Research Article

The Spectral Scale of a Self-Adjoint Operator in a Semifinite von Neumann Algebra

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We extend Akemann, Anderson, and Weaver’s Spectral Scale definition to include selfadjoint operators from semifinite von Neumann algebras. New illustrations of spectral scales in both the finite and semifinite von Neumann settings are presented. A counterexample to a conjecture made by Akemann concerning normal operators and the geometry of the their perspective spectral scales (in the finite setting) is offered.

1. Introduction

The notion of a “joint spectrum” for an $n$-tuple of operators was created and developed through the 1970s and work continues today. For an account of the various approaches taken and the achievements made see [1, page 3]. One attempt at a geometric interpretation of the “joint spectrum” of an $n$-tuple of self-adjoint operators is developed in spectral scale [2–6].

Definition 1.1. Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a finite, faithful, normal, trace $\tau$ such that $\tau(1) = 1$, and let $b = b_1 + ib_2 \in \mathcal{M}$ with $b_1$ and $b_2$ self-adjoint. Define

$$\Psi : \mathcal{M} \rightarrow \mathbb{R}^3$$

by $\Psi(c) = (\tau(c), \tau(b_1 c), \tau(b_2 c))$ for all $c \in \mathcal{M}$. The spectral scale of $b$ with respect to $\tau$ is the set

$$B(b) = \Psi(\mathcal{M}_1) = \{(\tau(c), \tau(b_1 c), \tau(b_2 c)) : 0 \leq c \leq 1\}.$$
The spectral scale of an \( n \)-tuple of self-adjoint operators is defined analogously (in the canonical fashion), becoming a subset of \( \mathbb{R}^{n+1} \).

The set \( B(b) \) has proven useful in gaining insight on problems in operator theory and operator algebras. In 2006, for a self-adjoint operator \( b \) in a \textit{finite} factor \( \mathcal{M} \) (i.e., \( \tau(1) < \infty \) and \( \mathcal{M} \) has trivial center), Akemann and Sherman [7] were able to use \( B(b) \) to characterize the \( \star \)-closure of the unitary orbit \( \mathcal{U}(b) = \{ ubu^* : u \text{ is unitary in } \mathcal{M} \} \) of \( b \). More precisely, they showed that \( \mathcal{U}(b) = C(b) = \{ d \in \mathcal{M} : B(d) \subset B(b) \} \). Unfortunately, the spectral scale has only been developed for \textit{finite} von Neumann algebras. In 2010, Wills [6] was able to extend the definition of \( B(b) \) to include the class of unbounded operators.

The purpose of the work that follows to develop \( B(b) \) for \( b = b^* \in \mathcal{M} \), where \( \mathcal{M} \) is a \textit{semifinite} von Neumann algebra, equipped with a semifinite, faithful, normal trace \( \tau \) such that \( \tau(1) = \infty \). We will show that \( B(b) \) is an unbounded subset of \( \mathbb{R}^2 \) which may or may not be closed, and that \( B(b) \) gives a complete description of elements in the spectrum \( \sigma(b) \) of \( b \) that lie outside the open interval determined by the minimum and maximum elements of the essential spectrum \( \mathcal{E} \) of \( b \).

We begin with a summary of “how to read” \( B(b) \) in the \textit{finite} von Neumann algebra setting.

2. Geometric Interpretations of \( B(b) \) When \( \mathcal{M} \) Is a Finite von Neumann Algebra

The following is a summary of the main results concerning the geometry of \( B(b) \) [2–5].

**Theorem 2.1.** Let \( \mathcal{M} \), \( \tau \), \( b \), \( \Psi \), and \( B(b) \) be as given in Definition 1.1. Then the following hold.

(i) \( B(b) \) is a compact, convex subset of \( \mathbb{R}^3 \) which is symmetric about the point \( (1/2, b_1/2, b_2/2) \). If \( b = b^* \), then \( B(b) \) can be identified with a subset of \( \mathbb{R}^2 \).

(ii) \( B(b) \) gives a complete description of \( \sigma(t_1 b_1 + t_2 b_2) \) for all \( t_1, t_2 \in \mathbb{R} \).

(iii) \( \tau(b) \) can be read from the point \( \Psi(1) \).

(iv) If \( b = b^* \), then the set of all slopes of lines tangent to the boundary of \( B(b) \) is equal to \( \sigma(b) \).

(v) If \( b = b^* \), then corners in the boundary of \( B(b) \) correspond to gaps in \( \sigma(b) \).

(vi) If \( b = b^* \), then the extreme points of \( B(b) \) are images under \( \Psi \) of spectral projections of \( b \).

(vii) If \( b \neq b^* \), then the extreme points of \( B(b) \) are images under \( \Psi \) of spectral projections of the operators \( t_1 b_1 + t_2 b_2 \) for various \( t_1, t_2 \in \mathbb{R} \).

(viii) If \( b \) is normal, then the set of all complex slopes of 1-dimensional faces of \( B(b) \) (i.e., all the 1-dimensional “edges” in the boundary) is equal to \( \sigma_p(b) \).

(ix) The Numerical Range \( W(b) \) is equal to the set of all slopes (complex when \( b \neq b^* \)) of line segments emanating from the origin to points in \( B(b) \).

(x) If \( \dim(\mathcal{M}) = n \), then for each \( 1 \leq k \leq n \) the cross-section of \( B(b) \) taken at \( x = k/n \) is isomorphic to the \( k \)-numerical range

\[
W_k(b) = \left\{ \frac{1}{k} \sum_{i=1}^{k} (bx_i, x_i) : \{ x_i \} \text{ is an orthonormal set} \right\}, \quad 1 \leq k \leq \dim(\mathcal{M}).
\]
We finish this section with a few examples of spectral scales in the finite setting.

**Example 2.2.** Let $\mathcal{M} = M_5(\mathbb{C})$ with $\tau(x) = (1/5) \operatorname{tr}(x)$ for all $x \in \mathcal{M}$, where $\operatorname{tr}$ is the canonical trace on $M_5(\mathbb{C})$. Let

$$b = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -10
\end{pmatrix}.$$  \hspace{1cm} (2.2)

$B(b)$ is given in Figure 1. The MATLAB program SpecGUI [8] was used to generate Figure 1, and by Theorem 2.1 one can see that $\sigma(b) = \{-10, -1, 0, 1\}$, $\|b\| = 10$, 1 has multiplicity 2, $W(b) = [-10, 1]$, and $\tau(b) = -9/5$.

**Example 2.3.** Let $\mathcal{M} = L^\infty([0, 1], m)$, where $m$ is Lebesgue measure on the interval $[0, 1]$, let $\tau(c) = \int_0^1 c \, dm$ for all $c \in \mathcal{M}$, and let $b \in \mathcal{M}$ be defined by $b(x) = x$ for all $x \in [0, 1]$. Then $b$ is positive, with no eigenvalues, and $\sigma(b) = [0, 1]$. $B(b)$ is given in Figure 2.

**Example 2.4.** Let $\mathcal{M} = M_3(\mathbb{C})$ with $\tau(c) = (1/3) \operatorname{tr}(c)$ for all $c \in \mathcal{M}$, and let

$$b = \begin{pmatrix}
1 & 0 & 0 \\
0 & i & 0 \\
0 & 0 & 2i
\end{pmatrix}. \hspace{1cm} (2.3)$$

$B(b)$ is given in Figure 3 [8].

Since $b$ is normal, by Theorem 2.1 we know that the eigenvalues of $b$ can be read from $B(b)$ as the slopes (complex) of 1-dimensional faces of $B(b)$ (i.e., slopes of 1-dimensional “edges” in the boundary of $B(b)$). Alternatively, if we were given $B(b)$ without knowing
what $b$ actually was, we could conclude by [3, Theorem 6.1] that $b$ is normal. The numerical range $W(b)$ of $b$ can be seen as the slopes (complex) of secant lines between the origin and points in $B(b)$.

Our last example is a counterexample to conjecture made by Akemann in 2005 which stated: $b$ is normal if the cross-section of $B(b)$ taken at $x = 1/2$ is a polygon. The following is a spectral scale that has no 1-dimensional faces, and the cross-section taken $x = 1/2$ is not a polygon, yet the operator it corresponds to is normal.

**Example 2.5.** Let $C$ be the unit circle, and let $m$ denote Lebesgue measure on $C$. Put $\mathcal{M} = L^\infty(C)$, and note that $\mathcal{M}$ is a finite von Neumann algebra equipped with the finite, faithful, normal trace $\tau : L^\infty(C) \rightarrow \mathbb{R}$ defined by $\tau(c) = (1/2\pi) \int_C c \, dm$ for all $c \in L^\infty(C)$.

