Research Article

Approximate Quartic and Quadratic Mappings in Quasi-Banach Spaces

M. Eshaghi Gordji,¹ H. Khodaei,¹ and Hark-Mahn Kim²

¹Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran
²Department of Mathematics, Chungnam National University, 220 Yuseong-Gu, Daejeon 305-764, Republic of Korea

Correspondence should be addressed to Hark-Mahn Kim, hmkim@cnu.ac.kr

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we establish the general solution for a mixed type functional equation of a quartic and a quadratic mapping in linear spaces. In addition, we investigate the generalized Hyers-Ulam stability in p-Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let (G₁, ·) be a group, and let (G₂, *) be a metric group with the metric d(·, ·). Given ε > 0, does there exist a δ > 0 such that if a mapping h : G₁ → G₂ satisfies the inequality d(h(x · y), h(x) * h(y)) < δ for all x, y ∈ G₁, then there exists a homomorphism H : G₁ → G₂ with d(h(x), H(x)) < ε for all x ∈ G₁? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let f : X → X' be a mapping between Banach spaces such that

\[ \| f(x + y) - f(x) - f(y) \| \leq \delta, \]  

(1.1)
for all \( x, y \in X \), and for some \( \delta > 0 \). Then, there exists a unique additive mapping \( T : X \to X' \) such that

\[
\|f(x) - T(x)\| \leq \delta,
\]

(1.2)

for all \( x \in X \).

The result of Hyers was generalized by Aoki [3] for approximate additive function and by Rassias [4] for approximate linear function by allowing the difference Cauchy equation \( \|f(x + y) - f(x) - f(y)\| \) to be controlled by \( \epsilon(\|x\|^p + \|y\|^p) \). Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias may be called the Hyers-Ulam-Rassias stability (see [5, 6]). In 1994, a generalization of Rassias theorem was obtained by Gavruta [7], who replaced \( \epsilon(\|x\|^p + \|y\|^p) \) by a general control function \( \phi(x, y) \).

The functional equation

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]

(1.3)

is related to a symmetric biadditive function [8–10]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.3) is said to be a quadratic function. It is well known that a function \( f \) between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function \( B_1 \) such that \( f(x) = B_1(x, x) \) for all \( x \) in the vector space. The biadditive function \( B_1 \) is given by

\[
B_1(x, y) = \frac{1}{4} (f(x + y) - f(x - y)).
\]

(1.4)

A Hyers-Ulam stability problem for the quadratic functional equation (1.3) was proved by Skof for functions \( f : X \to Y \), where \( X \) is normed space and \( Y \) is Banach space (see [11]). In the paper [12], Czerwik proved the Hyers-Ulam-Rassias stability of (1.3).

Lee et al. [13] considered the following functional equation:

\[
f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y).
\]

(1.5)

In fact, they proved that a function \( f \) between two real vector spaces \( X \) and \( Y \) is a solution of (1.5) if and only if there exists a unique symmetric biquadratic function \( B_2 : X \times X \to Y \) such that \( f(x) = B_2(x, x) \) for all \( x \in X \). The biquadratic function \( B_2 \) is given by

\[
B_2(x, y) = \frac{1}{12} (f(x + y) + f(x - y) - 2f(x) - 2f(y)).
\]

(1.6)

It is easy to show that the function \( f(x) = ax^4 \) satisfies the functional equation (1.5), which is called the quartic functional equation (see also [14]).

Jun and Kim [15] have obtained the generalized Hyers-Ulam stability for a mixed type of cubic and additive functional equation. In addition, the generalized Hyers-Ulam stability for a mixed type of cubic, quadratic, and additive functional equation has been investigated by Gordji and Khodaei [16] (see also [17, 18]). The stability problems for several mixed types
of functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [19–27].

In this paper, we deal with the following functional equation derived from quartic and quadratic functions:

\[
f(kx + y) + f(kx - y) = k^2f(x + y) + k^2f(x - y) + \frac{k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) - 2(k^2 - 1)f(y)
\]

(1.7)

for fixed integers \(k \neq 0, \pm 1\). It is easy to see that the function \(f(x) = ax^4 + bx^2\) is a solution of the functional equation (1.7). In the sequel, we investigate the general solution of functional equation (1.7) when \(f\) is a function between vector spaces, and then we prove the generalized Hyers-Ulam stability of (1.7) in the spirit of Hyers, Ulam, and Rassias using the direct method.

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

**Definition 1.1** (see [28, 29]). Let \(X\) be a real linear space. A quasinorm is a real-valued function on \(X\) satisfying the following:

1. \(\|x\| \geq 0\) for all \(x \in X\) and \(\|x\| = 0\) if and only if \(x = 0\),
2. \(\|\lambda \cdot x\| = |\lambda| \cdot \|x\|\) for all \(\lambda \in \mathbb{R}\) and all \(x \in X\),
3. there is a constant \(M \geq 1\) such that \(\|x + y\| \leq M(\|x\| + \|y\|)\) for all \(x, y \in X\).

The pair \((X, \| \cdot \|)\) is called a quasinormed space if \(\| \cdot \|\) is a quasinorm on \(X\).

