Research Article

Division Problem of a Regular Form: The Case $x^2u = \lambda xv$

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Received 13 December 2010; Revised 25 February 2011; Accepted 17 March 2011

Academic Editor: Heinrich Begehr

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We present a systematic study of a regular linear functional v to find all regular forms u which satisfy the equation $x^2u = \lambda xv$, $\lambda \in \mathbb{C} - \{0\}$. We also give the second-order recurrence relation of the orthogonal polynomial sequence with respect to u and study the semiclassical character of the found families. We conclude by treating some examples.

1. Introduction

In the present paper, we intend to study the following problem: let v be a regular form (linear functional), and R and D nonzero polynomials. Find all regular forms u satisfying

$$Ru = Dv. \quad (1.1)$$

This problem has been studied in some particular cases. In fact the product of a linear form by a polynomial ($R(x) = 1$) is studied in [1–3] and the inverse problem ($D(x) = \lambda$, $\lambda \in \mathbb{C} - \{0\}$) is considered in [4–7]. More generally, when R and D have nontrivial common factor the authors of [8] found necessary and sufficient conditions for u to be a regular form. The case where $R = D$ is treated in [4, 9–11]. The aim of this contribution is to analyze the case in which $R(x) = x^2$ and $D(x) = \lambda x$, $\lambda \in \mathbb{C} - \{0\}$. We remark that R and D have a common factor and $R \neq D$ (see also [7]). In fact, the inverse problem is studied in [12]. On the other hand, this situation generalize the case treated in [13] (see (2.9)). In Section 1, we will give the regularity conditions and the coefficients of the second-order recurrence relation satisfied by the monic orthogonal polynomial sequence (MOPS) with respect to u . We will study the case where v is a symmetric form; thus regularity conditions become simpler. The particular case when v is a symmetric positive definite form is analyzed. The second section is devoted to the case where v is semi-classical form. We will prove that u is also semi-classical and some

results concerning the class of u are given. In the last section, some examples will be treated. The regular forms u found in these examples are semi-classical of class $s \in \{1, 2, 3\}$ [14]. The integral representations of these regular forms and the coefficients of the second-order recurrence satisfied by the MOPS with respect to u are given.

2. The Problem $x^2u = \lambda xv$

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and $\mathcal{P}'$ its algebraic dual. We denote by $\langle u, f \rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we designate by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$, the moments of u . For any form u , any polynomial g , any $c \in \mathbb{C}$, $a \in \mathbb{C} - \{0\}$, let u' , $h_a u$, gu , and $(x - c)^{-1}u$ be the forms defined by duality:

$$\begin{aligned} \langle u', p \rangle &:= -\langle u, p' \rangle; & \langle h_a u, p \rangle &:= \langle u, h_a p \rangle; & \langle gu, p \rangle &:= \langle u, gp \rangle; \\ \langle (x - c)^{-1}u, p \rangle &:= \langle u, \theta_c p \rangle, & p &\in \mathcal{P}, \end{aligned} \quad (2.1)$$

where $(\theta_c p)(x) = (p(x) - p(c))/(x - c)$; $(h_a p)(x) = p(ax)$.

We define a left multiplication of a form u by a polynomial p as

$$(up)(x) := \left\langle u, \frac{xp(x) - \xi p(\xi)}{x - \xi} \right\rangle, \quad u \in \mathcal{P}', \quad p \in \mathcal{P}. \quad (2.2)$$

Let us recall that a form u is called regular if there exists a monic polynomial sequence $\{P_n\}_{n \geq 0}$, $\deg P_n = n$, such that

$$\langle u, P_n P_m \rangle = r_n \delta_{n,m}, \quad n, m \geq 0, \quad r_n \neq 0, \quad n \geq 0. \quad (2.3)$$

We have the following result.

Lemma 2.1 (see [15]). *Let $u \in \mathcal{P}'$, $f \in \mathcal{P}$, and $c \in \mathbb{C}$. The following formulas hold:*

$$(vf)'(x) = (v'f)(x) + (vf')(x) + (v\theta_0 f)(x), \quad f \in \mathcal{P}. \quad (2.4)$$

$$(\delta f)(x) = f(x), \quad f \in \mathcal{P}. \quad (2.5)$$

$$(x - c)^{-1}((x - c)u) = u - (u)_0 \delta_c, \quad (2.6)$$

where $\langle \delta_c, p \rangle = p(c)$, $p \in \mathcal{P}$.