Consider the function $b(z) = z$ on $C$. The spectral scale of $b$ with respect to $\tau$ is the collection of points

$$B(b) = \{ (\tau(c), \tau(bc)) : c \in (L^\infty(C))^+ \}$$

in $\mathbb{R}^3$. From [2, Theorem 2.3] we know that $\Psi$ applied to spectral projections of the operators $b_t = t_1 \text{Re}(b) + t_2 \text{Im}(b)$ for various $t = (t_1, t_2) \in \mathbb{R}^2$ are extreme points of $B(b)$. 
Let $\mu \in [0, 2\pi)$, and let $\theta \in [0, \pi]$. If $\chi_{C_{\mu,\theta}}$ is the characteristic function on the arc $C_{\mu,\theta}$ from $e^{i(\mu-\theta)}$ to $e^{i(\mu+\theta)}$, then,

$$
\chi_{C_{\mu,\theta}}(z) = \chi_{[\cos(\theta), \infty)} \left( \text{Re} \left( e^{-i\mu} z \right) \right) = \chi_{[\cos(\theta), \infty)} \left( (\cos(\mu)) x - (\sin(\mu)) y \right)
$$

for every $z = x + iy \in \mathbb{C}$ (to see this think of the case where $\mu = 0$). But

$$
\chi_{[\cos(\theta), \infty)} \left( (\cos(\mu)) x - (\sin(\mu)) y \right) = \chi_{[\cos(\theta), \infty)}(t_1 \text{Re}(b) + t_2 \text{Im}(b))
$$

for $t = (t_1, t_2) = (\cos(\mu), -\sin(\mu))$. Therefore, every $\chi_{C_{\mu,\theta}}$ is a spectral projection of some $b_1$. Conversely, if $t = (t_1, t_2) \in \mathbb{R}^2$ and $s \in \mathbb{R}$, then for if $\mu$ is chosen so that $\cos(\mu) = t_1 / \sqrt{t_1^2 + t_2^2}$ and $\sin(\mu) = -t_2 / \sqrt{t_1^2 + t_2^2}$ and $\theta$ is chosen so that $||t|| \cos(\theta) = s$, then we have

$$
\chi([s, \infty))(b_1) = \chi([||t|| \cos(\theta), \infty)) \left( ||t|| \cos(\mu) \text{Re}(b) - ||t|| \sin(\mu) \text{Im}(b) \right) = \chi_{[\cos(\theta), \infty)}(b_1)
$$

Since the extreme points of $B(b)$ come from the spectral projections described above, we can calculate the extreme points of $B(b)$ explicitly:

$$
\tau \left( \chi_{C_{\mu,\theta}} \right) = \left( \int_{C_{\mu,\theta}} 1 \, dm, \int_{C_{\mu,\theta}} b \, dm \right) = \left( \frac{\theta}{\pi}, \int_{C_{\mu,\theta}} e^{iz} \, dz \right)
$$

$$
= \left( \frac{\theta}{\pi}, \frac{1}{2\pi i} \left( e^{i(\mu+\theta)} - e^{i(\mu-\theta)} \right) \right) = \left( \frac{\theta}{\pi}, \frac{1}{\pi} e^{i \mu} \sin(\theta) \right).
$$

Since $B(b)$ is the convex hull of its extreme points, we see that $B(b)$ can be recovered by rotating the function $f(x) = (1/\pi \sin(\pi x))$ $0 \leq x \leq 1$ about the $x$-axis (see Figure 4).

### 3. The Spectral Scale of a Self-Adjoint Operator in a Semifinite von Neumann Algebra

For the remainder of what follows, let $\mathcal{M}$ be a semifinite von Neumann algebra, equipped with a faithful, semifinite, normal trace $\tau$, and let $b \in \mathcal{M}$ be self-adjoint. In order to avoid
the finite setting (for which the spectral scale has already been developed), we also assume that $\tau(1) = \infty$.

Define $n_\tau = \{x \in \mathcal{M} : \tau(x^*x) < +\infty\}$ and $m_\tau = \{\sum_{i=1}^n x_i y_i : x_i, y_i \in n_\tau\}$. The by [9, Lemma 2.16, page 318], we can define the function $\Psi : m_\tau \to \mathbb{R}^2$ by

$$\Psi(c) = (\tau(c), \tau(bc)) \quad \text{for every } c \in m_\tau.$$  \hfill (3.1)

**Definition 3.1.** The spectral scale of $b$ (with respect to $\tau$) is the set

$$B(b) = \Psi((m_\tau)_1^+) = \{(\tau(c), \tau(bc)) : c \in (m_\tau)_1^+\}. \quad \hfill (3.2)$$

Since $(m_\tau)_1^+$ is convex, $\Psi$ is linear, and $\tau$ is positive, we have that $B(b)$ is a convex subset of $\mathbb{R}^2$ contained in the right half-plane. Since $\tau$ is semifinite and $\tau(1) = \infty$, $B(b)$ is an unbounded subset of $\mathbb{R}^2$. We will see that $B(b)$ may or may not be closed.

**Note 1.** If $p$ and $q$ are projections in $\mathcal{M}$ and $p \leq q$ then we write

$$[p, q] = \{a : p \leq a \leq q\} \quad \hfill (3.3)$$

for the order interval determined by $p$ and $q$. Given a real number $s$, we let $p_s^+$ denote the spectral projection of $b$ corresponding to the interval $(-\infty, s]$. Similarly, we let $p_s^-$ denote the spectral projection of $b$ corresponding to the interval $(-\infty, s)$. We use $p_s^+$ to indicate either of these projections and we write $p_s$ when $p_s^+ = p_s^-$. We let $q_s^+$ and $q_s^-$ denote the spectral projections corresponding to the intervals $[s, \infty)$ and $(s, \infty)$, respectively. We also use $q_s^+\tau$ to indicate either of these projections and we write $q_s$ when $q_s^+ = q_s^-$. Note that $q_s^+ = 1 - p_s^-$ for every $s \in \mathbb{R}$. The spectral projections $p_s^+$ and $q_s^-$ will be used throughout the remainder of this text, and will be instrumental in deciphering $B(b)$.
The following lemma was proven in [2, Lemma 1.2] for any type of von Neumann algebra.

**Lemma 3.2.** Let \( b \) be a self-adjoint element in \( \mathcal{M} \), \( s \in \mathbb{R} \), \( c \in [p_s^-, p_s^+] \), and \( a \in \mathcal{M}_1^+ \). Then the following statements hold:

1. \((b-s1)(1-c) = (b-s1)(1-p_s^+) \geq 0 \) and the range projection of \((b-s1)(1-p_s^+)\) is \( 1-p_s^+ \);
2. if \( a^{1/2}(b-s1)(1-c)a^{1/2} = 0 \), then \( a \leq p_s^+ \);
3. \((s1-b)c = (s1-b)p_s^- \geq 0 \) and the range projection of \((s1-b)p_s^-\) is \( p_s^- \);
4. if \((1-a)^{1/2}(s1-b)c(1-a)^{1/2} = 0 \) then \( a \geq p_s^- \).

The following two lemmas tell us that points of the form \((\tau(c), \tau(cb))\) lie on the boundary when \( c \in [p_s^-, p_s^+] \) (or \( c \in [q_s^-, q_s^+] \)) for some \( s \in \mathbb{R} \). A slight variation of these lemmas was proved in [2, Lemma 1.3].

**Lemma 3.3.** If \( b \in \mathcal{M} \) is self-adjoint and \( s \in \mathbb{R} \), then the following statements hold for each \( c \in [p_s^-, p_s^+] \cap m_r \) and \( a \in (m_r)_1^+ \):

1. If \( \tau(a) = \tau(c) \), then \( \tau(ba) \geq \tau(bc) \).
2. If \( \tau(a) = \tau(c) \) and \( \tau(ba) = \tau(bc) \), then \( a \in [p_s^-, p_s^+] \); moreover, if \( p_s^+ \), then equality in the second equation obtains only for \( = c \).

**Proof.** (1) From the previous lemma we have that

\[
(b-s1)(1-c) = (b-s1)(1-p_s^+) \geq 0, \quad \text{so that } b(1-c) \geq s(1-c). \tag{3.4}
\]

Therefore

\[
a^{1/2}b(1-c)a^{1/2} \geq a^{1/2}s(1-c)a^{1/2}. \tag{3.5}
\]

Note that \( \tau(b(1-c)a) < \infty \) since \( a \in m_r \) and \( m_r \) is an ideal. Since \( a^{1/2} \in n_r \) and \( \tau \) is order preserving on \( m_r \) [6], we have that

\[
\tau(b(1-c)a) = \tau\left(a^{1/2}b(1-c)a^{1/2}\right)
\]

\[
\geq \tau\left(a^{1/2}s(1-c)a^{1/2}\right) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
It follows from (●), (●●), and our hypothesis that

$$
\tau(ba) - \tau(bc) = \tau(ba - bca - bc + bca) \\
= \tau(b(1 - c)a) - \tau(bc(1 - a)) \\
\geq \tau(s(1 - c)a) - \tau(sc(1 - a)) \\
= \tau(sa) - \tau(sc) \\
= s(\tau(a) - \tau(c)) \\
= 0,
$$

(3.6)

and so $\tau(ba) \geq \tau(bc)$.