The smallest possible \(M\) is called the modulus of concavity of \(\| \cdot \|\). A quasi-Banach space is a complete quasinormed space. A quasinorm \(\| \cdot \|\) is called a \(p\)-norm \((0 < p \leq 1)\) if

\[
\|x + y\|^p \leq \|x\|^p + \|y\|^p,
\]

(1.8)

for all \(x, y \in X\). In this case, a quasi-Banach space is called a \(p\)-Banach space.

Given a \(p\)-norm, the formula \(d(x, y) := \|x - y\|^p\) gives us a translation invariant metric on \(X\). By the Aoki-Rolewicz Theorem [29], each quasinorm is equivalent to some \(p\)-norm (see also [28]). Since it is much easier to work with \(p\)-norms, henceforth we restrict our attention mainly to \(p\)-norms.

**Lemma 1.2** (see [17]). Let \(x_1, x_2, \ldots, x_n\) be nonnegative real numbers. Then, one has

\[
\left( \sum_{i=1}^{n} x_i \right)^p \leq \sum_{i=1}^{n} x_i^p,
\]

(1.9)

for a positive real number \(p\) with \(p \leq 1\).

2. **General Solution**

We here present the general solution of (1.7).
Theorem 2.1. Let both $X$ and $Y$ be real vector spaces. A function $f : X \to Y$ satisfies (1.7) for all $x, y \in X$ if and only if there exists a unique symmetric biquadratic function $B_2 : X \times X \to Y$ and a unique symmetric biadditive function $B_1 : X \times X \to Y$ such that

$$f(x) = B_2(x, x) + B_1(x, x),$$

for all $x \in X$.

Proof. Let $f$ satisfy (1.7) and let $g, h : X \to Y$ be functions defined by

$$g(x) := f(2x) - 16f(x), \quad h(x) := f(2x) - 4f(x),$$

for all $x \in X$. We claim that the functions $g$ and $h$ are quadratic and quartic, respectively.

Letting $x = y = 0$ in (1.7), we have $f(0) = 0$. By putting $x = 0$ in (1.7), one leads to the evenness $f(-y) = f(y)$ of $f$. Replacing $y$ by $x + y$ in (1.7), we have

$$f((k + 1)x + y) + f((k - 1)x - y)$$

$$= k^2f(2x + y) + k^2f(-y) + \frac{k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) + 2(1 - k^2)f(x + y),$$

for all $x, y \in X$. Replacing $y$ by $-y$ in (2.3), we obtain

$$f((k + 1)x - y) + f((k - 1)x + y)$$

$$= k^2f(2x - y) + k^2f(y) + \frac{k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) + 2(1 - k^2)f(x - y),$$

for all $x, y \in X$. Adding (2.3) to (2.4), we get by evenness of $f$,

$$f((k + 1)x + y) + f((k + 1)x - y) + f((k - 1)x + y) + f((k - 1)x - y)$$

$$= k^2(2f(2x + y) + f(2x - y)) + \frac{2k^2(k^2 - 1)}{6}(f(2x) - 4f(x))$$

$$+ 2(1 - k^2)(f(x + y) + f(x - y)) + 2k^2f(y),$$

for all $x, y \in X$. From the substitution $y = kx$ in (1.7), we have by evenness of $f$,

$$f(2kx + y) + f(y) = k^2f((k + 1)x + y) + k^2f((k - 1)x + y)$$

$$+ \frac{k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) + 2(1 - k^2)f(kx + y),$$

for all $x, y \in X$. From the substitution $y = kx$ in (1.7), we have by evenness of $f$,
for all $x, y \in X$. Replacing $y$ by $-y$ in (2.6), we get

$$f(2kx - y) + f(-y) = k^2 f((k + 1)x - y) + k^2 f((k - 1)x - y)$$
$$+ \frac{k^2(k^2 - 1)}{6} (f(2x) - 4f(x)) + 2(1 - k^2) f(kx - y),$$

(2.7)

for all $x, y \in X$. Adding (2.6) to (2.7), we get by evenness of $f$,

$$f(2kx + y) + f(2kx - y)$$
$$= k^2(f((k + 1)x + y) + f((k + 1)x - y) + f((k - 1)x + y) + f((k - 1)x - y))$$
$$+ \frac{2k^2(k^2 - 1)}{6} (f(2x) - 4f(x)) + 2(1 - k^2) (f(kx + y) + f(kx - y)) - 2f(y),$$

(2.8)

for all $x, y \in X$. By using (1.7) and (2.5), it follows from (2.8) that

$$f(2kx + y) + f(2kx - y)$$
$$= k^2 \left[ k^2 (f(2x + y) + f(2x - y)) + \frac{2k^2(k^2 - 1)}{6} (f(2x) - 4f(x))$$
$$+ 2(1 - k^2) (f(x + y) + f(x - y)) + 2k^2 f(y) \right]$$
$$+ 2(1 - k^2) \left[ k^2 f(x + y) + k^2 f(x - y) + \frac{k^2(k^2 - 1)}{6} (f(2x) - 4f(x)) + 2(1 - k^2) f(y) \right]$$
$$+ \frac{2k^2(k^2 - 1)}{6} (f(2x) - 4f(x)) - 2f(y),$$

(2.9)

for all $x, y \in X$. If we replace $x$ by $2x$ in (1.7), then we get that

$$f(2kx + y) + f(2kx - y)$$
$$= k^2 f(2x + y) + k^2 f(2x - y) + \frac{k^2(k^2 - 1)}{6} (f(4x) - 4f(2x)) + 2(1 - k^2) f(y),$$

