Research Article

Commutative Pseudo Valuations on BCK-Algebras

Myung Im Doh and Min Su Kang

1 Department of Mathematics, Gyeongsang National University, Chinju 660-701, Republic of Korea
2 Department of Mathematics, Hanyang University, Seoul 133-791, Republic of Korea

Correspondence should be addressed to
Myung Im Doh, sansudo6@hanmail.net and Min Su Kang, sinchangmyun@hanmail.net

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The notion of a commutative pseudo valuation on a BCK-algebra is introduced, and its characterizations are investigated. The relationship between a pseudo valuation and a commutative pseudo valuation is examined.

1. Introduction


In this paper, we introduce the notion of a commutative pseudo valuation on a BCK-algebra, and investigate its characterizations. We discuss the relationship between a pseudo valuation and a commutative pseudo valuation. We provide conditions for a pseudo valuation to be a commutative pseudo valuation.

2. Preliminaries

A BCK-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.
An algebra \((X; *, 0)\) of type \((2,0)\) is called a \textit{BCI-algebra} if it satisfies the following axioms:

(i) \((\forall x, y, z \in X) \ ( ((x * y) * (x * z)) * (z * y) = 0), \)

(ii) \((\forall x, y \in X) \ ( (x * (x * y)) * y = 0), \)

(iii) \((\forall x \in X) \ (x * x = 0), \)

(iv) \((\forall x, y \in X) \ (x * y = 0, \ y * x = 0 \Rightarrow x = y). \)

If a BCI-algebra \(X\) satisfies the following identity:

(v) \((\forall x \in X) \ (0 * x = 0), \)

then \(X\) is called a \textit{BCK-algebra}. Any BCK/BCI-algebra \(X\) satisfies the following conditions:

(a1) \((\forall x \in X) \ (x * 0 = x), \)

(a2) \((\forall x, y, z \in X) \ (x * y = 0 \Rightarrow (x * z) * (y * z) = 0, \ (z * y) * (z * x) = 0), \)

(a3) \((\forall x, y, z \in X) \ ( (x * y) * z = (x * z) * y), \)

(a4) \((\forall x, y, z \in X) \ (((x * z) * (y * z)) * (x * y) = 0). \)

We can define a partial ordering \(\leq\) by \(x \leq y\) if and only if \(x * y = 0\).

A BCK-algebra \(X\) is said to be \textit{commutative} if \(x \land y = y \land x\) for all \(x, y \in X\) where \(x \land y = y * (y * x)\).

A subset \(A\) of a BCK/BCI-algebra \(X\) is called an \textit{ideal} of \(X\) if it satisfies the following conditions:

(b1) \(0 \in A, \)

(b2) \((\forall x, y \in X) \ (x * y \in A, \ y \in A \Rightarrow x \in A). \)

A subset \(A\) of a BCK-algebra \(X\) is called a \textit{commutative ideal} of \(X\) (see [6]) if it satisfies (b1) and

(b3) \((\forall x, y, z \in X) \ ((x * y) * z \in A, \ z \in A \Rightarrow x * (y \land x) \in A). \)

We refer the reader to the book in [7] for further information regarding BCK-algebras.

3. \textbf{Commutative Pseudo Valuations on BCK-Algebras}

In what follows let \(X\) denote a BCK-algebra unless otherwise specified.

\textit{Definition 3.1} (see [4]). A real-valued function \(\varphi\) on \(X\) is called a \textit{weak pseudo valuation} on \(X\) if it satisfies the following condition:

\( (c1) \ (\forall x, y \in X)(\varphi(x * y) \leq \varphi(x) + \varphi(y)). \)

\textit{Definition 3.2} (see [4]). A real-valued function \(\varphi\) on \(X\) is called a \textit{pseudo valuation} on \(X\) if it satisfies the following two conditions:

\( (c2) \ \varphi(0) = 0, \)

\( (c3) \ (\forall x, y \in X)(\varphi(x) \leq \varphi(x * y) + \varphi(y)). \)
Table 1: ∗-operation.

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Proposition 3.3 (see [4]). For any pseudo valuation ϕ on X, one has the following assertions:

1. ϕ(x) ≥ 0 for all x ∈ X.
2. ϕ is order preserving,
3. ϕ(x ∗ y) ≤ ϕ(x * z) + ϕ(z * y) for all x, y, z ∈ X.

Definition 3.4. A real-valued function ϕ on X is called a commutative pseudo valuation on X if it satisfies (c2) and

(c4) (∀x, y, z ∈ X) (ϕ(x ∗ (y ∧ x)) ≤ ϕ((x ∗ y) * z) + ϕ(z)).

Example 3.5. Let X = {0, a, b, c} be a BCK-algebra with the ∗-operation given by Table 1. Let θ be a real-valued function on X defined by

$$\theta = \begin{pmatrix} 0 & a & b & c \\ 0 & 7 & 9 & 9 \end{pmatrix}.$$ (3.1)

Routine calculations give that θ is a commutative pseudo valuation on X.

Theorem 3.6. In a BCK-algebra, every commutative pseudo valuation is a pseudo valuation.

Proof. Let ϕ be a commutative pseudo valuation on X. For any x, y, z ∈ X, we have

$$\varphi(x) = \varphi(x ∗ (0 ∧ x)) ≤ \varphi((x ∗ 0) * z) + \varphi(z) = \varphi(x * z) + \varphi(z).$$ (3.2)

This completes the proof. □

Combining Theorem 3.6 and [4, Theorem 3.9], we have the following corollary.

