Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 748159, 6 pages
doi:10.1155/2011/748159

Research Article

Generalized Derivations and Bilocal Jordan Derivations of Nest Algebras

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Received 19 December 2010; Accepted 13 January 2011

Academic Editor: Jianming Zhan

Let H be a complex Hilbert space and $B(H)$ the collection of all linear bounded operators, $\mathfrak{A}$ is the closed subspace lattice including 0 an H, then $\mathfrak{A}$ is a nest, accordingly alg $\mathfrak{A} = \{T \in B(H) : TN \subseteq N, \forall N \in \mathfrak{A}\}$ is a nest algebra. It will be shown that of nest algebra, generalized derivations are generalized inner derivations, and bilocal Jordan derivations are inner derivations.

1. Introduction

The concept of local derivations was introduced by Kadison [1] who showed that on a Vonnemann algebra all norm-continuous local derivations are derivations. Larson and Sourour [2] proved that on the algebra $B(X)$ local derivations are derivations. M. Brešar and P. Šemrl [3, 4] generalized the results of the three authors above under a weaker condition. Shulman [5] showed that all local derivations on C^*-algebra are derivations.

Based on a great deal of research works of many mathematicians, some scholars paid more interests in similar kind of problems under more generalized conditions, such as considering local derivations on nest algebras and generalized derivation. Zhu and Xiong [6, 7] proved that local derivations of nest algebra and standard operator algebra are derivations, Zhang [8] considered the Jordan derivations of nest algebras, Lee [9] discussed generalized derivations of left faithful rings. Recently, some scholars discussed some new types of derivations, as Li and Zhou [10] and Majieed and Zhou [11] investigated some new types of generalized derivations associated with Hochschild 2 cocycles, other examples are in [12–15]. In fact, under appropriate conditions, local derivations are derivations.

In this paper, we will show that of nest algebra, a generalized derivation is a generalized inner derivation, and bilocal Jordan derivations are inner derivations.

2. Some Notations and Definitions

In what follows, some notations and basic definitions are introduced.

Let H be a complex Hilbert space and $B(H)$ the collection of all linear bounded operators on H, $\mathfrak{A}$ is the closed subspace lattice including 0 an H, then $\mathfrak{A}$ is a nest, correspondingly the Nest algebra is $\operatorname{alg}\mathfrak{A} = \{T \in B(H) : TN \subseteq N,\ \forall N \in \mathfrak{A}\}$.

If $N \neq 0$, we denote $N_- = \vee\{M \in \mathfrak{A} : M \subsetneq N\}$ and if $N \neq H$, denote $N_+ = \wedge\{M \in \mathfrak{A} : M \supsetneq N\}$, where $\subsetneq$ is real inclusion, and we define $0_- = 0$, $H_+ = H$.

For all $N \in \mathfrak{A}$, $P(N)$ represent the project operator from H to N, and $N^{\perp} = \{f \in H : \langle x, f\rangle = 0,\ \forall x \in N\}$.

Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{A}_1$ a subalgebra of $\mathfrak{A}$, we call the linear map $\varphi : \mathfrak{A}_1 \to \mathfrak{A}$ a generalized inner derivation if and only if for all $T \in \mathfrak{A}_1$, there exist operators A and B in $\mathfrak{A}$ such that $\varphi(T) = AT + TB$; if for all $T \in \mathfrak{A}_1$, we have $\varphi(T^2) = \varphi(T)T + T\varphi(T)$, then φ is called a Jordan derivation; if for all $T \in \mathfrak{A}_1$, there is a Jordan derivation $\varphi_T : \mathfrak{A}_1 \to \mathfrak{A}$, such that $\varphi(T) = \varphi_T(T)$, then φ is said to be a local Jordan derivation.

Definition 2.1. Let $\varphi : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ be an additive mapping, if there exists a derivation $\delta : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ that $\varphi(ST) = \varphi(S)T + S\delta(T)$, for all $S, T \in \operatorname{alg}\mathfrak{A}$, then φ is called a generalized derivation.

Definition 2.2. We call the linear mapping $\varphi : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ a bilocal Jordan derivation, if for every $u \in H$, there is a Jordan derivation $\delta_{T,u} : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$, such that $\varphi(T)u = \delta_{T,u}(T)u$.

3. Main Results

Next to give out the main conclusions.