(2.10)
for all \( x, y \in X \). It follows from (2.9) and (2.10) that

\[
k^2 \left[ k^2 (f(2x + y) + f(2x - y)) + \frac{2k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) \right.
\]
\[
+ 2 \left( 1 - k^2 \right) (f(x + y) + f(x - y)) + 2k^2f(y) \left. \right] + 2 \left( 1 - k^2 \right) \left[ k^2f(x + y) + k^2f(x - y) + \frac{k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) + 2 \left( 1 - k^2 \right) f(y) \right]
\]
\[
+ \frac{2k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) - 2f(y)
\]
\[
= k^2f(2x + y) + k^2f(2x - y) + \frac{k^2(k^2 - 1)}{6}(f(4x) - 4f(2x)) + 2 \left( 1 - k^2 \right) f(y),
\]  
(2.11)

for all \( x, y \in X \). On the other hand, putting \( y = 0 \) in (1.7), we get

\[
f(kx) = k^2f(x) + \frac{k^2(k^2 - 1)}{12}(f(2x) - 4f(x)),
\]  
(2.12)

for all \( x \in X \). Putting \( y = x \) in (1.7), we get

\[
f((k + 1)x) + f((k - 1)x) = k^2f(2x) + \frac{k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) + 2 \left( 1 - k^2 \right) f(x),
\]  
(2.13)

for all \( x \in X \). Putting \( y = kx \) in (1.7) and using the evenness of \( f \), we obtain

\[
f(2kx) = k^2(f((k + 1)x) + f((k - 1)x)) + \frac{k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) + 2 \left( 1 - k^2 \right) f(kx),
\]  
(2.14)

for all \( x \in X \). Letting \( y = 0 \) in (2.10), we have

\[
f(2kx) = k^2f(2x) + \frac{k^2(k^2 - 1)}{12}(f(4x) - 4f(2x)),
\]  
(2.15)

for all \( x \in X \). It follows from (2.14) and (2.15) that

\[
\frac{k^2(k^2 - 1)}{12}(f(4x) - 4f(2x)) = k^2(f((k + 1)x) + f((k - 1)x)) + \frac{k^2(k^2 - 1)}{6}(f(2x) - 4f(x))
\]
\[
+ 2 \left( 1 - k^2 \right) f(kx) - k^2f(2x),
\]  
(2.16)
for all $x \in X$. Now, by using (2.12), (2.13) and (2.16), we lead to

$$
\frac{k^2(k^2-1)}{12}(f(4x) - 4f(2x)) = k^2 \left[ k^2 f(2x) + \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) + 2\left(1-k^2\right)f(x) \right] \\
+ 2\left(1-k^2\right) \left[ k^2 f(x) + \frac{k^2(k^2-1)}{12}(f(2x) - 4f(x)) \right] \\
+ \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) - k^2 f(2x),
$$

(2.17)

for all $x \in X$. Finally, comparing (2.11) with (2.17), then we conclude that

$$
f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 2(f(2x) - 4f(x)) - 6f(y),
$$

(2.18)

for all $x, y \in X$. Replacing $y$ by $2y$ in (2.18), we get

$$
f(2x + 2y) + f(2x - 2y) = 4f(x + 2y) + 4f(x - 2y) + 2(f(2x) - 4f(x)) - 6f(2y),
$$

(2.19)

for all $x, y \in X$. Interchanging $x$ with $y$ in (2.18), one gets

$$
f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) + 2(f(2y) - 4f(y)) - 6f(x),
$$

(2.20)

for all $x, y \in X$. It follows from (2.19) and (2.20) that

$$
f(2(x + y)) - 16f(x + y) + f(2(x - y)) - 16f(x - y) \\
= 2(f(2x) - 16f(x)) + 2(f(2y) - 16f(y)),
$$

(2.21)

for all $x, y \in X$. This means that

$$
g(x + y) + g(x - y) = 2g(x) + 2g(y),
$$

(2.22)

for all $x, y \in X$. So the function $g : X \to Y$ defined by $g(x) := f(2x) - 16f(x)$ is quadratic.

To prove that $h : X \to Y$ defined by $h(x) := f(2x) - 4f(x)$ is quartic, we need to show that

$$
h(2x + y) + h(2x - y) = 4h(x + y) + 4h(x - y) + 24h(x) - 6h(y),
$$

(2.23)

for all $x, y \in X$. Replacing $x$ and $y$ by $2x$ and $2y$ in (2.18), respectively, we obtain

$$
f(2(2x + y)) + f(2(2x - y)) = 4f(2(x + y)) + 4f(2(x - y)) + 2(f(4x) - 4f(2x)) - 6f(2y),
$$