To prove (2), suppose that $\tau(a) = \tau(c)$ and $\tau(ba) = \tau(bc)$. Then

$$
\tau((1 - c)a) = \tau(a - ca) \\
= \tau(c - ca) \\
= \tau(c(1 - a)),
$$

(3.7)

$$
\tau(b(1 - c)a) = \tau(ba - bca) \\
= \tau(bc - bca) \\
= \tau(bc(1 - a)).
$$

Combining these two results with (●) and (●●) we get that

$$
\tau(bc(1 - a)) = \tau(b(1 - c)a) \geq s\tau((1 - c)a) = s\tau(c(1 - a)) \geq \tau(bc(1 - a)).
$$

(3.8)

Therefore, all the above terms must be equal, and so

$$
\tau(b(1 - c)a) = s\tau((1 - c)a), \\
\tau(bc(1 - a)) = s\tau(c(1 - a)).
$$

(3.9)

From this it follows that

$$
\tau\left(a^{1/2}(b - s1)(1 - c)a^{1/2}\right) = 0,
$$

(3.10)

$$
\tau\left((1 - a)^{1/2}(b - s1)c(1 - a)^{1/2}\right) = 0.
$$

Since $\tau$ is faithful we get that

$$
a^{1/2}(b - s1)(1 - c)a^{1/2} = 0, \\
(1 - a)^{1/2}(b - s1)c(1 - a)^{1/2} = 0.
$$

(3.11)

By parts (2) and (4) of the previous lemma, $a \in [p_s^*, p_s^*]$. 
If it is the case that $c = p_s^+, \tau(a) = \tau(p_s^+)$, and $\tau(ba) = \tau(bc)$ then by what we just showed, $a \in [p_s^-, p_s^+]$, so $\tau(p_s^+ - a) = 0$, which implies that $a = p_s^+$ since $\tau$ is faithful. 

Remark 3.4. Notice the importance of our choice of $a$ and $c$ as elements of ideal $m_\tau$ in the above proof. Without this assumption, many of the equalities and inequalities above would have been either trivial or undefined because $\tau$ is not necessarily finite on all of $\mathcal{M}$.

Lemma 3.5. If $b \in \mathcal{M}$ is self-adjoint and $t \in \mathbb{R}$, then the following statements hold for each $d \in [q_t^-, q_t^+] \cap m_\tau$ and $a \in (m_\tau)_1^+$:

1. If $\tau(a) = \tau(d)$, then $\tau(ba) \leq \tau(bd)$.
2. If $\tau(a) = \tau(d)$ and $\tau(ba) = \tau(bd)$, then $a \in [q_t^-, q_t^+]$; moreover, if $q_t^+$, then equality in the second equation obtains only for $a = d$.

Proof. (1) Similarly, by Lemma 3.2 (since $1 - d \in [p_t^-, p_t^+]$), we have that

$$ bd \geq td, $$
$$ t(1 - d) \geq b(1 - d). $$

Therefore

$$ (1 - a)^{1/2}bd(1 - a)^{1/2} \geq (1 - a)^{1/2}td(1 - a)^{1/2}, $$
$$ a^{1/2}td(1 - d)a^{1/2} \geq b(1 - d)a^{1/2}. $$

Using the fact that the trace is faithful and finite on $m_\tau$, we get that

$$ \tau(bd(1 - a)) \geq \tau((1 - a)^{1/2}bd(1 - a)^{1/2}) = \tau(td(1 - a)), $$
$$ \tau(t(1 - d)a) \geq \tau(a^{1/2}td(1 - d)a^{1/2}) = \tau(b(1 - d)a). $$

It follows that

$$ \tau(bd) - \tau(ba) \geq \tau(bd(1 - a)) - \tau(b(1 - d)a) $$
$$ \geq \tau(td(1 - a)) - \tau(t(1 - d)a) $$
$$ = \tau(ta) $$
$$ = 0. $$
To prove (2), suppose that $\tau(a) = \tau(d)$ and $\tau(ba) = \tau(bd)$. Then

$$\tau((1 - d)a) = \tau(a - da)$$
$$= \tau(d - da)$$
$$= \tau(d(1 - a)), \quad (3.15)$$

$$\tau(b(1 - d)a) = \tau(ba - bda)$$
$$= \tau(bd - bda)$$
$$= \tau(bd(1 - a)).$$

From these two observations, ($\ast$), and ($\ast\ast$) we get that

$$\tau(bd(1 - a)) = \tau(b(1 - d)a) \leq t\tau((1 - d)a) = t\tau(d(1 - a)) \leq \tau(bd(1 - a)). \quad (3.16)$$

But then all these terms must be equal. Therefore $\tau(b(1 - d)a) = t\tau((1 - d)a)$ and $\tau(bd(1 - a)) = t\tau(d(1 - a))$. It follows that

$$\tau\left(a^{1/2}(b - t1)(1 - d)a^{1/2}\right) = 0, \quad (3.17)$$

$$\tau\left((1 - a)^{1/2}(b - t1)d(1 - a)^{1/2}\right) = 0.$$

But since $(b - t1)(1 - d) \geq 0$ and $(b - t1)d \geq 0$ we get

$$a^{1/2}(b - t1)(1 - d)a^{1/2} = 0,$$
$$d(1 - a)^{1/2}(b - t1)d(1 - a)^{1/2} = 0. \quad (3.18)$$

By Lemma 3.2 we get $a \in [q_i^-, q_i^+]$. If $d = q_i^+$, then since $\tau$ is faithful we must have $a = d = q_i^+$. \hfill $\square$

Notice that this is one major difference between $B(b)$ in the case where $\mathcal{M}$ is finite, and $B(b)$ when $\mathcal{M}$ is semifinite. In the finite setting $B(b)$ was convex and compact. Next, we define the boundary of $B(b)$, and then study some of its properties.

**Definition 3.6.** Let $\overline{B(b)}$ denote the closure of $B(b)$ in $\mathbb{R}^2$. The lower boundary of $B(b)$ is defined as

$$\partial_\mathcal{L}B(b) = \left\{ (x, y) \in \overline{B(b)} : (x, y') \in B(b) \implies y' \geq y \right\}. \quad (3.19)$$

The upper boundary of $B(b)$ is defined analogously as

$$\partial_\mathcal{U}B(b) = \left\{ (x, y) \in \overline{B(b)} : (x, y') \in B(b) \implies y' \leq y \right\}. \quad (3.20)$$
Since $\overline{B(b)}$ is a closed convex set, and $\tau$ takes on every value in $[0, \infty]$, we let

$$f : [0, \infty) \rightarrow \mathbb{R}$$ (3.21)

defined by $f(x) = y$ for each $(x, y) \in \partial B(b)$ denote the lower boundary function determined by $b$. Similarly, let

$$g : [0, \infty) \rightarrow \mathbb{R}$$ (3.22)

defined by $g(x) = y$ for all $(x, y) \in \partial B(b)$ denote the upper boundary function determined by $b$.

It is clear that $f$ is a convex function and $g$ is a concave function since $\overline{B(b)}$ is convex.

Theorems 3.7 and 3.8 describe the possibilities for the boundary of $B(b)$. These theorems describe the extreme points $B(b)$, the line segments in $\partial B(b) \cap B(b)$ and $\partial B(b) \cap B(b)$, respectively, and the rays in $\partial B(b) \cap B(b)$ and $\partial B(b) \cap B(b)$ (resp.). Rays contained in $\partial B(b) \setminus B(b)$ and $\partial B(b) \setminus B(b)$ will be described in Theorems 3.9 and 3.10 (resp.). Differentiability of the functions $f$ and $g$ will be discussed in Theorems 3.12 and 3.13 (resp.).

**Note 2.** For a convex set $C$ we let $\text{Ext}(C)$ denote the set of extreme points of $C$.

**Theorem 3.7.** Let $a \in (m_r)^+_1$. Then one has the following.

1. If $s \in \sigma(b)$ with $p^+_s \in m_r$, then $\Psi(p^+_s) \in \text{Ext}(B(b)) \cap \partial B(b)$.
2. If $s \in \sigma_p(b)$ with $p^+_s \in m_r$, then $\Psi([p^-_s, p^+_s])$ is a line segment in $\partial B(b) \cap B(b)$ having slope $s$.
3. If $\Psi(a) \in \partial B(b) \cap B(b)$ and $\Psi(a) \neq \Psi(p^+_s)$ for all $s \in \mathbb{R}$, then $a \in [p^-_s, p^+_s]$ for some $s \in \sigma(b)$ with $\tau(p^-_s) < \infty$.
4. If $\Psi(a) \in \text{Ext}(B(b)) \cap \partial B(b)$, then $a = p^+_s$ for some $s \in \sigma_p(b)$ with $p^+_s \in m_r$.
5. $R$ is a ray in $\partial B(b) \cap B(b)$ having slope $s$ if and only if $R = \Psi([p^-_s, p^+_s])$ and $s \in \sigma_p(b)$ with $\tau(p^-_s) < \tau(p^+_s) = \infty$.

