Research Article

A Suzuki Type Fixed-Point Theorem

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We present a fixed-point theorem for a single-valued map in a complete metric space using implicit relation, which is a generalization of several previously stated results including that of Suzuki (2008).

1. Introduction

There are a lot of generalizations of Banach fixed-point principle in the literature. See [1–5]. One of the most interesting generalizations is that given by Suzuki [6]. This interesting fixed-point result is as follows.

**Theorem 1.1.** Let $(X,d)$ be a complete metric space, and let $T$ be a mapping on $X$. Define a non-decreasing function $\theta$ from $[0,1]$ into $(1/2,1]$ by

$$\theta(r) = \begin{cases} 1, & 0 \leq r \leq \frac{\sqrt{5} - 1}{2}, \\ \frac{1-r}{r^2}, & \frac{\sqrt{5} - 1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

(1.1)

Assume that there exists $r \in [0,1)$, such that

$$\theta(r)d(x,Tx) \leq d(x,y) \quad \text{implies} \quad d(Tx,Ty) \leq rd(x,y),$$

(1.2)

for all $x, y \in X$, then there exists a unique fixed-point $z$ of $T$. Moreover, $\lim_{n \to \infty} T^n x = z$ for all $x \in X$. 
Like other generalizations mentioned above in this paper, the Banach contraction principle does not characterize the metric completeness of $X$. However, Theorem 1.1 does characterize the metric completeness as follows.

**Theorem 1.2.** Define a nonincreasing function $\theta$ as in Theorem 1.1, then for a metric space $(X, d)$ the following are equivalent:

(i) $X$ is complete,

(ii) Every mapping $T$ on $X$ satisfying (1.2) has a fixed point.

In addition to the above results, Kikkawa and Suzuki [7] provide a Kannan type version of the theorems mentioned before. In [8], it is provided a Chatterjea type version. Popescu [9] gives a Ciric type version. Recently, Kikkawa and Suzuki also provide multivalued versions which can be found in [10, 11]. Some fixed-point theorems related to Theorems 1.1 and 1.2 have also been proven in [12, 13].

The aim of this paper is to generalize the above results using the implicit relation technique in such a way that

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0,$$  \hspace{1cm} (1.3)

for $x, y \in X$, where $F : [0, \infty)^6 \to \mathbb{R}$ is a function as given in Section 2.

### 2. Implicit Relation

Implicit relations on metric spaces have been used in many papers. See [1, 14–16].

Let $\mathbb{R}_+$ denote the nonnegative real numbers, and let $\Psi$ be the set of all continuous functions $F : [0, \infty)^6 \to \mathbb{R}$ satisfying the following conditions:

$F_1$: $F(t_1, \ldots, t_6)$ is nonincreasing in variables $t_2, \ldots, t_6$,

$F_2$: there exists $r \in [0, 1)$, such that

$$F(u, v, v, u, u + v, 0) \leq 0 \hspace{1cm} (2.1)$$

or

$$F(u, v, 0, u + v, u, v) \leq 0 \hspace{1cm} (2.2)$$

or

$$F(u, v, v, v, v, v) \leq 0 \hspace{1cm} (2.3)$$

implies $u \leq rv$,

$F_3$: $F(u, 0, 0, u, u, 0) > 0$, for all $u > 0$.

**Example 2.1.** $F(t_1, \ldots, t_6) = t_1 - rt_2$, where $r \in [0, 1)$. It is clear that $F \in \Psi$. 

Example 2.2. $F(t_1, \ldots, t_6) = t_1 - \alpha[t_3 + t_4]$, where $\alpha \in [0, 1/2]$.

Let $F(u, v, v, u, u + v, 0) = u - \alpha[u + v] \leq 0$, then we have $u \leq (\alpha/(1-\alpha)v$. Similarly, let $F(u, v, 0, u + v, u, v) \leq 0$, then we have $u \leq (\alpha/(1-\alpha))v$. Again, let $F(u, v, v, v, v, 0) \leq 0$, then $u \leq 2av$. Since $\alpha/(1-\alpha) \leq 2a < 1$, $F_2$ is satisfied with $r = 2a$. Also $F(u, 0, 0, u, u, 0) = (1-\alpha)u > 0$, for all $u > 0$. Therefore, $F \in \Psi$.

Example 2.3. $F(t_1, \ldots, t_6) = t_1 - \alpha \max\{t_3, t_4\}$, where $\alpha \in [0, 1/2]$.