(2.24)
from this point on, assume that \( X \) for all \( x \in X \), where \( g : X \to Y \) is a quadratic function defined above, we see that

\[
    f(4x) = 20f(2x) - 64f(x),
    \tag{2.25}
\]

for all \( x \in X \). Hence, according to (2.24) and (2.25), we get

\[
    f(2(2x + y)) + f(2(2x - y)) = 4f(2(x + y)) + 4f(2(x - y)) + 32(f(2x) - 4f(x)) - 6f(2y),
    \tag{2.26}
\]

for all \( x, y \in X \). By multiplying 4 on both sides of (2.18), we get that

\[
    4f(2x + y) + 4f(2x - y) = 16f(x + y) + 16f(x - y) + 8(f(2x) - 4f(x)) - 24f(y),
    \tag{2.27}
\]

for all \( x, y \in X \). If we subtract the last equation from (2.26), then we arrive at

\[
    f(2(2x + y)) - 4f(2x + y) + f(2(2x - y)) - 4f(2x - y)
    \]

\[
    = 4(f(2(x + y)) - 4f(x + y)) + 4(f(2(x - y)) - 4f(x - y))
    \]

\[
    + 24(f(2x) - 4f(x)) - 6(f(2y) - 4f(y)),
    \tag{2.28}
\]

for all \( x, y \in X \). This means that \( h \) satisfies (2.23) and, therefore, the function \( h : X \to Y \) is quartic. Thus, there exists a unique symmetric biquadratic function \( B_2 : X \times X \to Y \) and a unique symmetric biadditive function \( B_1 : X \times X \to Y \) such that \( h(x) = 12B_2(x, x) \) and \( g(x) = -12B_1(x, x) \) for all \( x \in X \) (see [8, 13]). Therefore, we obtain from (2.2) that

\[
    f(x) = \frac{1}{12}h(x) - \frac{1}{12}g(x) = B_2(x, x) + B_1(x, x),
    \tag{2.29}
\]

for all \( x \in X \).

The proof of the converse is trivial.

\[\square\]

3. Generalized Hyers-Ulam Stability

From this point on, assume that \( X \) is a quasinormed space with quasinorm \( \| \cdot \|_X \) and that \( Y \) is a \( p \)-Banach space with \( p \)-norm \( \| \cdot \|_Y \). Let \( M \) be the modulus of concavity of \( \| \cdot \|_Y \).

Before taking up the main subject, given a mapping \( f : X \to Y \), we define the difference operator \( D_f : X \times X \to Y \) by

\[
    D_f(x, y) := f(kx + y) + f(kx - y) - k^2f(x + y) - k^2f(x - y)
    \]

\[
    - \frac{k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) + 2(k^2 - 1)f(y),
    \tag{3.1}
\]

for all \( x, y \in X \). Let \( \varphi^p(x, y) := (\varphi(x, y))^p \) for notational convenience.
Theorem 3.1. Let $j \in \{-1,1\}$ be fixed and let $\varphi_q : X \times X \to [0, \infty)$ be a function such that
\[
\lim_{n \to \infty} 4^{nj} \varphi_q \left( \frac{x}{2^{nj}}, \frac{y}{2^{nj}} \right) = 0,
\]
for all $x, y \in X$ and
\[
\sum_{i=1+j/2}^{\infty} 4^{nj} \varphi_q^p \left( \frac{u}{2^{i}}, \frac{v}{2^{i}} \right) < \infty,
\]
for all $(u, v) \in \{(x, 0), (2x, 0), (x, x), (x, kx) : x \in X\}$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality
\[
\|D_f(x, y)\|_Y \leq \varphi_q(x, y),
\]
for all $x, y \in X$. Then, there exists a unique quadratic function $Q : X \to Y$ such that
\[
\|f(2x) - 16f(x) - Q(x)\|_Y \leq \frac{M^2}{4} (\varphi_q(x))^\frac{1}{p},
\]
for all $x \in X$, where
\[
\varphi_q(x) := \sum_{i=(1+j)/2}^{\infty} \frac{4^{nj}}{k^{2p-1}} \left[ (12k^2)^p \varphi_q^p \left( \frac{x}{2^{i}}, \frac{x}{2^{i}} \right) + (12(k^2-1))^p \varphi_q^p \left( \frac{x}{2^{i}}, 0 \right) \right. \\
+ \left. 6p \varphi_q^p \left( \frac{2x}{2^{i}}, 0 \right) + 12p \varphi_q^p \left( \frac{x}{2^{i}}, \frac{kx}{2^{i}} \right) \right].
\]

Proof. Let $j = 1$. Setting $y = 0$ in (3.4), we have
\[
\left\| 2f(kx) - 2k^{2}f(x) - \frac{k^{2}(k^{2}-1)}{6} (f(2x) - 4f(x)) \right\|_Y \leq \varphi_q(x, 0),
\]
for all $x \in X$. Putting $y = x$ in (3.4), we obtain
\[
\left\| f((k+1)x) + f((k-1)x) - k^{2}f(2x) - \frac{k^{2}(k^{2}-1)}{6} (f(2x) - 4f(x)) + 2(k^{2}-1)f(x) \right\|_Y \\
\leq \varphi_q(x, x),
\]
for all $x \in X$. Replacing $x$ by $2x$ in (3.7), we see that
\[
\left\| 2f(2kx) - 2k^{2}f(2x) - \frac{k^{2}(k^{2}-1)}{6} (f(4x) - 4f(2x)) \right\|_Y \leq \varphi_q(2x, 0),
\]
for all \( x \in X \). Setting \( y \) by \( kx \) in (3.4) and using the evenness of \( f \), we get

\[
\left\| f(2kx) - k^2 f((k + 1)x) - k^2 f((k - 1)x) + 2\left(k^2 - 1\right)f(kx) - \frac{k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) \right\|_Y 
\leq \varphi_q(x, kx),
\]  
(3.10)