**Proof.** (1) Fix $s \in \sigma(b)$ such that $\tau(p^+_s) < \infty$. Then by part (1) of Lemma 3.3 we have that

$$\Psi(p^+_s) \in \partial B(b).$$ (3.23)

By part (2) of Lemma 3.3 we have that

$$\Psi^{-1}(\Psi(p^+_s)) \cap (m_r)^+_1 = \{p^+_s\}. \quad (3.24)$$

To see that $\Psi(p^+_s)$ are extreme in $B(b)$, suppose that

$$\Psi(p^+_s) = \lambda \Psi(a_1) + (1 - \lambda) \Psi(a_2)$$ (3.25)

for some $a_1, a_2 \in (m_r)^+_1$ and $0 < \lambda < 1$. But then

$$\Psi(p^+_s) = \Psi(\lambda a_1 + (1 - \lambda)a_2),$$ (3.26)

for example.
and by part (2) of Lemma 3.3 we have that \( p_s^\pm = \lambda a_1 + (1 - \lambda) a_2 \). Since projections are extreme points of \( \mathcal{M}_s \), we must have \( a_1 = a_2 = p_s^\pm \). Therefore, \( \Psi(p_s^\pm) \in \text{Ext}(B(b)) \cap \partial_L B(b) \).

(2) Fix \( s \in \sigma_p(b) \) so that \( p_s^+ \in m_+ \). Then we have that \( p_s^- < p_s^+ \), \( \Psi(p_s^-) \neq \Psi(p_s^+) \), and \( b(p_s^+ - p_s^-) = s(p_s^+ - p_s^-) \). Note that if \( c \in [p_s^-, p_s^+] \), then \( \Psi(c) \in \partial_L B(b) \) by part (1) of Lemma 3.3, so

\[
\Psi([p_s^-, p_s^+]) \subset \partial_L B(b) \cap B(b). \tag{3.27}
\]

For each \( \lambda \in (0, 1) \) let \( p_1 = \lambda p_s^+ + (1 - \lambda) p_s^- \). Then, each \( p_1 \) is in \([p_s^-, p_s^+]\), and so \( \Psi(p_1) \in \partial_B(b) \cap B(b) \). Furthermore, \( \Psi(p_1) \) is a typical point on the line segment between \( \Psi(p_s^-) \) and \( \Psi(p_s^+) \) and the slope of this line segment is

\[
\frac{\tau(b p_s^+ - \tau(b p_s^-))}{\tau(p_s^+) - \tau(p_s^-)} = \frac{\tau(s p_s^+ - \tau(p_s^-))}{\tau(p_s^+) - \tau(p_s^-)} = s. \tag{3.28}
\]

Therefore \( \Psi([p_s^-, p_s^+]) \) is a line segment in \( \partial_L B(b) \cap B(b) \) with slope \( s \). Note that since \( \Psi(p_s^+) \) are extreme points of \( B(b) \) contained in \( \partial_L B(b) \), we know that the endpoints of any line segment containing \( \Psi([p_s^-, p_s^+]) \) must be \( \Psi(p_s^-) \) and \( \Psi(p_s^+) \).

(3) Suppose \( \Psi(a) \in \partial_L B(b) \cap B(b) \) and \( \Psi(a) \neq \Psi(p_s^+) \) for all \( s \in \sigma(b) \). We will show that \( p_s^- < a < p_s^+ \) for some \( s \in \sigma(b) \) with \( \tau(p_s^+) < \infty \). Let

\[
\begin{align*}
\tau(p_s^-) & \leq \tau(a) & \tau(p_s^+) & \leq \tau(a), \\
\tau(p_s^-) & \leq \tau(a) & \tau(p_s^+) & \leq \tau(a),
\end{align*}
\]

We want to show that \( r_1 = r_2 \). Note that \( r_1 \) and \( r_2 \) are in \( \sigma(b) \) since \( \sigma(b) \) is closed, and \( \tau(p_s^-) \leq \tau(a) \) by lower semicontinuity of \( \tau \).

We first show that \( r_1 \leq r_2 \). If \( s, t \in \sigma(b) \) with \( \tau(p_s^-) \leq \tau(a) \leq \tau(p_t^+) \), and \( t < s \), then by the positivity of \( \tau \) we must also have \( \tau(p_t^+) \leq \tau(p_s^-) \), so \( \tau(p_s^-) = \tau(a) = \tau(p_t^+) \). This fact, along with Lemma 3.3, contradicts our assumption about \( \Psi(a) \). Thus, everything in \( \{ s \in \sigma(b) : \tau(p_s^-) \leq \tau(a) \} \) is less than or equal to every thing in \( \{ s \in \sigma(b) : \tau(a) \leq \tau(p_s^+) \} \), hence \( r_1 \leq r_2 \).

Now, either \( \tau(a) \leq \tau(p_r^+) \) or not. Suppose \( \tau(p_r^+) < \tau(a) \). Since \( \tau(p_r^-) \leq \tau(p_r^+) < \tau(a) \), and since \( r_1 \) is an upper bound for \( \{ s \in \sigma(b) : \tau(p_s^-) \leq \tau(a) \} \), we must have \( r_2 \leq r_1 \). But \( r_1 \leq r_2 \). Therefore \( r_1 = r_2 \) when \( \tau(p_r^+) < \tau(a) \).

Assume that \( \tau(a) \leq \tau(p_r^+) \). Then, \( \tau(p_r^-) \leq \tau(a) \leq \tau(p_r^+) \). But we are assuming that \( \Psi(a) \neq \Psi(p_s^+) \) for all \( s \in \sigma(b) \), and that \( \Psi(a) \in \partial_L B(b) \cap B(b) \). Therefore, by Lemma 3.3 we get

\[
\tau(p_r^-) < \tau(a) < \tau(p_r^+). \tag{3.30}
\]

Observe that, since \( r_2 \) is a lower bound for the set \( \{ s \in \sigma(b) : \tau(a) \leq \tau(p_s^+) \} \), if \( s < r_2 \) and \( s \in \sigma(b) \) then

\[
\tau(p_s^+) < \tau(a) < \tau(p_r^+). \tag{3.31}
\]
Similarly, if \( s \in \sigma(b) \) with \( r_1 < s \), then
\[
\tau(p^+_r) < \tau(a) < \tau(p^-_s). \tag{3.32}
\]

Suppose that \( r_1 < r_2 \). Then since \( r_1, r_2 \in \sigma(b) \) we have that
\[
\tau(p^+_r) < \tau(a) < \tau(p^-_r). \tag{3.33}
\]

On the other hand if \( s \in (r_1, r_2) \cap \sigma(b) \) then by (3.32) and (3.33) we would have \( \tau(p^+_r) < \tau(a) < \tau(p^-_s) \). This is not possible, so it must be that \( (r_1, r_2) \cap \sigma(b) = \emptyset \). Therefore \( p^+_r = p^-_r \), and we have \( \tau(p^+_r) = \tau(p^-_r) \), contradicting (3.33). Hence, it must be that \( r_1 = r_2 = r \) and so \( \tau(p^+_r) < \tau(a) < \tau(p^-_r) \). Therefore \( r \in \sigma_p(b) \) and \( a \in [p^-_r, p^+_r] \) by Lemma 3.3.

(4) Since \( \overline{B(b)} \) is convex, every point in the boundary of \( B(b) \) must either be an extreme point, or lie on a flat portion (face). Therefore, (1), and (2), (3), imply (4).

(5) \( \Rightarrow \) Suppose \( \tau(p^-_r) < \tau(p^+_s) = \infty \) for some \( s \in \sigma_p(b) \). We claim that \( R = \Psi([p^-_r, p^+_s]) \) is a ray in \( \partial_2 B(b) \cap B(b) \) with slope \( s \), emanating from the point \( \Psi(p^-_r) = (\tau(p^-_r), \tau(bp^-_r)) \). We will first show that every point in \( \Psi([p^-_r, p^+_s]) \) lies on the ray through \( \Psi(p^-_r) \) with slope \( s \).

Let \( c \in [p^-_r, p^+_s] \cap m_r \), and write \( c = p^-_r + d \) for some \( d \in (p^-_r - p^+_s) \mathbb{R}(p^-_r - p^+_s) \). Then \( b(c - p^-_r) = bd = sd = s(c - p^-_r) \), so \( \tau(b(c - p^-_r)) = \tau(s(c - p^-_r)) \). This implies that
\[
\tau(bc) = \tau(s(c - p^-_r)) + \tau(bp^-_r) = s \cdot \tau(c) + \tau((b - s1)p^-_r). \tag{3.34}
\]

Therefore \( \Psi(c) \) lies on the ray with slope \( s \) passing through the point \( (\tau(p^-_r), \tau(bp^-_r)) \). Furthermore, by the convexity of \( B(b) \), we know there can only be one ray having slope \( s \) in the lower boundary of \( B(b) \), and so if \( \Psi(x) \) is any other point in \( B(b) \) that lies on a ray with slope \( s \) in the lower boundary, then we must have \( \Psi(x) \in R = \Psi([p^-_r, p^+_s]) \).