Let $F(u, v, v, u, u + v, 0) = u - \alpha \max\{u, v\} \leq 0$, then we have $u \leq av \leq (\alpha/(1-\alpha))v$. Similarly, let $F(u, v, 0, u + v, u, v) \leq 0$, then we have $u \leq (\alpha/(1-\alpha))v$. Again, let $F(u, v, v, v, v, 0) \leq 0$, then $u \leq av \leq (\alpha/(1-\alpha))v$. Thus, $F_2$ is satisfied with $r = \alpha/(1-\alpha)$. Also $F(u, 0, 0, u, u, 0) = (1-\alpha)u > 0$, for all $u > 0$. Therefore, $F \in \Psi$.

Example 2.4. $F(t_1, \ldots, t_6) = t_1 - \alpha[t_3 + t_4]$, where $\alpha \in [0, 1/2]$.

Let $F(u, v, v, u, u + v, 0) = u - \alpha \max\{u, v\} \leq 0$, then we have $u \leq (\alpha/(1-\alpha))v$. Similarly, let $F(u, v, 0, u + v, u, v) \leq 0$, then we have $u \leq (\alpha/(1-\alpha))v$. Again, let $F(u, v, v, v, v, 0) \leq 0$, then $u \leq 2av$. Since $\alpha/(1-\alpha) \leq 2a < 1$, $F_2$ is satisfied with $r = 2a$. Also $F(u, 0, 0, u, u, 0) = (1-\alpha)u > 0$, for all $u > 0$. Therefore, $F \in \Psi$.

Example 2.5. $F(t_1, \ldots, t_6) = t_1 - at_3 - bt_4$, where $a, b \in [0, 1/2]$.

Let $F(u, v, v, u, u + v, 0) = u - av - bu \leq 0$, then we have $u \leq (a/(1-b)v$. Similarly, let $F(u, v, 0, u + v, u, v) \leq 0$, then we have $u \leq (b/(1-b)v$. Again, let $F(u, v, v, v, v, 0) \leq 0$, then $u \leq (a + b)v$. Thus, $F_2$ is satisfied with $r = \max\{a/(1-b), b/(1-b), a + b\}$. Also $F(u, 0, 0, u, u, 0) = (1-b)u > 0$, for all $u > 0$. Therefore, $F \in \Psi$.

3. Main Result

Theorem 3.1. Let $(X, d)$ be a complete metric space, and let $T$ be a mapping on $X$. Define a nonincreasing function $\theta$ from $[0, 1]$ into $(1/2, 1]$ as in Theorem 1.1. Assume that there exists $F \in \Psi$, such that $\theta(r)d(x, Tx) \leq d(x, y)$ implies

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0, \quad (3.1)$$

for all $x, y \in X$, then $T$ has a unique fixed-point $z$ and $\lim_{n \to \infty} T^n x = z$ holds for every $x \in X$.

Proof. Since $\theta(r) \leq 1$, $\theta(r)d(x, Tx) \leq d(x, Tx)$ holds for every $x \in X$, by hypotheses, we have

$$F\left(d(Tx, T^2x), d(x, Tx), d(x, Tx), d\left(Tx, T^2x\right), d\left(x, T^2x\right), 0\right) \leq 0, \quad (3.2)$$

and so from $(F_1)$,

$$F\left(d(Tx, T^2x), d(x, Tx), d(x, Tx), d\left(Tx, T^2x\right), d\left(x, T^2x\right), d\left(Tx, T^2x\right), 0\right) \leq 0. \quad (3.3)$$

By $(F_2)$, we have

$$d(Tx, T^2x) \leq rd(x, Tx), \quad (3.4)$$
for all \( x \in X \). Now fix \( u \in X \) and define a sequence \( \{u_n\} \) in \( X \) by \( u_n = T^n u \). Then from (3.4), we have

\[
d(u_n, u_{n+1}) = d(Tu_{n-1}, T^2u_{n-1}) \leq rd(u_{n-1}, Tu_{n-1}) \leq \cdots \leq r^n d(u, Tu).
\]

This shows that \( \sum_{n=1}^{\infty} d(u_n, u_{n+1}) < \infty \), that is, \( \{u_n\} \) is Cauchy sequence. Since \( X \) is complete, \( \{u_n\} \) converges to some point \( z \in X \). Now, we show that

\[
d(Tx, z) \leq rd(x, z) \quad \forall x \in X \setminus \{z\}.
\]

For \( x \in X \setminus \{z\} \), there exists \( n_0 \in \mathbb{N} \), such that \( d(u_n, z) \leq d(x, z)/3 \) for all \( n \geq n_0 \). Then, we have

\[
\theta(r)d(u_n, Tu_n) \leq d(u_n, Tu_n) = d(u_n, u_{n+1}) \\
\leq d(u_n, z) + d(z, u_{n+1}) \\
\leq \frac{2}{3}d(x, z) = d(x, z) - \frac{1}{3}d(x, z) \\
\leq d(x, z) - d(u_n, z) \leq d(u_n, x).
\]