for all \( x \in X \). It follows from (3.9) and (3.10) that

\[
\left\| k^2 f(2x) + \frac{k^2(k^2 - 1)}{12}(f(4x) - 4f(2x)) - k^2 f((k + 1)x) - k^2 f((k - 1)x) 
+ 2\left(k^2 - 1\right)f(kx) - \frac{k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) \right\|_Y 
\leq M \left[ \frac{1}{2}\varphi_q(2x, 0) + \varphi_q(x, kx) \right],
\]  
(3.11)

for all \( x \in X \). Also, it follows from (3.7) and (3.8) that

\[
\left\| k^2 f((k + 1)x) + k^2 f((k - 1)x) - 2\left(k^2 - 1\right)f(kx) - k^4 f(2x) 
- \frac{k^2(k^2 - 1)}{6}(f(2x) - 4f(x)) + 4k^2\left(k^2 - 1\right)f(x) \right\|_Y 
\leq M \left[ k^2 \varphi_q(x, x) + \left(k^2 - 1\right)\varphi_q(x, 0) \right],
\]  
(3.12)

for all \( x \in X \). Finally, using (3.11) and (3.12), we obtain that

\[
\left\| f(4x) - 20f(2x) + 64f(x) \right\|_Y 
\leq \frac{M^2}{k^2(k^2 - 1)} \left[ 12k^2\varphi_q(x, x) + 12\left(k^2 - 1\right)\varphi_q(x, 0) + 6\varphi_q(2x, 0) + 12\varphi_q(x, kx) \right]
\leq M^2 \varphi_q(x),
\]  
(3.13)

where

\[
\varphi_q(x) := \frac{1}{k^2(k^2 - 1)} \left[ 12k^2\varphi_q(x, x) + 12\left(k^2 - 1\right)\varphi_q(x, 0) + 6\varphi_q(2x, 0) + 12\varphi_q(x, kx) \right].
\]  
(3.14)

for all \( x \in X \). Let \( g : X \to Y \) be a function defined by \( g(x) := f(2x) - 16f(x) \) for all \( x \in X \). From (3.13), we conclude that

\[
\left\| g(2x) - 4g(x) \right\|_Y \leq M^2 \varphi_q(x),
\]  
(3.15)
for all \( x \in X \). If we replace \( x \) in (3.15) by \( x/2^{n+1} \) and multiply both sides of (3.15) by \( 4^n \), then we get

\[
\left\| 4^{n+1} g\left( \frac{x}{2^{n+1}} \right) - 4^n g\left( \frac{x}{2^n} \right) \right\|_Y \leq M^2 4^n \varphi_q\left( \frac{x}{2^{n+1}} \right),
\]

(3.16)

for all \( x \in X \) and all non-negative integers \( n \). Since \( Y \) is a \( p \)-Banach space, the inequality (3.16) gives

\[
\left\| 4^{n+1} g\left( \frac{x}{2^{n+1}} \right) - 4^m g\left( \frac{x}{2^m} \right) \right\|_Y^p \leq \sum_{i=m}^{n} \left\| 4^{i+1} g\left( \frac{x}{2^{i+1}} \right) - 4^i g\left( \frac{x}{2^i} \right) \right\|_Y^p \leq M^{2p} \sum_{i=m}^{n} 4^p \varphi_q^p\left( \frac{x}{2^{i+1}} \right),
\]

(3.17)

for all nonnegative integers \( n \) and \( m \) with \( n \geq m \) and all \( x \in X \). Since \( 0 < p \leq 1 \), by Lemma 1.2 and (3.14), we conclude that

\[
\varphi_q^p(x) \leq \frac{1}{k^{2p} (k^2 - 1)^p} \left[ \left( 12k^2 \right)^p \varphi_q^p(x, x) + \left( 12 \left( k^2 - 1 \right) \right)^p \varphi_q^p(x, 0) + 6^p \varphi_q^p(2x, 0) + 12^p \varphi_q^p(kx, x) \right],
\]

(3.18)

for all \( x \in X \). Therefore, it follows from (3.3) and (3.18) that

\[
\sum_{i=1}^{\infty} 4^p \varphi_q^p\left( \frac{x}{2^i} \right) < \infty,
\]

(3.19)

for all \( x \in X \). It follows from (3.17) and (3.19) that the sequence \( \{4^n g(x/2^n)\} \) is a Cauchy for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{4^n g(x/2^n)\} \) converges for all \( x \in X \). So one can define a function \( Q : X \to Y \) by

\[
Q(x) = \lim_{n \to \infty} 4^n g\left( \frac{x}{2^n} \right),
\]

(3.20)

for all \( x \in X \). Letting \( m = 0 \) and passing the limit \( n \to \infty \) in (3.17), we get

\[
\left\| g(x) - Q(x) \right\|_Y^p \leq M^{2p} \sum_{i=0}^{\infty} 4^p \varphi_q^p\left( \frac{x}{2^{i+1}} \right) = \frac{M^{2p}}{4^p} \sum_{i=1}^{\infty} 4^p \varphi_q^p\left( \frac{x}{2^i} \right),
\]