(5) \( \Leftarrow \) If \( R \) is a ray in \( \partial_2 B(b) \cap B(b) \) with slope \( s \), emanating from \( \Psi(x) \), then \( \Psi(x) \) is an extreme point of \( B(b) \) (if not, then \( R \) does not emanate from \( \Psi(x) \) since \( \Psi(x) \) would then lie on a flat portion of \( \partial_2 B(b) \cap B(b) \)), so we must have \( x = p^+_r \in m_r \) for some \( r \in \sigma(b) \) by part (4). But \( R \) is a ray, so \( R \) cannot contain any extreme points of \( B(b) \). So by part (3) we know that if \( z \in (m^-_r)_1 \) with \( \Psi(z) \in R \), then \( z \in [p^-_s, p^+_s] \) for some \( s \in \sigma_p(b) \) with \( \tau(p^-_s) < \infty \). Furthermore, since any such \( \Psi([p^-_s, p^+_s]) \) is a line segment (or ray) with slope \( s \), and since \( \Psi([p^-_s, p^+_s]) \subset R \), we must have \( s = s \). Since there can only be one ray with slope \( s \) in the lower boundary of \( B(b) \), we may conclude that \( r = s \) and \( R = \Psi([p^-_s, p^+_s]) \).

**Theorem 3.8.** Let \( a \in (m^-_r)_1 \). Then one has the following.

1. If \( t \in \sigma(b) \) with \( q^+_t \in m_r \), then \( \Psi(q^+_t) \in \text{Ext}(B(b)) \cap \partial_R B(b) \).
2. If \( t \in \sigma_p(b) \) with \( q^+_t \in m_r \), then \( \Psi([q^-_t, q^+_t]) \) is a line segment in \( \partial_R B(b) \cap B(b) \) having slope \( t \).
3. If \( \Psi(a) \in \partial_R B(b) \cap B(b) \) and \( \Psi(a) \neq \Psi(q^+_t) \) for all \( t \in \mathbb{R} \), then \( a \in [q^-_t, q^+_t] \) for some \( t \in \sigma(b) \) with \( \tau(q^-_t) < \infty \).
4. If \( \Psi(a) \in \text{Ext}(B(b)) \cap \partial_R B(b) \) then \( a = q^+_t \) for some \( t \in \sigma_p(b) \) with \( q^+_t \in m_r \).
5. \( R \) is a ray in \( \partial_R B(b) \cap B(b) \) having slope \( t \) if and only if \( R = \Psi([q^-_t, q^+_t]) \) and \( t \in \sigma_p(b) \) with \( \tau(q^-_t) < \tau(q^+_t) = \infty \).
Proof. Note that for each \( s \in \mathbb{R} \) we have that
\[
q_s = \chi_{(s,\infty)}(b) = \chi_{(-\infty,-s)}(-b).
\] (3.35)

Since \(-b\) is self-adjoint, the previous theorem applies. Alternatively, substitute Lemma 3.5 for Lemma 3.3 in the proof of Theorem 3.7, the claim follows immediately. \(\square\)

Until this point we have focused on points that lie in \(B(b)\). As mentioned earlier, it may be the case that \(B(b)\) is not closed. The next four results describe this phenomenon, and tell us exactly when \(B(b)\) is closed, when it is not, and, when we get a “dotted” (open) portion in the boundary of \(B(b)\).

**Theorem 3.9.** Let \(\mathcal{E}_\lambda = \{ s \in \sigma(b) : \tau(p^*_s) = \infty \}\) and let \(\lambda = \inf \mathcal{E}_\lambda\). Then \(\partial_\lambda B(b) \cap B(b) = \partial_\lambda B(b)\) if and only if \(\tau(p^*_\lambda) = \infty\). Furthermore, if \(\tau(p^*_\lambda) < \infty\) then
\[
y = \lambda x + \tau((b - \lambda)p^*_1), \quad x > \tau(p^*_1)
\] (3.36)
is a ray in \(\partial_\lambda B(b) \setminus B(b)\) emanating from \(\Psi(p^*_\lambda)\).

**Proof.** Suppose that \(\partial_\lambda B(b) \cap B(b) = \partial_\lambda B(b)\). Theorem 3.7 tells us that every \(\Psi(c) \in \partial_\lambda B(b)\) has the property that either \(c \in [p^-_r, p^+_r]\) for some \(s \in \sigma(b)\) with \(\tau(p^*_s) < \infty\), or \(c = p^+_s\) for some \(s \in \sigma(b)\) with \(p^+_s \in m_r\). Note that by the definition of \(\lambda\), \(\tau(p^*_s) = \infty\) for all \(s > \lambda\). Furthermore, for every \(s > \lambda\) we can choose \(\lambda < s' < s\), so \(\infty = \tau(p^*_s) \leq \tau(p^*_{s'})\). Therefore, for every \(s > \lambda\) we have \(\tau(p^*_{s'}) = \infty\). Suppose, for the sake of contradiction, that \(\tau(p^*_\lambda) < \infty\). Then, if \(d \in (m_r\lambda)^1\) with \(\Psi(d) \in \partial_\lambda B(b)\) and \(\tau(d) > \tau(p^*_\lambda)\), then \(d \in [p^-_r, p^+_r]\) for some \(r \in \sigma(b)\) with \(\tau(p^*_r) < \infty\). Therefore it must be that \(r \leq \lambda\). Clearly this is a contradiction since \(\tau(d) > \tau(p^*_\lambda)\). Hence, it must be that \(\tau(p^*_\lambda) = \infty\).

Suppose \(\tau(p^*_\lambda) = \infty\), and let \((x,y) \in \partial_\lambda B(b)\). We want to show that \((x,y) \in B(b)\). Now, either \(\tau(p^*_{\lambda_{n+1}}) = \infty\) or not. If \(\tau(p^*_{\lambda_{n+1}}) = \infty\) then since \(\tau(p^*_\lambda) < \infty\) for all \(s < \lambda\), we get that \(\tau(x(1-\epsilon,\lambda)) = \infty\) for every \(\epsilon > 0\). For each \(n \in \mathbb{N}\) let \(\lambda_n = \lambda - 1/n\). Then \(\tau(p^*_{\lambda_n}) \leq \tau(p^*_{\lambda_{n+1}}) < \infty\) for every \(n\), and \(\tau(p^*_{\lambda_n}) \to \infty\) as \(n \to \infty\). Therefore, we can find \(s \in \sigma(b)\) such that \(\tau(p^*_s) > x\). Let
\[
r_1 = \sup \{ s \in \sigma(b) : \tau(p^*_s) \leq x \},
\]
\[
r_2 = \inf \{ s \in \sigma(b) : x \leq \tau(p^*_s) \}. \tag{3.37}
\]
To show that \((x,y) \in B(b)\), we will show that \(r_1 = r_2\), which will force \((x,y) = \Psi(c)\) for some \(c \in [p^-_{\lambda_n}, p^+_r]\) by Theorem 3.7. Note that \(\tau(p^*_{\lambda_n}) \leq x\) by lower semicontinuity of \(\tau\). We will first show that \(r_1 \leq r_2\).

If \(s,t \in \sigma(b)\) with \(\tau(p^*_{s}) \leq x \leq \tau(p^*_{r})\), and \(t < s\), then by the positivity of \(\tau\) we must also have \(\tau(p^*_{r}) \leq \tau(p^*_{t})\), which implies \(\tau(p^*_s) = \tau(p^*_t)\). Thus, everything in \(\{ s \in \sigma(b) : \tau(p^*_s) \leq x \}\) is less than or equal to everything in \(\{ s \in \sigma(b) : x \leq \tau(p^*_s) \}\), hence \(r_1 \leq r_2\).