Hence, by hypotheses, we have

\[
F(d(Tu_n, Tx), d(u_n, x), d(u_n, Tu_n), d(x, Tx), d(u_n, Tx), d(x, Tu_n)) \leq 0, \tag{3.8}
\]

and so

\[
F(d(u_{n+1}, Tx), d(u_n, x), d(u_n, u_{n+1}), d(x, Tx), d(u_n, Tx), d(x, u_{n+1})) \leq 0. \tag{3.9}
\]

Letting \( n \to \infty \), we have

\[
F(d(z, Tx), d(z, x), 0, d(x, Tx), d(z, Tx), d(x, z)) \leq 0, \tag{3.10}
\]

and so

\[
F(d(z, Tx), d(z, x), 0, d(x, z) + d(z, Tx), d(z, Tx), d(x, z)) \leq 0. \tag{3.11}
\]

By \( (F_2) \), we have

\[
d(z, Tx) \leq rd(x, z), \tag{3.12}
\]

and this shows that (3.6) is true.
Now, we assume that $T_m z \neq z$ for all $m \in \mathbb{N}$, then from (3.6), we have
\[ d(T^{m+1}z, z) \leq r^m d(Tz, z), \quad (3.13) \]
for all $m \in \mathbb{N}$.

Case 1. Let $0 \leq r \leq (\sqrt{5} - 1)/2$. In this case, $\theta(r) = 1$. Now, we show by induction that
\[ d(T^n z, Tz) \leq rd(z, Tz), \quad (3.14) \]
for $n \geq 2$. From (3.4), (3.14) holds for $n = 2$. Assume that (3.14) holds for some $n$ with $n \geq 2$.

Since
\[ d(z, Tz) \leq d(z, T^n z) + d(T^n z, Tz) \]
\[ \leq d(z, T^n z) + rd(z, Tz), \quad (3.15) \]
we have
\[ d(z, Tz) \leq \frac{1}{1-r} d(z, T^n z), \quad (3.16) \]
and so
\[ \theta(r) d(T^n z, T^{n+1} z) = d(T^n z, T^{n+1} z) \leq r^n d(z, Tz) \]
\[ \leq \frac{r^n}{1-r} d(z, T^n z) \leq \frac{r^2}{1-r} d(z, T^n z) \]
\[ \leq d(z, T^n z). \quad (3.17) \]

Therefore, by hypotheses, we have
\[ F(d(T^{n+1} z, Tz), d(T^n z, z), d(T^n z, T^{n+1} z), d(z, Tz), d(T^n z, Tz), d(z, T^{n+1} z)) \leq 0, \quad (3.18) \]
and so
\[ F(d(T^{n+1} z, Tz), r^{-1} d(Tz, z), r^n d(z, Tz), d(z, Tz), rd(z, Tz), r^n d(z, Tz)) \leq 0, \quad (3.19) \]
then
\[ F(d(T^{n+1} z, Tz), d(Tz, z), d(z, Tz), d(z, Tz), d(z, Tz), d(z, Tz)) \leq 0, \quad (3.20) \]
and by $(F_2)$, we have

$$d(T^{n+1}z, Tz) \leq rd(Tz, z). \quad (3.21)$$

Therefore, (3.14) holds.

Now, from (3.6), we have

$$d(T^{n+1}z, Tz) \leq rd(T^n z, z) \leq r^n d(Tz, z). \quad (3.22)$$

This shows that $T^n z \to z$, which contradicts (3.14).

**Case 2.** Let $(\sqrt{5} - 1)/2 \leq r < \sqrt{2}/2$. In this case, $\theta(r) = (1-r)/r^2$. Again we want to show that (3.14) is true for $n \geq 2$. From (3.4), (3.14) holds for $n = 2$. Assume that (3.14) holds for some $n$ with $n \geq 2$. Since

$$d(z, Tz) \leq d(z, T^n z) + d(T^n z, Tz) \leq d(z, T^n z) + rd(z, Tz), \quad (3.23)$$

we have

$$d(z, Tz) \leq \frac{1}{1-r} d(z, T^n z), \quad (3.24)$$

and so

$$\theta(r) d(T^n z, T^{n+1}z) = \frac{1-r}{r^2} d(T^n z, T^{n+1}z) \leq \frac{1-r}{r^n} d(T^n z, T^{n+1}z) \leq (1-r) d(z, Tz) \leq d(z, T^n z). \quad (3.25)$$

Therefore, as in the previous case, we can prove that (3.14) is true for $n \geq 2$. Again from (3.6), we have

$$d(T^{n+1}z, z) \leq rd(T^n z, z) \leq r^n d(Tz, z). \quad (3.26)$$

This shows that $T^n z \to z$, which contradicts (3.14).