(3.21)

for all \( x \in X \). Thus (3.5) follows from (3.18) and (3.21). Now we show that \( Q \) is quadratic. It follows from (3.16), (3.19) and (3.20) that

\[
\left\| Q(2x) - 4Q(x) \right\|_Y = \lim_{n \to \infty} \left\| 4^n g\left( \frac{x}{2^{n-1}} \right) - 4^{n+1} g\left( \frac{x}{2^n} \right) \right\|_Y = 4 \lim_{n \to \infty} \left\| 4^{n-1} g\left( \frac{x}{2^{n-1}} \right) - 4^n g\left( \frac{x}{2^n} \right) \right\|_Y \leq M^2 \lim_{n \to \infty} 4^n \varphi_q\left( \frac{x}{2^n} \right) = 0,
\]

(3.22)
for all\( x \in X \). So,

\[
Q(2x) = 4Q(x),
\]

(3.23)

for all \( x \in X \). On the other hand, it follows from (3.2), (3.4) and (3.20) that

\[
\|D_Q(x, y)\|_Y = \lim_{n \to \infty} 4^n \|D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\|_Y = \lim_{n \to \infty} 4^n \|D_f\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) - 16D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\|_Y
\]

\[
\leq M \lim_{n \to \infty} 4^n \left\{ \|D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\|_Y + 16\|D_f\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right)\|_Y \right\}
\]

\[
\leq M \lim_{n \to \infty} 4^n \left\{ \varphi_q\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) + 16\varphi_q\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\} = 0,
\]

(3.24)

for all \( x, y \in X \). Hence the function \( Q \) satisfies (1.7). Thus, by Theorem 2.1, the function \( x \mapsto Q(2x) - 16Q(x) \) is quadratic. Therefore, (3.23) implies that the function \( Q \) is quadratic.

Now, to prove the uniqueness property of \( Q \), let \( Q' : X \to Y \) be another quadratic function satisfying (3.5). It follows from (3.3) that

\[
\lim_{n \to \infty} 4^n \sum_{i=1}^{2^n} 4^n \varphi_q^p\left(\frac{u_i}{2^n}, \frac{y_i}{2^n}\right) = \lim_{n \to \infty} \sum_{i=n+1}^{2^n} 4^n \varphi_q^p\left(\frac{u_i}{2^n}, \frac{y_i}{2^n}\right) = 0,
\]

(3.25)

for all \((u, y) \in \{(x, 0), (2x, 0), (x, x), (x, kx) : x \in X\}\). Hence,

\[
\lim_{n \to \infty} 4^n \varphi_q\left(\frac{x}{2^n}\right) = 0,
\]

(3.26)

for all \( x \in X \). It follows from (3.5), (3.20) and (3.26) that

\[
\|Q(x) - Q'(x)\|_Y^p = \lim_{n \to \infty} 4^n \\|g\left(\frac{x}{2^n}\right) - Q\left(\frac{x}{2^n}\right)\|_Y^p \leq \frac{M_{2^n}}{4^n} \lim_{n \to \infty} 4^n \varphi_q\left(\frac{x}{2^n}\right) = 0,
\]

(3.27)

for all \( x \in X \). So \( Q = Q' \).

For \( j = -1 \), we can prove the theorem by a similar argument. \( \square \)

**Corollary 3.2.** Let \( \theta, r, s \) be nonnegative real numbers such that \( r, s > 2 \) or \( r, s < 2 \). Suppose that an even function \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality

\[
\|D_f(x, y)\|_Y \leq \theta(\|x\|_X + \|y\|_X),
\]

(3.28)

for all \( x, y \in X \). Then there exists a unique quadratic function \( Q : X \to Y \) satisfying

\[
\|f(2x) - 16f(x) - Q(x)\|_Y \leq \frac{M^2\theta}{k^2(k^2 - 1)} \varphi_q(x),
\]

(3.29)
for all \( x \in X \), where
\[
γ_q(x) = \left( \frac{12p \left[ k^{2p} + (k^2 - 1)^p + (2r - 1)^p + 1 \right]}{|4p - 2^sp|} \|x\|_X^p + \frac{12p (k^{2p} + k^p)}{|4p - 2^sp|} \|x\|_X^{sp} \right)^{1/p}.
\]

**Proof.** In Theorem 3.1, putting \( q(x, y) := θ(\|x\|_X^p + \|y\|_X^p) \) for all \( x, y \in X \), we get the desired result. □

**Corollary 3.3.** Let \( θ \geq 0 \) and \( r, s > 0 \) be real numbers such that \( λ := r + s \neq 2 \). Suppose that an even function \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality
\[
\|Df(x, y)\|_Y \leq θ\|x\|_X^p \|y\|_X^s,
\]
for all \( x, y \in X \). Then there exists a unique quadratic function \( Q : X \to Y \) satisfying
\[
\|f(2x) - 16f(x) - Q(x)\|_Y \leq \frac{M^2 θ}{k^2(k^2 - 1)} \left( \frac{12p (k^{2p} + k^p)}{|4p - 2^sp|} \right)^{1/p} \|x\|_X^3,
\]
for all \( x \in X \).