We consider the following problem: given a regular form v , find all regular forms u satisfying

$$x^2u = \lambda xv, \quad \lambda \in \mathbb{C} - \{0\}, \quad (2.7)$$

with constraints $(u)_0 = 1, (v)_0 = 1$. From (2.6) we can deduce that

$$xu = ((u)_1 - \lambda)\delta + \lambda v, \quad (2.8)$$

$$u = \delta + (\lambda - (u)_1)\delta' + \lambda x^{-1}v. \quad (2.9)$$

Then the form u depends on two arbitrary parameters $(u)_1$ and λ .

We notice that when $(u)_1 = \lambda$, we encounter the problem studied in [13] again.

We suppose that the form v has the following integral representation:

$$\langle v, f \rangle = \int_{-\infty}^{+\infty} V(x)f(x)dx, \quad \text{for each polynomial } f, \quad (2.10)$$

where V is a locally integrable function with rapid decay, continuous at the origin; then the form u is represented by

$$\langle u, f \rangle = \left(1 - \lambda P \int_{-\infty}^{+\infty} \frac{V(x)}{x} dx\right) f(0) + ((u)_1 - \lambda) f'(0) + \lambda P \int_{-\infty}^{+\infty} \frac{V(x)f(x)}{x} dx, \quad (2.11)$$

where [16, 17]

$$P \int_{-\infty}^{+\infty} \frac{V(x)}{x} dx = \lim_{\epsilon \rightarrow 0^+} \left(\int_{-\infty}^{-\epsilon} \frac{V(x)}{x} dx + \int_{\epsilon}^{+\infty} \frac{V(x)}{x} dx \right). \quad (2.12)$$

Let $\{S_n\}_{n \geq 0}$ denote the sequence of monic orthogonal polynomials with respect to v ; we have

$$\begin{aligned} S_0(x) &= 1, & S_1(x) &= x - \xi_0, \\ S_{n+2}(x) &= (x - \xi_{n+1})S_{n+1}(x) - \sigma_{n+1}S_n(x), & n &\geq 0, \end{aligned} \quad (2.13)$$

with

$$\xi_n = \frac{\langle v, xS_n^2(x) \rangle}{\langle v, S_n^2 \rangle}, \quad \sigma_{n+1} = \frac{\langle v, S_{n+1}^2 \rangle}{\langle v, S_n^2 \rangle}, \quad n \geq 0. \quad (2.14)$$

When u is regular, let $\{Z_n\}_{n \geq 0}$ be the corresponding MOPS:

$$\begin{aligned} Z_0(x) &= 1, & Z_1(x) &= x - \beta_0, \\ Z_{n+2}(x) &= (x - \beta_{n+1})Z_{n+1}(x) - \gamma_{n+1}Z_n(x), & n &\geq 0. \end{aligned} \quad (2.15)$$

From (2.7), we know that the existence of the sequence $\{Z_n\}_{n \geq 0}$ is among all the strictly quasi-orthogonal sequences of order two with respect to $\lambda x v = w$ (w is not necessarily a regular form) [15, 18–20]. That is,

$$\begin{aligned} xZ_0(x) &= S_1(x) + c_0, & xZ_1(x) &= S_2(x) + c_1S_1(x) + b_0. \\ xZ_{n+2}(x) &= S_{n+3}(x) + c_{n+2}S_{n+2}(x) + b_{n+1}S_{n+1}(x) + a_nS_n(x), \quad n \geq 0, \end{aligned} \quad (2.16)$$

with $a_n \neq 0, n \geq 0$.