Theorem 3.1. *If $\varphi : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ is a generalized derivation, then there are operators A and B in $\operatorname{alg}\mathfrak{A}$, such that $\varphi(T) = AT + TB$, for all $T \in \operatorname{alg}\mathfrak{A}$.*

Proof. From the definition of generalized derivation, we can find a derivation: $\delta : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$, such that $\varphi(ST) = \varphi(S)T + S\delta(T)$, for all $S, T \in \operatorname{alg}\mathfrak{A}$, so when $S = I$, we have $\varphi(T) = \varphi(I)T + \delta(T)$, for all $T \in \operatorname{alg}\mathfrak{A}$, denote $\varphi(I) = C$, apparently $C \in \operatorname{alg}\mathfrak{A}$ and $\varphi(T) = CT + \delta(T)$, for all $T \in \operatorname{alg}\mathfrak{A}$.

Since $\delta : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ is a derivation, by [6], it is an inner derivation, namely, there exists $D \in \operatorname{alg}\mathfrak{A}$, such that $\delta(T) = DT - TD$, consequently

$$\varphi(T) = CT + DT - TD = (C + D)T - TD. \tag{3.1}$$

Denote $A = C + D$, $B = -D$, then $\varphi(T) = AT + TB$, for all $T \in \operatorname{alg}\mathfrak{A}$. □

Theorem 3.2. *If $\varphi : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ is a local Jordan derivation, then φ is an inner derivation.*

Proof. Since φ is a local Jordan derivation, there exists a Jordan derivation $\varphi_T : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$, such that $\varphi(T) = \varphi_T(T)$, from Theorem 2.12 in [8], we know that the Jordan derivation of nest algebra $\operatorname{alg}\mathfrak{A}$ is an inner derivation, so there exists $A_T \in \operatorname{alg}\mathfrak{A}$, such that $\varphi_T(T) = TA_T - A_T T$, by imitating the proof in [6], we can conclude that $\varphi_T(T) = TA - AT$, so $\varphi(T) = TA - AT$, namely, φ is an inner derivation. □

The following is the main result.

Theorem 3.3. *If $\varphi : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ is a bilocal Jordan derivation, then it is an inner derivation.*

Proof. We will prove this proposition by the following three steps.

(1) $\varphi(T)(\ker T) \subseteq \operatorname{ran}T$, where $\ker T$ and $\operatorname{ran}T$ are the kernal of T and range of T, respectively. In fact, since $\varphi(T)u = \delta_{T,u}(T)u$ for all $T \in \operatorname{alg}\mathfrak{A}$, for all $u \in H$, and $\delta_{T,u}$ is a Jordan derivation, by Theorem 2.12 in [8], there is an $A_{T,u} \in \operatorname{alg}\mathfrak{A}$, such that $\varphi(T)u = (TA_{T,u} - A_{T,u}T)u$, so if $u \in \ker T$, we have $\varphi(T)u = TA_{T,u}u \in \operatorname{ran}T$.

(2) For all $N \in \mathfrak{A}$, $\{o\} \subset N \subset H$, there exists $C_N \in B(H)$ and $B_N \in B(H)$, such that $\varphi(x \otimes f) = x \otimes C_N f + B_N x \otimes f$, for all $x \in N$, $f \in N_-^{\perp}$.

For arbitrary fixed $f \in N_-^{\perp}$, $f \neq 0$, and for all $x \in N$, we know that $x \otimes y \in \operatorname{alg}\mathfrak{A}$, from step (1), we have $\varphi(x \otimes f)\{f\}^{\perp} \subseteq \operatorname{span}\{x\}$, so there exists a linear function $\lambda_{x,f}$over$\{f\}^{\perp}$, such that $\varphi(x \otimes f)(u) = \langle u, \lambda_{x,f}\rangle x$, for all $u \in \{f\}^{\perp}$, in succession we will prove that $\lambda_{x,f}$ is independent of x. Take a $z \in N$ which is linear independent of x, we have $z \otimes f \in \operatorname{alg}\mathfrak{A}$, then

$$\varphi\big((x+z)\otimes f\big)(u) = \varphi\big(x \otimes f\big)(u) + \varphi\big(z \otimes f\big)(u) = \langle u, \lambda_{x,f}\rangle x + \langle u, \lambda_{z,f}\rangle z. \tag{3.2}$$

