3.24}$$

Taking $p \in \text{Fix}(\mathcal{S})$, we have

$$\begin{aligned}
 \|z_{n+1} - p\| &= \|T(t_n)y_n - T(t_n)p\| \leq \|y_n - p\| \\
 &= \|\alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n - (I - \alpha_n F)p - \alpha_n Fp\| \\
 &\leq \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|z_n - p\| + \alpha_n \|\gamma f(z_n) - F(p)\| \\
 &= \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|z_n - p\| + \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) \frac{\|\gamma f(z_n) - F(p)\|}{\left(1 - \sqrt{(1-\delta)/\lambda}\right)}.
 \end{aligned} \tag{3.25}$$

It follows from induction that

$$\|z_{n+1} - p\| \leq \max \left\{ \|z_0 - p\|, \frac{\|\gamma f(z_0) - F(p)\|}{1 - \sqrt{(1-\delta)/\lambda}} \right\}, \quad n \geq 0. \tag{3.26}$$

Thus, both $\{z_n\}$ and $\{y_n\}$ are bounded. We observe that

$$y_{n+1} = \alpha_{n+1} \gamma f(z_{n+1}) + (I - \alpha_{n+1} F)z_{n+1} = \beta_n \gamma f(T(t_n)y_n) + (I - \beta_n F)T(t_n)y_n. \tag{3.27}$$

Thus, Corollary 3.2 implies that $\{y_n\}$ converges strongly to some point $\tilde{x}$. In this case, we also have

$$\|z_n - \tilde{x}\| \leq \|z_n - y_n\| + \|y_n - \tilde{x}\| = \alpha_n \|\gamma f(z_n) - Fz_n\| + \|y_n - \tilde{x}\| \longrightarrow 0. \tag{3.28}$$

Hence, the sequence $\{z_n\}$ converges strongly to some point $\tilde{x}$. This complete the proof. $\square$

Using Theorem 3.1, Lemma 2.1, and Example 2.2, we have the following result.

Corollary 3.4. *Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J . Let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a nonexpansive semigroup on E such that $\text{Fix}(\mathcal{S}) \neq \emptyset$. Suppose that the real sequences $\{\alpha_n\} \subset [0, 1]$, $\{t_n\} \subset (0, \infty)$ satisfy the conditions*

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n \rightarrow \infty} t_n = \infty. \tag{3.29}$$

Let F be δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \rightarrow E$ a contraction mapping with coefficient $\alpha \in (0, 1)$, and γ a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{x_n\}$ defined by

$$\begin{aligned}
 x_0 &= x \in E, \\
 x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n F) \frac{1}{t_n} \int_0^{t_n} T(t)x_n ds, \quad n \geq 0
 \end{aligned} \tag{3.30}$$

converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\text{Fix}(\mathcal{S})$ of the variational inequality

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x}) \rangle \geq 0, \quad x \in \text{Fix}(\mathcal{S}) \quad (3.31)$$

or equivalently $\tilde{x} = Q_{\text{Fix}(\mathcal{S})}((I - F + \gamma f)\tilde{x})$, where $Q_{\text{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\text{Fix}(\mathcal{S})$.

Corollary 3.5. Let H be a real Hilbert space. Let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a nonexpansive semigroup on H such that $\text{Fix}(\mathcal{S}) \neq \emptyset$. Suppose that the real sequences $\{\alpha_n\} \subset [0, 1]$, $\{t_n\} \subset (0, \infty)$ satisfy the conditions

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n \rightarrow \infty} t_n = \infty. \quad (3.32)$$

Let $f : E \rightarrow E$ be a contraction mapping with coefficient $\alpha \in (0, 1)$ and A a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 1/2$ and $0 < \gamma < (1 - \sqrt{2 - 2\bar{\gamma}})/\alpha$. Then, the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_0 &= x \in E, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(t)x_n ds, \quad n \geq 0 \end{aligned} \quad (3.33)$$

converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\text{Fix}(\mathcal{S})$ of the variational inequality

$$\langle (A - \gamma f)\tilde{x}, j(x - \tilde{x}) \rangle \geq 0, \quad x \in \text{Fix}(\mathcal{S}) \quad (3.34)$$

or equivalently $\tilde{x} = Q_{\text{Fix}(\mathcal{S})}((I - A + \gamma f)\tilde{x})$, where $Q_{\text{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\text{Fix}(\mathcal{S})$.