**Case 3.** Let $\sqrt{2}/2 \leq r < 1$. In this case, $\theta(r) = 1/(1+r)$. Note that for $x, y \in X$, either

$$\theta(r) d(x, Tx) \leq d(x, y) \quad (3.27)$$

or

$$\theta(r) d(Tx, T^2 x) \leq d(Tx, y) \quad (3.28)$$
then we have
\[ \theta(r) d(x, Tx) > d(x, y), \]
\[ \theta(r) d(Tx, T^2 x) > d(Tx, y), \] (3.29)

then we have
\[ d(x, Tx) \leq d(x, y) + d(Tx, y) < \theta(r) \left[ d(x, Tx) + d(Tx, T^2 x) \right] \]
\[ \leq \theta(r) \left[ d(x, Tx) + r d(x, Tx) \right] = d(x, Tx), \] (3.30)

which is a contradiction. Therefore, either
\[ \theta(r) d(u_{2n}, Tu_{2n}) \leq d(u_{2n}, z) \] (3.31)
or
\[ \theta(r) d(u_{2n+1}, Tu_{2n+1}) \leq d(u_{2n+1}, z) \] (3.32)
holds for every \( n \in \mathbb{N} \). If
\[ \theta(r) d(u_{2n}, Tu_{2n}) \leq d(u_{2n}, z) \] (3.33)
holds, then by hypotheses we have
\[ F(d(Tu_{2n}, Tu_{2n}^z, d(u_{2n}, z), d(u_{2n}, Tu_{2n}), d(z, Tu_{2n}^z), d(u_{2n}, Tu_{2n}), d(z, Tu_{2n}^z)) \leq 0, \] (3.34)
and so
\[ F(d(u_{2n+1}, Tu_{2n+1}, d(u_{2n}, z), d(u_{2n}, u_{2n+1}), d(z, Tu_{2n+1}), d(u_{2n}, Tu_{2n}), d(z, u_{2n+1})) \leq 0. \] (3.35)

Letting \( n \to \infty \), we have
\[ F(d(z, Tu_{2n}), 0, 0, d(z, Tu_{2n}), d(z, Tu_{2n})) \leq 0, \] (3.36)
which contradicts (\( F_3 \)). If
\[ \theta(r) d(u_{2n+1}, Tu_{2n+1}) \leq d(u_{2n+1}, z) \] (3.37)
holds, then by hypotheses we have
\[ F(d(Tu_{2n+1}, Tu_{2n+1}), d(u_{2n+1}, z), d(u_{2n+1}, Tu_{2n+1}), d(z, Tu_{2n+1}), d(u_{2n+1}, Tu_{2n+1}), d(z, Tu_{2n+1})) \leq 0, \] (3.38)
and so

\[ F(d(u_{2n+2}, Tz), d(u_{2n+1}, z), d(u_{2n+1}, u_{2n+2}), d(z, Tz), d(u_{2n+1}, Tz), d(z, u_{2n+2})) \leq 0. \]  \hfill (3.39)

Letting \( n \to \infty \), we have

\[ F(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz), 0) \leq 0, \]  \hfill (3.40)

which contradicts \( (F_3) \).

Therefore, in all the cases, there exists \( m \in \mathbb{N} \), such that \( T^m z = z \). Since \( \{T^n z\} \) is Cauchy sequence, we obtain \( Tz = z \). That is, \( z \) is a fixed point of \( T \). The uniqueness of fixed point follows easily from (3.6).

**Remark 3.2.** If we combine Theorem 3.1 with Examples 2.1, 2.2, 2.3, and 2.4, we have Theorem 2 of [6], Theorem 2.2 of [7], Theorem 3.1 of [7], and Theorem 4 of [8], respectively.

Using Example 2.5, we obtain the following result.

**Corollary 3.3.** Let \((X, d)\) be a complete metric space, and let \( T \) be a mapping on \( X \). Define a nonincreasing function \( \theta \) from \([0, 1)\) into \([1/2, 1]\) as in Theorem 1.1. Assume that

\[ \theta(r)d(x, Tx) \leq d(x, y) \]  \hfill (3.41)

implies

\[ d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty), \]  \hfill (3.42)

for all \( x, y \in X \), where \( a, b \in [0, 1/2) \), then there exists a unique fixed point of \( T \).

**Remark 3.4.** We obtain some new results, if we combine Theorem 3.1 with some examples of \( F \).

**References**


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