On the other hand, $\varphi((x+z)\otimes f)(u) = \langle u, \lambda_{x+z,f}\rangle(x+z)$, so

$$\langle u, \lambda_{x+z,f} - \lambda_{x,f}\rangle x = \langle u, \lambda_{z,f} - \lambda_{x+z,f}\rangle z. \tag{3.3}$$

Since x is linear independent of z, we know that $\lambda_{x,f} = \lambda_{x+z,f}$, that is, $\lambda_{x,f}$ is independent of x, so $\lambda_{x,f}$ can be denoted by λ_f, and

$$\varphi\big(x \otimes f\big)(u) = \langle u, \lambda_f\rangle x, \quad \forall u \in \{f\}^{\perp}. \tag{3.4}$$

Let g_f be the linear continuous span on H of λ_f, we define $B_{u,f} : N \to N$ as follows:

$$B_{u,f}(x) = \frac{1}{\langle u, f\rangle}\{\varphi\big(x \otimes f\big)(u) - \langle u, g_f\rangle x\}, \quad \text{where } u \in \{f\}^{\perp}. \tag{3.5}$$

Obviously, $B_{u,f}$ is linear and $\varphi(x \otimes f)(u) = \langle u, g_f\rangle x + \langle u, f\rangle B_{u,f}x$, $x \in N$, $u \in H$.

Next $B_{u,f}$ is independent of u, which reduce to show (i) $B_{au,f} = B_{u,f}$, $a \in C$; (ii) $B_{u,f} = B_{v,f}$, where $v \neq u$, $v \in H$.

In fact, (i) is evident. For (ii), since for all $x \in N$, $\varphi(x \otimes f)(v) = \langle v, g_f\rangle x + \langle v, f\rangle B_{v,f}x$ and $\varphi(x \otimes f)(u+v) = \langle u+v, g_f\rangle x + \langle u+v, f\rangle B_{u+v,f}x = \langle u+v, g_f\rangle x + \langle v, f\rangle B_{v,f}x + \langle u, f\rangle B_{u,f}x$, we have $\langle u, f\rangle B_{u+v,f} + \langle v, f\rangle B_{u+v,f} = \langle u, f\rangle B_{u,f} + \langle v, f\rangle B_{v,f}$, namely, $\langle u, f\rangle(B_{u+v,f} - B_{u,f}) = \langle v, f\rangle(B_{v,f} - B_{u+v,f})$, on account of $u \neq v$, so $B_{u+v,f} = B_{u,f}$, that is, $B_{u,f}$ is independent of u, so we can mark $B_{u,f}$ by B_f, as a result, we have

$$\varphi\big(x \otimes f\big)(u) = \langle u, g_f\rangle x + \langle u, f\rangle B_f x = x \otimes g_f(u) + B_f x \otimes f(u), \quad \forall u \in H. \tag{3.6}$$

Consequently

$$\varphi(x \otimes f) = x \otimes g_f + B_f x \otimes f. \tag{3.7}$$

Define $C_N : N_-^\perp \to N_-^\perp : C_N f \to g_f$, now we will show that C_N is a linear bounded operator. Because g_f is a continuous linear function, so g_f is bounded, consequently C_N is bounded, according to (3.4), we know

$$\varphi(x \otimes af)(u) = \langle u, \lambda_{af} \rangle x = \overline{a}\varphi(x \otimes f)(u) = \overline{a}\langle u, \lambda_f \rangle x = \langle u, a\lambda_f \rangle x, \tag{3.8}$$

so $\lambda_{a,f} = a\lambda_f$; on the other hand,

$$\begin{aligned} &\varphi(x \otimes (f_1 + f_2))(u) \\ &\quad = \langle u, \lambda_{f_1+f_2} \rangle x = \varphi(x \otimes f_1)(u) + \varphi(x \otimes f_2)(u) = \langle u, \lambda_{f_1} \rangle x + \langle u, \lambda_{f_2} \rangle x = \langle u, \lambda_{f_1} + \lambda_{f_2} \rangle x, \end{aligned} \tag{3.9}$$

so $\lambda_{f_1+f_2} = \lambda_{f_1} + \lambda_{f_2}$, this is enough to show that $g_{a_f} = ag_f$, $g_{f_1+f_2} = g_{f_1} + g_{f_2}$, so when $af_1 + f_2 \in N_-^\perp$,

$$C_N(af_1 + f_2) = g_{af_1+f_2} = g_{af_1} + g_{f_2} = aC_N f_1 + C_N f_2. \tag{3.10}$$