From (2.16), we have

$$Z_1(x) = (\theta_0 S_2)(x) + c_1, \quad (2.17)$$

$$Z_{n+2}(x) = (\theta_0 S_{n+3})(x) + c_{n+2}(\theta_0 S_{n+2})(x) + b_{n+1}(\theta_0 S_{n+1})(x) + a_n(\theta_0 S_n)(x), \quad n \geq 0. \quad (2.18)$$

Lemma 2.2. Let $\{Z_n\}_{n \geq 0}$ be a sequence of polynomials satisfying (2.16) where a_n, b_n , and c_n are complex numbers such that $a_n \neq 0$ for all $n \geq 0$. The sequence $\{Z_n\}_{n \geq 0}$ is orthogonal with respect to u if and only if

$$\begin{aligned} \langle u, Z_n \rangle &= 0, \quad n \geq 1, \\ \langle u, xZ_n(x) \rangle &= 0, \quad n \geq 2, \quad \langle u, xZ_1(x) \rangle \neq 0. \end{aligned} \quad (2.19)$$

Proof. The conditions (2.19) are necessary from the definition of the orthogonality of $\{Z_n\}_{n \geq 0}$ with respect to u .

For $k \geq 2$, we have (by (2.7))

$$\langle u, x^k Z_{n+2}(x) \rangle = \langle x^2 u, x^{k-2} Z_{n+2}(x) \rangle = \lambda \langle v, x^{k-1} Z_{n+2}(x) \rangle, \quad n \geq 0, \quad (2.20)$$

and from (2.16), we get

$$\begin{aligned} \langle u, x^k Z_{n+2}(x) \rangle &= \lambda \langle v, x^{k-2} S_{n+3}(x) \rangle + \lambda c_{n+2} \langle v, x^{k-2} S_{n+2}(x) \rangle \\ &\quad + \lambda b_{n+1} \langle v, x^{k-2} S_{n+1}(x) \rangle + \lambda a_n \langle v, x^{k-2} S_n(x) \rangle, \quad n \geq 0. \end{aligned} \quad (2.21)$$

Taking into account the orthogonality of $\{S_n\}_{n \geq 0}$, we obtain

$$\begin{aligned} \langle u, x^k Z_{n+2}(x) \rangle &= 0, \quad 2 \leq k \leq n+1, \quad n \geq 1, \\ \langle u, x^{n+2} Z_{n+2}(x) \rangle &= \lambda a_n \langle v, S_n^2 \rangle \neq 0, \quad n \geq 0. \end{aligned} \quad (2.22)$$

By (2.19), it follows that

$$\begin{aligned}\langle u, Z_1 \rangle &= 0, & \langle u, xZ_1(x) \rangle &\neq 0, \\ \langle u, Z_{n+2} \rangle &= \langle u, xZ_{n+2}(x) \rangle = 0, & n &\geq 0.\end{aligned}\tag{2.23}$$

Consequently, the previous relations and (2.22) prove that $\{Z_n\}_{n \geq 0}$ is orthogonal with respect to u , which proves the Lemma. $\square$

Remark 2.3. When u is regular, from Theorem 5.1 in [21], there exist complex numbers $r_{n+2} \neq 0$, t_{n+2} and $v_{n+2} \neq 0$ such that

$$Z_{n+2}(x) + r_{n+2}Z_{n+1}(x) = S_{n+2}(x) + t_{n+2}S_{n+1}(x) + v_{n+2}S_n(x), \quad n \geq 0.\tag{2.24}$$

From (2.16), (2.24), and (2.15) we obtain the following relations:

$$\begin{aligned}r_{n+2} - t_{n+2} + c_{n+2} - \xi_{n+2} &= 0, & n &\geq 0, \\ r_{n+2}c_{n+1} - t_{n+2}\xi_{n+1} - v_{n+2} + b_{n+1} - \sigma_{n+2} &= 0, & n &\geq 0, \\ r_{n+2}b_n - t_{n+2}\sigma_{n+1} - v_{n+2}\xi_n + a_n &= 0, & n &\geq 0, \\ r_{n+2}a_{n-1} - v_{n+2}\sigma_n &= 0, & n &\geq 1.\end{aligned}\tag{2.25}$$