Suppose \(r_1 < r_2\). Either \(x \leq \tau(p^*_{r_2})\) or not. Suppose \(\tau(p^*_{r_2}) < x\). Since \(\tau(p^*_{r_2}) \leq \tau(p^*_{r_1}) < x\), and since \(r_1\) is an upper bound for \(\{ s \in \sigma(b) : \tau(p^*_{r_1}) \leq x \}\), we get \(s \leq r_1\). But this can not be, so it must be that \(x \leq \tau(p^*_{r_3})\). This implies \(\tau(p^*_{r_1}) \leq x \leq \tau(p^*_{r_2})\). If equality holds on either side, then we would get \((x,y) \in B(b)\) (again, by Theorem 3.7).
Suppose \( \tau(p_{r_2}^-) < x < \tau(p_{r_2}^+) \). Observe that, since \( r_2 \) is a lower bound for the set \( \{ s \in \sigma(b) : x \leq \tau(p_{s}^+) \} \), if \( s < r_2 \) and \( s \in \sigma(b) \) then

\[
\tau(p_{s}^+) < x < \tau(p_{r_2}^+). \tag{3.38}
\]

Similarly, if \( s \in \sigma(b) \) with \( r_1 < s \) then

\[
\tau(p_{r_1}^-) < x < \tau(p_{s}^+). \tag{3.39}
\]

Since \( r_1 < r_2 \), we have

\[
\tau(p_{r_1}^+) < x < \tau(p_{r_2}^+). \tag{3.40}
\]

Thus, if \( s \in (r_1, r_2) \cap \sigma(b) \) then by (3.32) and (3.33) we would have \( \tau(p_{s}^+) < x < \tau(p_{r_2}^+) \). This is not possible, so it must be that \( (r_1, r_2) \cap \sigma(b) = \emptyset \). Therefore \( p_{r_1}^+ = p_{r_2}^- \), and we have \( \tau(p_{r_1}^+) = \tau(p_{r_2}^-) \), contradicting (3.33). Hence, it must be that \( r_1 = r_2 = r \) and so \( \tau(p_{r_1}^+) < x < \tau(p_{r_2}^+) \), and our desired result holds. Therefore, in this case we have \( \partial_x \mathcal{B}(b) \cap B(b) = \partial_x \mathcal{B}(b) \).

If \( \tau(p_{r_1}^+) < \infty \) then by Theorem 3.7 \( \Psi([p_{r_1}^-, p_{r_1}^+]) \) will be a ray in \( \partial_x \mathcal{B}(b) \cap B(b) \). In both cases we have \( \partial_x \mathcal{B}(b) \cap B(b) = \partial_x \mathcal{B}(b) \).

For the second conclusion of the theorem, suppose \( \tau(p_{r_2}^+) < \infty \). Plugging \( x = \tau(p_{r_2}^+) \) into \( y = \lambda x + \tau((b-\lambda)p_{r_2}^+) \) we get \( y = \tau(bp_{r_2}^+) \), so the point \( \Psi(p_{r_2}^+) \) lies on the line \( y = \lambda x + \tau((b-\lambda)p_{r_2}^+) \). We will now show that, for \( x > \tau(p_{r_2}^+) \), the ray \( y = \lambda x + \tau((b-\lambda)p_{r_2}^+) \) forms the lower boundary of \( B(b) \), but no point on the ray lies in \( B(b) \).

Suppose \( d \in (m_\tau)^+ \) with \( \tau(d) > \tau(p_{r_2}^+) \) (see Figure 5).

We will show that \( \Psi(d) \) lies above the line \( y = \lambda x + \tau((b-\lambda)p_{r_2}^+) \), and, for any \( \epsilon > 0 \) there exists \( d' \in (m_\tau)^+ \) such that \( \tau(d') = \tau(d) \) and \( \tau(bd') \) is within \( \epsilon \) of \( y = \lambda \tau(d') + \tau((b-\lambda)p_{r_2}^+) \).

We will first show that \( \Psi(d) \) lies above \( y = \lambda x + \tau((b-\lambda)p_{r_2}^+) \) when \( \lambda = 0 \), then prove the general case (\( \lambda \neq 0 \)). Suppose \( \lambda = 0 \). We want to show that

\[
\tau(bd) > \lambda \tau(d) + \tau((b-\lambda)p_{r_2}^+) = \tau(bp_{r_2}^+). \tag{3.41}
\]

Suppose for the sake of contradiction that \( \tau(bd) \leq \tau(bp_{r_2}^+) \). Since \( B(b) \) is convex there must be \( d' \in (m_\tau)^+ \) such that \( \tau(d') = \tau(d) \) and \( \tau(bd') = \tau(bp_{r_2}^+) \) (see Figure 6).
Write $b = b^+ - b^-$ with $b^+$ and $b^-$ positive and $b^+b^- = 0$. Note that $\tau(bp^+_0) = -\tau(b^-)$, so we have $\tau(bd') = -\tau(b^-)$. On the other hand we have that

\[-\tau(b^-) = \tau(bd')
= \tau(b^+d') - \tau(b^-d')
= \tau(b^+d') - \tau(p_0^+bp_0^-d')
= \tau(b^+d') - \tau((b^-)^{1/2}p_0^-d'p_0^-(b^-)^{1/2})
\geq \tau(b^+d') - \tau((b^-)^{1/2}p_0^-(b^-)^{1/2})
= \tau(b^+d') - \tau(b^-).
\] (3.42)

Therefore, adding $\tau(b^-)$ to both sides we get $\tau(b^+d') \leq 0$. This implies $0 = \tau(b^+d') = \tau((d')^{1/2}b^+(d')^{1/2})$. Therefore, since $\tau$ is faithful,

\[0 = (d')^{1/2}b^+(d')^{1/2} = (d')^{1/2}b(1 - p_0^+)(d')^{1/2}.
\] (3.43)

By part (2) of Lemma 3.2 we have

\[d' \leq p_0^+.
\] (3.44)

But our assumption is that $\tau(d') = \tau(d) > \tau(p_0^+) = \tau(p_0^+)$. Therefore, we have reached a contradiction to our assumption that $\tau(bd) \leq \tau(bp_0^+)$. Thus it must be that $\tau(bd) > \tau(bp_0^+)$. Now suppose that $\lambda \neq 0$. As before, we want to show that

\[\tau(bd) > \lambda\tau(d) + \tau((b - \lambda)p_0^+).
\] (3.45)

Suppose that is not. Then as before we can find $d' \in (m_\tau)_1^+$ such that

\[\tau(bd') = \lambda \tau(d') + \tau((b - \lambda)p_0^+).
\] (3.46)
Corollary 3.11. Let $a = b - \lambda 1$ and consider $B(a)$. From (3.46) and (3.47) we have that

$$
\tau(a d') = \tau(a \chi_{(-\infty,0]}(a)).
$$

(3.48)

Using (3.47) and the argument concluded at (3.44) we get $d' \leq \chi_{(-\infty,0]}(a) = p_1^+$. But this is a contradiction since $\tau(d') = \tau(d) = \tau(p_1^+)$. Therefore $\tau(bd) > \lambda \tau(d) + \tau((b - \lambda)p_1^+)$. To prove $B(b)$ comes arbitrarily close to the line $y = \lambda x + \tau((b - \lambda)p_1^+)$, let $e > 0$. Using the definition of $\lambda$ and our assumption that $\tau(p_1^+) < \infty$, we have that $\tau(p_1^+) = \infty$. Since $\tau$ is semipositive we can find a nonzero, positive $e < p_{1+e/2} - p_1^+$ such that $\tau(e) < \infty$. Furthermore, we may scale $e$ so that $\tau(p_1^+ + e) = \tau(d)$. Let $d' = p_1^+ + e$. Then since $e < p_{1+e/2} - p_1^+ \leq 1 - p_1^+ = q_1^-$, we have that

$$
\tau(bd') - (\lambda \tau(d') + \tau((b - \lambda)p_1^+)) = \tau(bd') - \lambda \tau(d') - \tau((b - \lambda)p_1^+)
$$

$$
= \tau(b(p_1^+ + e)) - \lambda \tau(p_1^+ + e) - \tau((b - \lambda)p_1^+)
$$

$$
= \tau((b - \lambda)e)
$$

$$
= \tau(q_1^-(b - \lambda)e)
$$

$$
= \tau(q_1^- e)\leq \tau(q_1^- e^{1/2})
$$

$$
= \tau(e) < \frac{\epsilon}{2}.
$$

(3.49)

\[ \blacksquare \]

**Theorem 3.10.** Let $\mathcal{E}_U = \{ t \in \sigma(b) : \tau(q_t^p) = \infty \}$ and let $\rho = \sup \mathcal{E}_U$. Then $\partial_B B(b) \cap B(b) = \partial_B B(b)$ if and only if $\tau(q_\rho^p) = \infty$. Furthermore, if $\tau(q_\rho^p) < \infty$ then

$$
y = \rho x + \tau((b - \rho)q_\rho^p), \quad x > \tau(q_\rho^p)
$$

(3.50)

is a ray in $\partial_B B(b) \setminus B(b)$ emanating from $\Psi(q_\rho^p)$.

**Proof.** Substitute Theorem 3.8 for Theorem 3.7 in the proof of Theorem 3.9.  

\[ \blacksquare \]

**Corollary 3.11.** Let $\lambda$ and $\rho$ be as in Theorems 3.9 and 3.10 (resp.). Then $B(b) = B(b)$ if and only if $\tau(p_1^+) = \tau(q_\rho^p) = \infty$. 