Proof. Since A is a strongly positive bounded linear operator with coefficient $\bar{\gamma}$, we have

$$\langle Ax - Ay, x - y \rangle \geq \bar{\gamma} \|x - y\|^2. \quad (3.35)$$

Therefore, A is $\bar{\gamma}$ -strongly accretive. On the other hand,

$$\begin{aligned} \|(I - A)x - (I - A)y\|^2 &= \langle (x - y) - (Ax - Ay), (x - y) - (Ax - Ay) \rangle \\ &= \langle x - y, x - y \rangle - 2\langle Ax - Ay, x - y \rangle + \langle Ax - Ay, Ax - Ay \rangle \\ &= \|x - y\|^2 - 2\langle Ax - Ay, x - y \rangle + \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\langle Ax - Ay, x - y \rangle + \|A\|^2 \|x - y\|^2. \end{aligned} \quad (3.36)$$

Since A is strongly positive if and only if $(1/\|A\|)A$ is strongly positive, we may assume, without loss of generality, that $\|A\| = 1$, so that

$$\begin{aligned}\langle Ax - Ay, x - y \rangle &\leq \|x - y\|^2 - \frac{1}{2} \|(I - A)x - (I - A)y\|^2 \\ &= \|x - y\|^2 - \frac{1}{2} \|(x - y) - (Ax - Ay)\|^2.\end{aligned}\quad (3.37)$$

Hence, A is 12-strongly pseudocontractive. Applying Corollary 3.4, we conclude the result. $\square$

Theorem 3.6. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J . Let $\mathcal{S} = \{T(t) : 0 < t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that $\text{Fix}(\mathcal{S}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real number satisfying*

$$0 < \alpha_n < 1, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad t_n > 0, \quad \lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \frac{\alpha_n}{t_n} = 0. \quad (3.38)$$

Let F be δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \rightarrow E$ a contraction mapping with coefficient $\alpha \in (0, 1)$, and γ a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1 - \delta)/\lambda})$. Then, the sequence $\{x_n\}$ defined by

$$\begin{aligned}x_0 &= x \in E, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n, \quad n \geq 0\end{aligned}\quad (3.39)$$

converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\text{Fix}(\mathcal{S})$ of the variational inequality

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x}) \rangle \geq 0, \quad x \in \text{Fix}(\mathcal{S}) \quad (3.40)$$

or equivalently $\tilde{x} = Q_{\text{Fix}(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{\text{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\text{Fix}(\mathcal{S})$.

Proof. By the same argument as in the proof of Theorem 3.1, we can obtain that $\{x_n\}$, $\{f(x_n)\}$, and $\{FT(t_n)x_n\}$ are bounded and $Q_{\text{Fix}(\mathcal{S})}(I - F - \gamma f)$ is a contraction on E . Thus, by Banach contraction principle, $Q_{\text{Fix}(\mathcal{S})}(I - F - \gamma f)$ has a unique fixed point $\tilde{x}$. Then, using Lemma 2.3, $\tilde{x}$ is the unique solution in $\text{Fix}(\mathcal{S})$ of the variational inequality (3.40). Next, we show that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x}) \rangle \leq 0. \quad (3.41)$$

Indeed, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x}) \rangle = \lim_{k \rightarrow \infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n_k} - \tilde{x}) \rangle. \quad (3.42)$$

We may assume that $x_{n_k} \rightarrow p \in E$ as $k \rightarrow \infty$. Now, we show that $p \in \text{Fix}(\mathcal{S})$. Put

$$x_k = x_{n_k}, \quad \alpha_k = \alpha_{n_k}, \quad s_k = t_{n_k} \quad \forall k \in \mathbb{N}. \quad (3.43)$$

Fix $t > 0$, then we have

$$\begin{aligned} \|x_k - T(t)p\| &= \sum_{i=0}^{\lfloor t/s_k \rfloor - 1} \|T((i+1)s_k)x_k - T(is_k)x_k\| \\ &\quad + \left\| T\left(\left[\frac{t}{s_k}\right]s_k\right)x_k - T\left(\left[\frac{t}{s_k}\right]s_k\right)p \right\| + \left\| T\left(\left[\frac{t}{s_k}\right]s_k\right)p - T(t)p \right\| \\ &\leq \left[\frac{t}{s_k}\right] \|T(s_k)x_k - x_{k+1}\| + \|x_{k+1} - p\| + \left\| T\left(t - \left[\frac{t}{s_k}\right]s_k\right)p - p \right\| \\ &\leq \left[\frac{t}{s_k}\right] \alpha_k \|FT(s_k)x_k - f(x_k)\| + \|x_{k+1} - p\| + \left\| T\left(t - \left[\frac{t}{s_k}\right]s_k\right)p - p \right\| \\ &\leq \left(\frac{t\alpha_k}{s_k}\right) \|FT(s_k)x_k - f(x_k)\| + \|x_{k+1} - p\| + \max\{\|T(s)p - p\| : 0 \leq s \leq s_k\}. \end{aligned} \quad (3.44)$$

Thus, for all $k \in \mathbb{N}$, we obtain

$$\limsup_{k \rightarrow \infty} \|x_k - T(t)p\| \leq \limsup_{k \rightarrow \infty} \|x_{k+1} - p\| = \limsup_{k \rightarrow \infty} \|x_k - p\|. \quad (3.45)$$

Since Banach space E has a weakly sequentially continuous duality mapping satisfying Opial's condition [13], we can conclude that $T(t)p = p$ for all $t > 0$, that is, $p \in \text{Fix}(\mathcal{S})$. In view of the variational inequality (3.2) and the assumption that duality mapping J is weakly sequentially continuous, we conclude

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x}) \rangle &= \lim_{k \rightarrow \infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n_k} - \tilde{x}) \rangle \\ &\leq \langle \gamma f(\tilde{x}) - F\tilde{x}, J(p - \tilde{x}) \rangle \leq 0. \end{aligned} \quad (3.46)$$

By the same argument as in the proof of Theorem 3.1, we conclude that $x_n \rightarrow \tilde{x}$ as $n \rightarrow \infty$. This completes the proof. $\square$

Using Theorem 3.6 and the method as in the proof of Corollary 3.7, we have the following result.