Taking into account (2.16), (2.18) and (2.19), we get

$$\begin{aligned}0 &= \langle u, xZ_{n+2}(x) \rangle \\ &= \langle u, S_{n+3} \rangle + c_{n+2}\langle u, S_{n+2} \rangle + b_{n+1}\langle u, S_{n+1} \rangle + a_n\langle u, S_n \rangle = 0, & n &\geq 0, \\ 0 &= \langle u, Z_{n+2} \rangle \\ &= \langle u, \theta_0 S_{n+3} \rangle + c_{n+2}\langle u, \theta_0 S_{n+2} \rangle + b_{n+1}\langle u, \theta_0 S_{n+1} \rangle + a_n\langle u, \theta_0 S_n \rangle, & n &\geq 0, \\ 0 &= S_{n+3}(0) + c_{n+2}S_{n+2}(0) + b_{n+1}S_{n+1}(0) + a_nS_n(0), & n &\geq 0,\end{aligned}\tag{2.26}$$

with the initial conditions:

$$\begin{aligned}0 &= S_1(0) + c_0, \\ 0 &= S_2(0) + c_1S_1(0) + b_0, \\ 0 &= \langle u, Z_1 \rangle = \langle u, (\theta_0 S_2) \rangle + c_1, \\ 0 &\neq \langle u, xZ_1(x) \rangle = \langle u, S_2 \rangle + c_1\langle u, S_1 \rangle + b_0.\end{aligned}\tag{2.27}$$

If we denote

$$\Delta_n := \begin{vmatrix} S_{n+2}(0) & S_{n+1}(0) & S_n(0) \\ \langle u, S_{n+2} \rangle & \langle u, S_{n+1} \rangle & \langle u, S_n \rangle \\ \langle u, \theta_0 S_{n+2} \rangle & \langle u, \theta_0 S_{n+1} \rangle & \langle u, \theta_0 S_n \rangle \end{vmatrix}, \quad n \geq 0, \quad (2.28)$$

from the Cramer rule we have

$$\Delta_n a_n = -\Delta_{n+1}, \quad n \geq 0, \quad (2.29)$$

$$\Delta_n b_{n+1} = \begin{vmatrix} S_{n+2}(0) & -S_{n+3}(0) & S_n(0) \\ \langle u, S_{n+2} \rangle & -\langle u, S_{n+3} \rangle & \langle u, S_n \rangle \\ \langle u, \theta_0 S_{n+2} \rangle & -\langle u, \theta_0 S_{n+3} \rangle & \langle u, \theta_0 S_n \rangle \end{vmatrix}, \quad n \geq 0, \quad (2.30)$$

$$\Delta_n c_{n+2} = \begin{vmatrix} -S_{n+3}(0) & S_{n+1}(0) & S_n(0) \\ -\langle u, S_{n+3} \rangle & \langle u, S_{n+1} \rangle & \langle u, S_n \rangle \\ -\langle u, \theta_0 S_{n+3} \rangle & \langle u, \theta_0 S_{n+1} \rangle & \langle u, \theta_0 S_n \rangle \end{vmatrix}, \quad n \geq 0. \quad (2.31)$$

Lemma 2.4. *The following formulas hold:*

$$\langle u, S_n \rangle = S_n(0) + ((u)_1 - \lambda) S'_n(0) + \lambda S_{n-1}^{(1)}(0), \quad n \geq 0, \quad (2.32)$$

$$\langle u, x S_n(x) \rangle = ((u)_1 - \lambda) S_n(0), \quad n \geq 1, \quad (2.33)$$

$$\langle u, (\theta_0 S_n) \rangle = S'_n(0) + \frac{1}{2}((u)_1 - \lambda) S''_n(0) + \lambda (S_{n-1}^{(1)})'(0), \quad n \geq 0, \quad (2.34)$$

$$S_n^{(1)}(0) S_n(0) - S_{n-1}^{(1)}(0) S_{n+1}(0) = \langle v, S_n^2 \rangle, \quad n \geq 0, \quad (2.35)$$

where $S_n^{(1)}(x) := (v \theta_0 S_{n+1})(x)$, $n \geq 0$, and $S_{-1}^{(1)}(x) = 0$.

Proof. Equations (2.32) and (2.33) are deduced, respectively, from (2.9) and (2.8).