Proof. Note that \( \tau(1) = \infty \) if and only if \( \mathcal{E}_b \neq \emptyset \) if and only if \( \mathcal{E}_c \neq \emptyset \) (since \( q_s^t \in B(b) \) for each \( s \in \mathbb{R} \)). Thus, the claim follows immediately from Theorems 3.9 and 3.10.

Note that Theorems 3.7, 3.8, 3.9, and 3.10 tell us that the boundary of \( B(b) \) (which contains information about \( \sigma(b) \)) is completely determined by spectral projections of the form \( p_s^t \) and \( q_s^t = 1 - p_s^t \) where \( s, t \in \mathbb{R} \). To summarize, the following information is contained in the boundary of \( B(b) \):

(i) Extreme points contained in the lower boundary are of the form \( \Psi(p_s^t) \) with \( s \in \mathbb{R} \).

(ii) Line segments having slope \( s \) in the lower boundary are of the form \( \Psi([p_s^t, p_s^t]) \) with \( s \in \mathbb{R} \) and \( \tau(p_s^t) \in (-\infty, \infty) \).

(iii) Rays having slope \( s \) in the lower boundary are of the form \( \Psi([p_s^t, p_s^t]) \) with \( s \in \mathbb{R} \) and \( \tau(p_s^t) \in (-\infty, \infty) \).

(iv) A “dotted” ray in the lower boundary occurs if \( \tau(p_s^t) \in (-\infty, \infty) \). Moreover, this ray emanates from \( \Psi(p_s^t) \) and has as its equation \( y = \lambda x + \tau((b - \lambda)p_s^t) \).

(v) Extreme points contained in the upper boundary are of the form \( \Psi(q_s^t) \) with \( t \in \mathbb{R} \).

(vi) Line segments having slope \( t \) in the upper boundary are sets of the form \( \Psi([q_s^t, q_s^t]) \) with \( t \in \mathbb{R} \) and \( \tau(q_s^t) \in (-\infty, \infty) \).

(vii) Rays having slope \( t \) in the upper boundary are sets of the form \( \Psi([q_s^t, q_s^t]) \) with \( t \in \mathbb{R} \) and \( \tau(q_s^t) \in (-\infty, \infty) \).

(viii) A “dotted” ray in the upper boundary occurs if \( \tau(q_s^t) \in (-\infty, \infty) \). Moreover, this ray emanates from \( \Psi(q_s^t) \) and has as its equation \( y = \lambda x + \tau((b - \lambda)q_s^t) \).

Recall that for each \((x, y) \in \partial B(b)\) we let \( f : [0, \infty) \to \mathbb{R} \) defined by \( f(x) = y \) denote the lower boundary function generated by \( b \). Similarly, \( g \) denotes the upper boundary function generated by \( b \). For each \( x \in [0, \infty) \), let \( f'_-(x) \) \((g'_-(x))\) and \( f'_+(x) \) \((g'_+(x))\) denote the left-sided and right-sided, (respectively) derivatives of \( f \) and \( g \) at \( x \). Note that by Theorem 3.7, every \( \Psi(c) \) in \( \partial B(b) \) has the property that \( \Psi(c) \in \Psi([p_s^t, p_s^t]) \) for some \( s \in \sigma(b) \) with \( \tau(p_s^t) \in (-\infty, \infty) \).

**Theorem 3.12.** Let \( s \in \sigma(b) \), and let \( c \in [p_s^t, p_s^t] \cap (m_r)_1^+ \). Then

1. \( f'_-(\tau(c)) \) and \( f'_+(\tau(c)) \) exist,
2. \( f'_-(\tau(p_s^t)) = \sup \{s' \in \sigma(b) : s' < s\} \)
3. \( f'_+(\tau(p_s^t)) = \inf \{s' \in \sigma(b) : s < s'\} \),
4. \( f \) is not differentiable at \( \tau(p_s^t) \) if and only if \((s - e, s) \cap \sigma(b) \neq \emptyset \) (or \((s, s + e) \cap \sigma(b) \neq \emptyset \)) for some \( e > 0 \).

**Proof.** (1) Since \( B(b) \) is convex, \( f \) is a convex function, and so for any \( x \in (0, +\infty) \)

\[
\frac{f(x) - f(x - h)}{h}
\]  (3.51)

increases as \( h \) decreases to zero. Furthermore, if \( x = \tau(c) \) with \( c \in [p_s^t, p_s^t] \cap m_r \) for some \( s \in \sigma(b) \), then since \( \Psi([p_s^t, p_s^t]) \) is a line segment (or ray) in the lower boundary with slope equal to \( s \) (by Theorem 3.7), we know \( (f(x) - f(x - h))/h \) is bounded above by \( s \). Since
bounded increasing sequences converge, \( f'(x) \) exists at each \( x = \tau(c) \) with \( c \in [p^-_s, p^+_s] \cap m_r \) for some \( s \in \sigma(b) \). Using a similar argument considering the sequence \( (f(x+h) - f(x))/h \) for some fixed \( x = \tau(c) \) as \( h \) decreases to 0, and Theorem 3.7, we get that \( f'_+(x) \) also exists at every \( x \in [0, +\infty) \).

(2) Fix \( s \in \sigma(b) \) with \( s_{\min} < s \), where \( s_{\min} \) is the minimum of \( \sigma(b) \), and suppose \( x = \tau(p^-_s) < \infty \). Let

\[
\begin{align*}
  r &= \sup \{ s' \in \sigma(b) : s' < s \}. \quad (3.52)
\end{align*}
\]

Since \( \sigma(b) \) is closed, \( r \in \sigma(b) \). We want to show that \( r = f'_-(\tau(p^-_s)) \). Clearly \( r \leq s \).

First suppose that \( (r - \epsilon, r) \cap \sigma(b) \neq \emptyset \) for some \( \epsilon > 0 \). If we had \( r = s \) then we would have \( (s - \epsilon, s) \cap \sigma(b) \neq \emptyset \), which contradicts the definition of \( r \). Therefore, in this case we must have \( r < s \). But then again, by the definition of \( r \), we have \( (r, s) \cap \sigma(b) \neq \emptyset \). Therefore \( r \in \sigma_p(b) \), and by Theorem 3.7 we know that \( \Psi(p^-_s) = \Psi(p^+_s) \) is the right endpoint of a line segment in \( \partial S \cap B(b) \cap B(b) \) with slope equal to \( r \). Thus, \( r = f'_-(\tau(p^-_s)) \).

If \( (r - \epsilon, r) \cap \sigma(b) = \emptyset \) for every \( \epsilon > 0 \), then we can choose an increasing sequence \( (s_n) \) of distinct points in \( \sigma(b) \) that converge to \( r \). But then since \( \tau \) is normal, we know that \( \alpha_n = \tau(p^-_{s_n}) \) converges in the \( w^* \)-topology to \( \alpha = \tau(p^-_r) \). Since all the elements under consideration are distinct spectral projections of \( b \) we have

\[
\begin{align*}
  s_n(p^-_{s_n} - p^-_{s_n}) &\leq b(p^-_{s_n} - p^-_{s_n}) \leq r(p^-_{s_n} - p^-_{s_n}) \quad (3.53)
\end{align*}
\]

for each \( n \in \mathbb{N} \). Applying \( \tau \) to both sides of this we get

\[
\begin{align*}
  s_n(\alpha - \alpha_n) &\leq f(\alpha) - f(\alpha_n) \leq r(\alpha - \alpha_n) \quad (3.54)
\end{align*}
\]

for every \( n \in \mathbb{N} \). Therefore

\[
\begin{align*}
  s_n \leq \frac{f(\alpha) - f(\alpha_n)}{\alpha - \alpha_n} \leq r \quad (3.55)
\end{align*}
\]

for every \( n \). Since \( (s_n) \) converges to \( r \), we get

\[
\begin{align*}
  \lim_{n} \frac{f(\alpha) - f(\alpha_n)}{\alpha - \alpha_n} = r. \quad (3.56)
\end{align*}
\]

Since the function defined by

\[
\begin{align*}
  h(x) = \frac{f(\alpha) - f(x)}{\alpha - x} \quad \forall x \in [0, \infty] \quad (3.57)
\end{align*}
\]

is an increasing function. From this and (3.55) above, we see that if \( (r_n) \) is any sequence in \( \mathbb{R} \) increasing to \( \tau(p^-_r) \), then

\[
\begin{align*}
  \lim_{n} \frac{f(\alpha) - f(r_n)}{\alpha - r_n} = r. \quad (3.58)
\end{align*}
\]

Thus, \( f'_-(\tau(p^-_r)) = r \).
(3) Let \( r = \inf\{s' \in \sigma(b) : s < s'\} \). Clearly, by the definition of \( r \), we have \( s \leq r \). Using the same argument as in (2), we get

\[
f'_-(\tau(p_s^*)) = r = \inf\{s' \in \sigma(b) : s < s'\}.
\]