Corollary 3.7. In a BCK-algebra, every commutative pseudo valuation is a weak pseudo valuation.

The converse of Theorem 3.6 may not be true as seen in the following example.

Example 3.8. Let X = {0, a, b, c, d} be a BCK-algebra with the ∗-operation given by Table 2. Let θ be a real-valued function on X defined by

$$\theta = \begin{pmatrix} 0 & a & b & c & d \\ 0 & 5 & 8 & 8 & 8 \end{pmatrix}.$$ (3.3)
Then \( \vartheta \) is a pseudo valuation on \( X \). Since

\[
\vartheta(b \ast (c \land b)) = 8 \not\leq 0 = \vartheta((b \ast c) \ast 0) + \vartheta(0),
\]

\( \vartheta \) is not a commutative pseudo valuation on \( X \).

We provide conditions for a pseudo valuation to be a commutative pseudo valuation.

**Theorem 3.9.** For a real-valued function \( \varphi \) on \( X \), the following are equivalent:

1. \( \varphi \) is a commutative pseudo valuation on \( X \).
2. \( \varphi \) is a pseudo valuation on \( X \) that satisfies the following condition:

\[
(\forall x, y \in X) \quad (\varphi(x \ast (y \land x)) \leq \varphi(x \ast y)).
\]

**Proof.** Assume that \( \varphi \) is a commutative pseudo valuation on \( X \). Then \( \varphi \) is a pseudo valuation on \( X \) by Theorem 3.6. Taking \( z = 0 \) in (c4) and using (a1) and (c2) induce the condition (3.5).

Conversely let \( \varphi \) be a pseudo valuation on \( X \) satisfying the condition (3.5). Then

\[
\varphi(x \ast y) \leq \varphi((x \ast y) \ast z) + \varphi(z)
\]

for all \( x, y, z \in X \) so that \( \varphi \) is a commutative pseudo valuation on \( X \).

**Lemma 3.10** (see [8]). Every pseudo valuation \( \varphi \) on \( X \) satisfies the following implication:

\[
(\forall x, y, z \in X) \quad ((x \ast y) \ast z = 0 \implies \varphi(x) \leq \varphi(y) + \varphi(z)).
\]

**Theorem 3.11.** In a commutative BCK-algebra, every pseudo valuation is a commutative pseudo valuation.

**Proof.** Let \( \varphi \) be a pseudo valuation on a commutative BCK-algebra \( X \). Note that

\[
((x \ast (y \land x)) \ast ((x \ast y) \ast z)) \ast z = ((x \ast (y \land x)) \ast z) \ast ((x \ast y) \ast z)
\]

\[
\leq (x \ast (y \land x)) \ast (x \ast y)
\]

\[
= (x \land y) \ast (y \land x) = 0
\]
Theorem 3.16. For any ideal \( I \) of \( X \), we define a real-valued function \( \varphi_I \) on \( X \) by

\[
\varphi_I(x) = \begin{cases} 
0 & \text{if } x = 0, \\
 t_1 & \text{if } x \in I \setminus \{0\}, \\
 t_2 & \text{if } x \in X \setminus I
\end{cases}
\]
Table 3: $*$-operation.

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for all $x \in X$ where $0 < t_1 < t_2$. Then $\varphi_I$ is a pseudo valuation on $X$.

**Proof.** Let $x, y \in X$. If $x = 0$, then clearly $\varphi_I(x) \leq \varphi_I(x \ast y) + \varphi_I(y)$. Assume that $x \neq 0$. If $y = 0$, then $\varphi_I(x) \leq \varphi_I(x \ast y) + \varphi_I(y)$. If $y \neq 0$, we consider the following four cases:

(i) $x \ast y \in I$ and $y \in I$,
(ii) $x \ast y \notin I$ and $y \notin I$,
(iii) $x \ast y \in I$ and $y \notin I$,
(iv) $x \ast y \notin I$ and $y \in I$.

Case (i) implies that $x \in I$ because $I$ is an ideal of $X$. If $x \ast y = 0$, then $\varphi_I(x \ast y) = 0$ and so $\varphi_I(x) = t_1 = \varphi_I(x \ast y) + \varphi_I(y)$. If $x \ast y \neq 0$, then $\varphi_I(x \ast y) = t_1$ and thus $\varphi_I(x) = t_1 \leq \varphi_I(x \ast y) + \varphi_I(y)$. The second case implies that $\varphi_I(x \ast y) = t_2$ and $\varphi_I(y) = t_2$. Hence $\varphi_I(x) \leq t_2 < \varphi_I(x \ast y) + \varphi_I(y)$. Let us consider the third case. If $x \ast y = 0$, then $\varphi_I(x \ast y) = 0$ and thus $\varphi_I(x) \leq t_2 = \varphi_I(x \ast y) + \varphi_I(y)$. If $x \ast y \neq 0$, then $\varphi_I(x \ast y) = t_1$ and so $\varphi_I(x) \leq t_2 < t_1 + t_2 = \varphi_I(x \ast y) + \varphi_I(y)$. For the final case, the proof is similar to the third case. Therefore $\varphi_I$ is a pseudo valuation on $X$.

Before ending our discussion, we pose a question.

**Question 1.** If $I$ is commutative ideal of $X$, then is the function $\varphi_I$ in Theorem 3.16 a commutative pseudo valuation on $X$?

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**References**