**Proof.** In Theorem 3.1, taking \( q(x, y) := θ\|x\|_X^p \|y\|_X^s \), for all \( x, y \in X \), we arrive at the desired result. □

**Theorem 3.4.** Let \( j \in \{-1, 1\} \) be fixed and let \( q_j : X \times X \to [0, \infty) \) be a function such that
\[
\lim_{n \to ∞} 16^n q_j\left( \frac{x}{2^nj}, \frac{y}{2^nj} \right) = 0,
\]
for all \( x, y \in X \) and
\[
\sum_{i=(1+j)/2}^{∞} 16^i q_j\left( \frac{u}{2^ij}, \frac{y}{2^ij} \right) < ∞,
\]
for all \( (u, y) \in \{(x, 0), (2x, 0), (x, x), (x, kx) : x \in X\} \). Suppose that an even function \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality
\[
\|Df(x, y)\|_Y \leq q_j(x, y),
\]
for all \( x, y \in X \). Then there exists a unique quartic function \( V : X \to Y \) such that
\[
\|f(2x) - 4f(x) - V(x)\|_Y \leq \frac{M^2}{16} (\bar{q}_j(x))^1/p,
\]
for all \( x, y \in X \).
for all $x \in X$, where

$$
\tilde{\varphi}_v(x) := \sum_{i=1}^{\infty} \frac{16^{ip}}{2^{ip}(2-1)^{ip}} \left\{ \left( 12k^2 \right)^{ip} \varphi_0 \left( \frac{x}{2^{ip}}, \frac{x}{2^{ip}} \right) + \left( 12(k^2-1) \right)^{ip} \varphi_0 \left( \frac{x}{2^{ip}}, 0 \right) \right. \\
+ 6p \varphi_0 \left( \frac{2x}{2^{ip}}, 0 \right) + 12p \varphi_0 \left( \frac{x}{2^{ip}}, \frac{kx}{2^{ip}} \right) \right\}.
$$

(3.37)

Proof. Being similar to the proof of Theorem 3.1, we omit its proof. \qed

Corollary 3.5. Let $\theta, r, s$ be nonnegative real numbers such that $r, s > 4$ or $r, s < 4$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.28) for all $x, y \in X$. Then there exists a unique quartic function $V : X \to Y$ satisfying

$$
\| f(2x) - 4f(x) - V(x) \|_Y \leq \frac{M^2\theta}{k^2(k^2-1)} \gamma_v(x),
$$

(3.38)

for all $x \in X$, where

$$
\gamma_v(x) = \left( \frac{12p \left[k^{2p} + (k^2-1)^p + (2r^{-1})^{ip} + 1 \right]}{16^p - 2^{ip}} \|x\|_X^p + \frac{12p \left(k^{2p} + k^{sp} \right)}{16^p - 2^{ip}} \|x\|_X^{Sp} \right)^{1/p}.
$$

(3.39)

for all $x \in X$.

Corollary 3.6. Let $\theta \geq 0$ and $r, s > 0$ be real numbers such that $\lambda := r + s \neq 4$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.31) for all $x, y \in X$. Then, there exists a unique quartic function $V : X \to Y$ satisfying

$$
\| f(2x) - 4f(x) - V(x) \|_Y \leq \frac{M^2\theta}{k^2(k^2-1)} \left( \frac{12p \left[k^{2p} + k^{sp} \right]}{16^p - 2^{ip}} \right)^{1/p} \|x\|_X,
$$

(3.40)

for all $x \in X$.

Now, we are ready to prove the main theorem concerning the stability problem for (1.7).

Theorem 3.7. Let $j \in \{-1, 1\}$ be fixed and let $\varphi : X \times X \to [0, \infty)$ be a function such that

$$
\lim_{n \to \infty} \left( \left( \frac{1-j}{2} \right) 4^{ip} \varphi \left( \frac{x}{2^{ip}}, \frac{y}{2^{ip}} \right) + \left( \frac{1+j}{2} \right) 16^{ip} \varphi \left( \frac{x}{2^{ip}}, \frac{y}{2^{ip}} \right) \right) = 0,
$$

(3.41)

for all $x, y \in X$ and

$$
\sum_{i=1}^{\infty} \left( \left( \frac{1-j}{2} \right) 4^{ip} \varphi \left( \frac{u}{2^{ip}}, \frac{v}{2^{ip}} \right) + \left( \frac{1+j}{2} \right) 16^{ip} \varphi \left( \frac{u}{2^{ip}}, \frac{v}{2^{ip}} \right) \right) < \infty,
$$

(3.42)
for all \((u, y) \in \{(x, 0), (2x, 0), (x, x), (x, kx) : x \in X\}\). Suppose that an even function \(f : X \to \mathcal{Y}\) with \(f(0) = 0\) satisfies the inequality

\[
\|D_f(x, y)\|_{\mathcal{Y}} \leq \varphi(x, y),
\]

for all \(x, y \in X\). Then, there exists a unique quadratic function \(Q : X \to \mathcal{Y}\) and a unique quartic function \(V : X \to \mathcal{Y}\) such that

\[
\|f(x) - Q(x) - V(x)\|_{\mathcal{Y}} \leq \frac{M^3}{192} \left\{ 4\left(\tilde{q}_q(x)\right)^{1/p} + \left(\tilde{q}_v(x)\right)^{1/p} \right\},
\]

for all \(x \in X\), where

\[
\begin{align*}
\tilde{q}_q(x) := & \sum_{i=1+j/2} 4^{ij} rac{1}{k^2p(k^2-1)^p} \left\{ (12k^2)^p q^p \left( \frac{x}{2^{2i}}, \frac{x}{2^{2j}} \right) + (12(k^2-1))^p q^p \left( \frac{x}{2^{2i}}, 0 \right) \\
+ & 6p q^p \left( \frac{2x}{2^{2j}}, 0 \right) + 12p q^p \left( \frac{x}{2^{2i}}, \frac{mk}{2^{2i}} \right) \right\}, \\
\tilde{q}_v(x) := & \sum_{i=1+j/2} 16^{ij} \frac{1}{k^2p(k^2-1)^p} \left\{ (12k^2)^p q^p \left( \frac{x}{2^{2i}}, \frac{x}{2^{2j}} \right) + (12(k^2-1))^p q^p \left( \frac{x}{2^{2i}}, 0 \right) \\
+ & 6p q^p \left( \frac{2x}{2^{2j}}, 0 \right) + 12p q^p \left( \frac{x}{2^{2i}}, \frac{mk}{2^{2i}} \right) \right\}.
\end{align*}
\]