(4) From parts (2) and (3) above, we have

\[
f'_-(\tau(p_s^*)) = \sup\{s' \in \sigma(b) : s' < s\}, \quad f'_+(\tau(p_s^*)) = \inf\{s' \in \sigma(b) : s < s'\}.
\]

But then, \((s, s + \epsilon) \cap \sigma(b) \neq \emptyset\) for some \( \epsilon > 0 \) if and only if \( x < y \) whenever \( x \in \{s' \in \sigma(b) : s' < s\} \) and \( y \in \{s' \in \sigma(b) : s' > s\} \) if and only if \( f'_-(\tau(p_s^*)) < f'_+(\tau(p_s^*)) \) if and only if \( f \) is not differentiable at \( \tau(p_s^*) \) or \( f \) is not differentiable at \( \tau(p_s^*) \). An identical argument shows that \((s - \epsilon, s) \cap \sigma(b) \neq \emptyset\) for some \( \epsilon > 0 \) if and only if \( f \) is not differentiable at \( \tau(p_s^*) \) or \( f \) is not differentiable at \( \tau(p_s^*) \).

\[\blacksquare\]

**Theorem 3.13.** Let \( t \in \sigma(b) \), and let \( c \in [q_t^-, q_t^+] \cap (m_c)_t^+ \). Then

1. \( g'_-(\tau(c)) \) and \( g'_+(\tau(c)) \) exist,
2. \( g'_-(\tau(q_t^-)) = \sup\{t' \in \sigma(b) : t' < t\} \),
3. \( g'_+(\tau(q_t^+)) = \inf\{t' \in \sigma(b) : t < t'\} \),
4. \( g \) is not differentiable at \( \tau(q_t^+) \) if and only if \((t - \epsilon, t) \cap \sigma(b) = \emptyset \) for some \( \epsilon > 0 \).

**Proof.** Substitute (3.34) for (3.33) into the proof of Theorem 3.12. \(\blacksquare\)

Theorems 3.12 and 3.13 tell us that if \( p_s^* = p_s^- \) and \( s \) does not lie in a gap of the spectrum, then \( f \) is differentiable at \( \tau(p_s) \) and \( f'(\tau(p_s)) \in \sigma(b) \). Part (4) of these theorems also confirm that corners in the boundary of \( B(b) \) correspond to gaps in \( \sigma(b) \).

We finish with a few examples of spectral scales in the semifinite von Neumann algebra setting. Recall from Theorems 3.9 and 3.10 that we let

\[
\mathcal{E}_L = \{s \in \sigma(b) : \tau(p_s^*) = \infty\}, \quad \text{with } \lambda = \inf \mathcal{E}_L,
\]

\[
\mathcal{E}_U = \{t \in \sigma(b) : \tau(q_t^-) = \infty\}, \quad \text{with } \rho = \sup \mathcal{E}_U.
\]

We also let

\[
\mathcal{E} = \{\lambda \in \mathbb{C} : \tau(\chi(\lambda - \epsilon, \lambda + \epsilon))(b) = \infty \forall \epsilon > 0\}
\]

denote the essential spectrum of \( b \). We will see in the next section that \( B(b) \) describes the minimum and maximum elements of \( \mathcal{E} \). Note that all of the figures that follow are (supposed to resemble) unbounded subsets of \( \mathbb{R}^2 \).

**Example 3.14.** Let \( p \in \mathcal{M} \) be a projection. Then either \( \tau(p) < \infty \) and \( \tau(1 - p) = \infty \), or \( \tau(p) = \infty \) and \( \tau(1 - p) < \infty \), or \( \tau(p) = \infty \) and \( \tau(1 - p) = \infty \). In any case we have that \( \tau(p_s^+) = \tau(q_s^-) = \infty \). Therefore by Corollary 3.11 we have \( \overline{B(p)} = B(p) \). Furthermore, \( B(p) \) must be one of the following (see Figure 7).
Example 3.15. Let $\mathcal{M} = B(\mathcal{H})$ for some infinite-dimensional separable Hilbert Space $\mathcal{H}$ with $\tau$ the standard canonical trace on $\mathcal{M}$, and let

$$b = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$  

(3.63)

$B(b)$ is pictured in Figure 8.
Notice that in this case we have $\lambda = -1/3$ and $\rho = 1$ are the minimum and maximum of the essential spectrum $\mathcal{E}$ of $b$. Moreover, $\tau(p_p^+) = \infty$ and $\tau(q_q^-) < \infty$. By Theorem 3.9, $\partial_L B(b) \cap B(b) = \partial_L B(b)$, and by Theorem 3.10, $\partial_R B(b) \cap B(b) \neq \partial_R B(b)$. Corollary 3.11 then tells us that $B(b) \neq B(b)$. Note the closed portion of $\partial_R B(b)$ between $x = 0$ and $x = 1$, which tells us that $\rho = 1$ is an eigenvalue with multiplicity equal to 1.

**Example 3.16.** Suppose $\mathcal{M} = B(\mathcal{H})$ for some infinite-dimensional separable Hilbert Space $\mathcal{H}$, with $\tau$ the standard canonical trace, and let $b \in \mathcal{M}$ be the diagonal operator with $\text{diag}(b) = \mathbb{Q} \cap (0, 1)$. In this case we have $\lambda = 0$ and $\rho = 1$. Furthermore, $\lambda \notin \mathcal{E}_L$ and $\rho \notin \mathcal{E}_R$, so $\partial_L B(b) \cap B(b) \neq \partial_L B(b)$ and $\partial_R B(b) \cap B(b) \neq \partial_R B(b)$. Again, by Corollary 3.11 $B(b) \neq B(b)$. $B(b)$ is pictured in Figure 9.

**Example 3.17.** Let $\mathcal{M} = L^\infty([0, \infty), m)$ with $\tau(c) = \int_0^\infty c \, dm$ for all $c \in \mathcal{M}$, where $m$ is Lebesgue measure on $[0, \infty)$. Let $b \in \mathcal{M}$ be defined by $b(x) = e^{-x}$ on $[0, \infty)$. Then $\sigma(b)$ is equal to the essential range of $b$, which equals $[0, 1]$. For each $s \in [0, 1]$ we have

$$
p_s^- = p_s^+ = \chi_{(-\infty, s]}(b) = \chi_{[-\ln(s), \infty)}, \quad q_s^- = q_s^+ = 1 - p_s^+ = 1 - \chi_{[-\ln(s), \infty)} = \chi_{[0, -\ln(s)]}.
$$

(3.64)

Since points of the form $(\tau(p_p^+), \tau(bp_p^+))$ and $(\tau(q_q^-), \tau(bq_q^-))$ determine the extreme points of $B(b)$ (see Theorems 3.7 and 3.8), by performing a couple quick integrals we find $g$ and $f$ (the upper and lower boundary curves (resp.) generated by $b$):

$$
g(x) = 1 - e^{-x}, \quad f(x) = 0.
$$

(3.65)

Note that we have $\lambda = 0 \notin \mathcal{E}_L$ and $\rho = 0 \in \mathcal{E}_R$. Therefore $\partial_L B(b) \cap B(b) \neq B(b)$ and $\partial_R B(b) \cap B(b) = B(b)$. Thus $B(b) \neq B(b)$ by Corollary 3.11. $B(b)$ is pictured in Figure 10.
Since \( g(x) \) has a horizontal asymptote at \( y = 1 \), we know that \( \tau(b) < \infty \). The spectrum of \( b \) can be read from Figure 10 as slopes of lines tangent to \( g \) and \( f \). Since \( \lambda = 0 \in \sigma(b) \setminus \sigma_p(b) \) and \( \tau(p^*_i) < \infty \), \( f(x) \) is “dotted” (or “open”).

**Example 3.18.** Suppose \( \mathcal{M} = B(\mathcal{K}) \) for some infinite-dimensional separable Hilbert Space \( \mathcal{K} \), with \( \tau \) the standard canonical trace, and let

\[
b = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\] (3.66)

\( B(b) \) is pictured in Figure 11 (the black dots in the boundary are meant to emphasize the extreme points).

From Figure 11 we can see that the eigenvalue 1 has multiplicity three by noticing that the line segment \( \Psi([p_i^+, p_i^-]) \) extends from \( x = 0 \) to \( x = 3 \). We can also see that \( b \) is trace class since the upper and lower boundary curves \( g \) and \( f \) (resp.) have asymptotes. In fact, if we write \( b = b^+ - b^- \) with \( b^+ \geq 0 \) and \( b^- \geq 0 \), we see that the asymptote for \( g \) is \( y = \tau(b^+) \) and the asymptote for \( f \) is \( y = -\tau(b^-) \). Also note that we have \( \lambda = 0 \in \mathcal{L}_b \) and \( \rho = 0 \in \mathcal{L}_p \), so by Corollary 3.11 \( B(b) \) is closed. Since the slopes of the line segments in the boundary of \( B(b) \) tend to 0 as \( x \to \infty \), we know that \( b \) is compact.
References