We have

$$\begin{aligned} \langle v, \theta_0^2 S_n \rangle &= \langle v, \theta_0 S'_n - (\theta_0 S_n)' \rangle = \langle v, \theta_0 S'_n \rangle + \langle v', \theta_0 S_n \rangle \\ &= (x \theta_0 S'_n)(0) + (v' \theta_0 S_n)(0), \quad n \geq 0. \end{aligned} \quad (2.36)$$

Using (2.4), we get

$$\langle v, \theta_0^2 S_n \rangle = (v \theta_0 S_n)'(0) = (S_{n-1}^{(1)})'(0), \quad n \geq 0. \quad (2.37)$$

From (2.9), we obtain

$$\langle u, \theta_0 S_n \rangle = \langle \delta, \theta_0 S_n \rangle + ((u)_1 - \lambda) \langle \delta, (\theta_0 S_n)' \rangle + \lambda \langle v, \theta_0^2 S_n \rangle, \quad n \geq 0. \quad (2.38)$$

According to (2.5) and (2.37), we can deduce (2.34).

We have

$$\begin{aligned} S_0^{(1)}(x) &= 1, & S_1^{(1)}(x) &= x - \xi_2, \\ S_{n+2}^{(1)}(x) &= (x - \xi_{n+2}) S_{n+1}^{(1)}(x) - \sigma_{n+2} S_n^{(1)}(x), & n &\geq 0. \end{aligned} \quad (2.39)$$

Then (by (2.39))

$$\begin{aligned} S_n^{(1)}(0) S_n(0) - S_{n-1}^{(1)}(0) S_{n+1}(0) &= \sigma_n S_{n-1}^{(1)}(0) S_{n-1}(0) + S_n(0) \left(S_n^{(1)}(0) + \xi_n S_{n-1}^{(1)}(0) \right) \\ &= \sigma_n \left(S_{n-1}^{(1)}(0) S_{n-1}(0) - S_{n-2}^{(1)}(0) S_n(0) \right). \end{aligned} \quad (2.40)$$

It follows that

$$S_n^{(1)}(0) S_n(0) - S_{n-1}^{(1)}(0) S_{n+1}(0) = \prod_{\mu=0}^n \sigma_\mu = \langle v, S_n^2 \rangle, \quad n \geq 0, \quad (2.41)$$

hence (2.35). $\square$

Proposition 2.5. *One has*

$$\Delta_n = E_n \lambda^2 + F_n \lambda + G_n, \quad n \geq 0, \quad (2.42)$$

where

$$\begin{aligned} E_n &= S_{n+1}(0) \left\{ \mu_n(0) + \frac{1}{2} \chi_n'(0) \right\} + \left\{ S_n^{(1)}(0) - S_{n+1}'(0) \right\} \left\{ \chi_n(0) - \langle v, S_n^2 \rangle \right\}, \quad n \geq 0, \\ F_n &= -S_{n+1}(0) \left\{ (u)_1 (\mu_n(0) + \chi_n'(0)) + \langle v, S_n^2 \rangle \right\} \\ &\quad - (u)_1 \left\{ S_n^{(1)}(0) \chi_n(0) - 2S_{n+1}'(0) \chi_n(0) + S_{n+1}'(0) \langle v, S_n^2 \rangle \right\}, \quad n \geq 0, \\ G_n &= (u)_1^2 \left\{ \frac{1}{2} S_{n+1}(0) \chi_n'(0) - S_{n+1}'(0) \chi_n(0) \right\}, \quad n \geq 0, \end{aligned} \quad (2.43)$$

with

$$\begin{aligned} \chi_n(x) &= S_n(x) S_{n+1}'(x) - S_{n+1}(x) S_n'(x), \quad n \geq 0, \\ \mu_n(x) &= S_{n+1}(x) \left(S_{n-1}^{(1)} \right)'(x) - S_n(x) \left(S_n^{(1)} \right)'(x), \quad n \geq 0. \end{aligned} \quad (2.44)$$