That is to say C_N is linear, so (3.7) has the form of

$$\varphi(x \otimes f) = x \otimes C_N f + B_f x \otimes f. \tag{3.11}$$

In succession we will prove that B_f is independent of f, arbitrarily choose $y \in N_-^\perp$, where y is linear independent of f, then $y + f \in N_-^\perp$ and

$$\begin{aligned} \varphi(x \otimes (f + y)) &= x \otimes C_N(f + y) + B_{f+y} x \otimes (f + y) \\ &= x \otimes C_N f + x \otimes C_N y + B_{f+y} x \otimes f + B_{f+y} x \otimes y. \end{aligned} \tag{3.12}$$

On the other hand,

$$\varphi(x \otimes (f + y)) = \varphi(x \times \otimes f) + \varphi(x \otimes y) = x \otimes C_N f + B_f x \otimes f + x \otimes C_N y + B_y x \otimes y. \tag{3.13}$$

So $(B_{f+y} - B_f)x \otimes f = (B_y - B_{y+f})x \otimes f$, then $B_f = B_{f+y} = B_y$, that is, B_f is independent of f, which can be marked by B_N, so (3.11) has the form of

$$\varphi(x \otimes f) = x \otimes C_N f + B_N x \otimes f. \tag{3.14}$$

We proved that B_N is bounded, because $\|B_N x \otimes f\| \le \|\varphi(x \otimes f)\| + \|x \otimes C_N f\| \le M + \|C_N\| \|x\| \|f\|$.

On account of the boundary of C_N and $\varphi(x \otimes f) \in \operatorname{alg} N \subseteq B(H)$, we know that B_N is bounded, namely, $B_N \in B(N)$, $C_N \in B(N_-^{\perp})$.

(3) For arbitrary $x \otimes y \in \operatorname{alg} \mathfrak{A}$, there is $\varphi(x \otimes f) = x \otimes Cf + Bx \otimes f$.

For all $M, N \in \mathfrak{A}$, $\{0\} \subset N \subset M \subset H$, select $x \in N$, $f \in M_-^{\perp}$, then $x \in M$, $f \in N_-^{\perp}$ and $x \otimes f \in \operatorname{alg} \mathfrak{A}$, from the result of step (2), it is easy to know that

$$\varphi(x \otimes f) = x \otimes C_N f + B_N x \otimes f, \quad \varphi(x \otimes f) = x \otimes C_M f + B_M x \otimes f. \tag{3.15}$$

Consequently $x \otimes (C_N - C_M) f = (B_M - B_N) x \otimes f$, so there exists a scalar $\lambda(N, M)$, such that

$$(C_N - C_M)|_{M_-^{\perp}} = \lambda P\left(M_-^{\perp}\right), \qquad (B_N - B_M)|_N = -\lambda P(N). \tag{3.16}$$

By imitating Lemma 2 mentioned in [6], we can prove that

$$\varphi(x \otimes f) = x \otimes Cf + Bx \otimes f, \quad x \otimes y \in \operatorname{alg} N. \tag{3.17}$$

Since the collection of all rank one operators is dense in $\operatorname{alg} \mathfrak{A}$, so for every $T \in \operatorname{alg} \mathfrak{A}$, we have $\varphi(T) = TC^* + BT$, let $T = I$, then $\varphi(I) = B + C^*$, considering φ to be a bilocal Jordan derivation, namely, $\varphi(I)(u) = \delta_{I,u}(I^2)u = (I\delta_{I,u}(I) + \delta_{I,u}(I)I)u = 2\delta_{I,u}(I)u$, we can conclude that $\delta_{I,u}(I) = 0$, so $\varphi(I) = 0$, thereby $B + C^* = 0$ and $\varphi(T) = BT - TB$, which shows that φ is an inner derivation. □

Acknowledgments

The authors wish to thank the anonymous reviewers for their valuable suggestions. This work was supported by the Natural Science Foundation Project of CQ CSTC under Contract no. 2010BB2240.

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