Corollary 3.7. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J . Let $\mathcal{S} = \{T(t) : 0 < t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that $\text{Fix}(\mathcal{S}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real number satisfying*

$$0 < \alpha_n < 1, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad t_n > 0, \quad \lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \frac{\alpha_n}{t_n} = 0. \quad (3.47)$$

Let F be a δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \rightarrow E$ a contraction mapping with coefficient $\alpha \in (0, 1)$, and γ is a positive real number such that $\gamma < 1/\alpha(1 - \sqrt{(1 - \delta)/\lambda})$. Then, the sequence $\{y_n\}$ defined by

$$\begin{aligned} y_0 &= y \in E, \\ y_{n+1} &= \alpha_n \gamma f(T(t_n)y_n) + (I - \alpha_n F)T(t_n)y_n, \quad n \geq 0 \end{aligned} \quad (3.48)$$

converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\text{Fix}(\mathcal{S})$ of the variational inequality

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x}) \rangle \geq 0, \quad x \in \text{Fix}(\mathcal{S}) \quad (3.49)$$

or equivalently $\tilde{x} = Q_{\text{Fix}(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{\text{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\text{Fix}(\mathcal{S})$.

Using Theorem 3.6 and the method as in the proof of Corollary 3.8, we have the following result.

Corollary 3.8. Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J . Let $\mathcal{S} = \{T(t) : 0 < t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that $\text{Fix}(\mathcal{S}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real number satisfying

$$0 < \alpha_n < 1, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad t_n > 0, \quad \lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \frac{\alpha_n}{t_n} = 0. \quad (3.50)$$

Let F be a δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \rightarrow E$ a contraction mapping with coefficient $\alpha \in (0, 1)$, and γ is a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1 - \delta)/\lambda})$. Then, the sequence $\{z_n\}$ defined by

$$\begin{aligned} z_0 &= z \in E, \\ z_{n+1} &= T(t_n)(\alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n), \quad n \geq 0 \end{aligned} \quad (3.51)$$

converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\text{Fix}(\mathcal{S})$ of the variational inequality

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x}) \rangle \geq 0, \quad x \in \text{Fix}(\mathcal{S}) \quad (3.52)$$

or equivalently $\tilde{x} = Q_{\text{Fix}(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{\text{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\text{Fix}(\mathcal{S})$.

Using Theorem 3.6, Lemma 2.1, and Example 2.2, we have the following result.

Corollary 3.9. Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J . Let $\mathcal{S} = \{T(t) : 0 < t < \infty\}$ be a nonexpansive semigroup on E such that $\text{Fix}(\mathcal{S}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying

$$0 < \alpha_n < 1, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad t_n > 0, \quad \lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \frac{\alpha_n}{t_n} = 0. \quad (3.53)$$

Let F be δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \rightarrow E$ a contraction mapping with coefficient $\alpha \in (0, 1)$, and γ a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{x_n\}$ defined by

$$x_0 = x \in E, \quad (3.54)$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F) \frac{1}{t_n} \int_0^{t_n} T(t)x_n ds, \quad n \geq 0$$

converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\text{Fix}(\mathcal{S})$ of the variational inequality

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x}) \rangle \geq 0, \quad x \in \text{Fix}(\mathcal{S}) \quad (3.55)$$

or equivalently $\tilde{x} = Q_{\text{Fix}(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{\text{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\text{Fix}(\mathcal{S})$.

Corollary 3.10. Let H be a real Hilbert space. Let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a nonexpansive semigroup on H such that $\text{Fix}(\mathcal{S}) \neq \emptyset$. Suppose that the real sequences $\{\alpha_n\} \subset [0, 1]$, $\{t_n\} \subset (0, \infty)$ satisfy the conditions

$$0 < \alpha_n < 1, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad t_n > 0, \quad \lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \frac{\alpha_n}{t_n} = 0. \quad (3.56)$$

Let $f : E \rightarrow E$ be a contraction mapping with coefficient $\alpha \in (0, 1)$ and A a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 1/2$ and $0 < \gamma < (1 - \sqrt{2 - 2\bar{\gamma}})/\alpha$. Then, the sequence $\{x_n\}$ defined by

$$x_0 = x \in E, \quad (3.57)$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(t)x_n ds, \quad n \geq 0$$

converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\text{Fix}(\mathcal{S})$ of the variational inequality

$$\langle (A - \gamma f)\tilde{x}, j(x - \tilde{x}) \rangle \geq 0, \quad x \in \text{Fix}(\mathcal{S}) \quad (3.58)$$

or equivalently $\tilde{x} = Q_{\text{Fix}(\mathcal{S})}((I - A + \gamma f)\tilde{x})$, where $Q_{\text{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\text{Fix}(\mathcal{S})$.

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