Proof. By Theorems 3.1 and 3.4, there exists a quadratic function \(Q_0 : X \to \mathcal{Y}\) and a quartic function \(V_0 : X \to \mathcal{Y}\) such that

\[
\|f(2x) - 16f(x) - Q_0(x)\|_{\mathcal{Y}} \leq \frac{M^2}{4} \left[ \tilde{q}_q(x) \right]^{1/p}, \quad \|f(2x) - 4f(x) - V_0(x)\|_{\mathcal{Y}} \leq \frac{M^2}{16} \left[ \tilde{q}_v(x) \right]^{1/p},
\]

for all \(x \in X\). Therefore, it follows from (3.46) that

\[
\left\| f(x) + \frac{1}{12} Q_0(x) - \frac{1}{12} V_0(x) \right\|_{\mathcal{Y}} \leq \frac{M^3}{192} \left\{ 4\left[ \tilde{q}_q(x) \right]^{1/p} + \left[ \tilde{q}_v(x) \right]^{1/p} \right\},
\]

for all \(x \in X\). Thus we obtain (3.44) by letting \(Q(x) = -(1/12)Q_0(x)\) and \(V(x) = (1/12)V_0(x)\) for all \(x \in X\).

To prove the uniqueness property of \(Q\) and \(V\), let \(Q', V' : X \to \mathcal{Y}\) be another quadratic and quartic functions satisfying (3.44). Let \(\overline{Q} = Q - Q'\) and \(\overline{V} = V - V'\). Hence,

\[
\|\overline{Q}(x) + \overline{V}(x)\|_{\mathcal{Y}} \leq M \left\{ \|f(x) - Q(x) - V(x)\|_{\mathcal{Y}} + \|f(x) - Q'(x) - V'(x)\|_{\mathcal{Y}} \right\} \leq \frac{M^4}{96} \left\{ 4\left[ \tilde{q}_q(x) \right]^{1/p} + \left[ \tilde{q}_v(x) \right]^{1/p} \right\},
\]
for all $x \in X$. Since $\lim_{n \to \infty} 4^{\nu/n} \overline{q}_n(x/2^n) = \lim_{n \to \infty} 16^{\nu/n} \overline{q}_n(x/2^n) = 0$, for all $x \in X$, we figure out that

$$\lim_{n \to \infty} 16^n \| \overline{Q}(\frac{X}{2^n}) + \overline{V}(\frac{X}{2^n}) \|_Y = 0,$$

(3.49)

for all $x \in X$. Therefore, we get $\overline{V} = 0$ and then $\overline{Q} = 0$. $\square$

**Corollary 3.8.** Let $\theta, r, s$ be nonnegative real numbers such that $r, s > 4$ or $2 < r, s < 4$ or $r, s < 2$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.28), for all $x, y \in X$. Then, there exists a unique quadratic function $Q : X \to Y$ and a unique quartic function $V : X \to Y$ such that

$$\| f(x) - Q(x) - V(x) \|_Y \leq \frac{M^3\theta}{12k^2(k^2 - 1)} (\gamma_4(x) + \gamma_5(x)),$$

(3.50)

for all $x \in X$, where $\gamma_4(x)$ and $\gamma_5(x)$ are defined as in Corollaries 3.2 and 3.5.

**Corollary 3.9.** Let $\theta \geq 0$ and $r, s > 0$ be non-negative real numbers such that $\lambda := r + s \in (0, 2) \cup (2, 4) \cup (4, \infty)$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.31) for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique quartic function $V : X \to Y$ such that

$$\| f(x) - Q(x) - V(x) \|_Y \leq \frac{M^3\theta}{12k^2(k^2 - 1)} \left\{ \left( \frac{12p (k^2p + k^3p)}{|4p^2 - 2^{3p}|} \right)^{1/p} + \left( \frac{12p (k^2p + k^3p)}{|16p^2 - 2^{3p}|} \right)^{1/p} \right\} \| x \|_X^3,$$

(3.51)

for all $x \in X$.

**Corollary 3.10.** Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\| D_f(x, y) \|_X \leq \varepsilon,$$

(3.52)

for all $x, y \in X$ where $\varepsilon > 0$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique quartic function $V : X \to Y$ such that

$$\| f(x) - Q(x) - V(x) \|_Y \leq \frac{M^3\varepsilon}{k^2(k^2 - 1)} \left\{ \left( \frac{k^3p + (k^2 - 1)^p + 2^{-p} + 1}{4p^2 - 1} \right)^{1/p} + \left( \frac{k^3p + (k^2 - 1)^p + 2^{-p} + 1}{16p^2 - 1} \right)^{1/p} \right\},$$

(3.53)

for all $x \in X$. 

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References