Proof. Using (2.13), we, respectively, obtain

$$\begin{aligned} S_{n+2}(0) &= -\xi_{n+1}S_{n+1}(0) - \sigma_{n+1}S_n(0), \quad n \geq 0, \\ \langle u, S_{n+2} \rangle &= \langle u, xS_{n+1}(x) \rangle - \xi_{n+1}\langle u, S_{n+1} \rangle - \sigma_{n+1}\langle u, S_n \rangle, \quad n \geq 0, \\ \langle u, \theta_0 S_{n+2} \rangle &= \langle u, S_{n+1} \rangle - \xi_{n+1}\langle u, \theta_0 S_{n+1} \rangle - \sigma_{n+1}\langle u, \theta_0 S_n \rangle, \quad n \geq 0. \end{aligned} \quad (2.45)$$

Taking into account previous relations, we obtain for (2.28) the following:

$$\Delta_n = \begin{vmatrix} 0 & S_{n+1}(0) & S_n(0) \\ \langle u, xS_{n+1}(x) \rangle & \langle u, S_{n+1} \rangle & \langle u, S_n \rangle \\ \langle u, S_{n+1} \rangle & \langle u, \theta_0 S_{n+1} \rangle & \langle u, \theta_0 S_n \rangle \end{vmatrix}, \quad n \geq 0, \quad (2.46)$$

that is,

$$\Delta_n = -\langle u, xS_{n+1}(x) \rangle \begin{vmatrix} S_{n+1}(0) & S_n(0) \\ \langle u, \theta_0 S_{n+1} \rangle & \langle u, \theta_0 S_n \rangle \end{vmatrix} + \langle u, S_{n+1} \rangle \begin{vmatrix} S_{n+1}(0) & S_n(0) \\ \langle u, S_{n+1} \rangle & \langle u, S_n \rangle \end{vmatrix}, \quad n \geq 0. \quad (2.47)$$

Let $n \geq 0$; based on the relations (2.32)–(2.34), it follows that

$$\begin{aligned} \begin{vmatrix} S_{n+1}(0) & S_n(0) \\ \langle u, \theta_0 S_{n+1} \rangle & \langle u, \theta_0 S_n \rangle \end{vmatrix} &= \left\{ \mu_n(0) + \frac{1}{2}\chi'_n(0) \right\} \lambda - \chi_n(0) - \frac{1}{2}(u)_1 \chi'_n(0), \\ \begin{vmatrix} S_{n+1}(0) & S_n(0) \\ \langle u, S_{n+1} \rangle & \langle u, S_n \rangle \end{vmatrix} &= \left\{ \chi_n(0) - \langle v, S_n^2 \rangle \right\} \lambda - (u)_1 \chi_n(0). \end{aligned} \quad (2.48)$$

From (2.48) and (2.47), we obtain the desired results. $\square$

Proposition 2.6. *The form u is regular if and only if $\Delta_n \neq 0$, $n \geq 0$. Then, the coefficients of the three-term recurrence relation (2.15) are given by*

$$\gamma_1 = \Delta_0, \quad \gamma_2 = -\lambda \Delta_1 \Delta_0^{-2}, \quad (2.49)$$

$$\gamma_{n+3} = \frac{\Delta_n \Delta_{n+2}}{\Delta_{n+1}^2} \sigma_{n+1}, \quad n \geq 0, \quad (2.50)$$

$$\beta_0 = (u)_1, \quad \beta_1 = c_1 - \xi_0 - \xi_1 + \lambda b_0 \Delta_0^{-1}, \quad (2.51)$$

$$\beta_{n+2} = c_{n+2} - \xi_{n+1} - \xi_{n+2} - b_{n+1} \Delta_n \Delta_{n+1}^{-1} \sigma_{n+1}, \quad n \geq 0. \quad (2.52)$$

Proof

Necessity. From (2.27) and Lemma 2.4, we get

$$\langle u, xZ_1(x) \rangle = \langle u, S_2 \rangle + \langle u, \theta_0 S_2 \rangle (S_1(0) - \langle u, S_1 \rangle) - S_2(0) = \lambda S_1(0) - (u)_1^2, \quad (2.53)$$

and again with (2.27) and (2.42), we can deduce that

$$\Delta_0 = \langle u, S_2 \rangle + \langle u, \theta_0 S_2 \rangle (S_1(0) - \langle u, S_1 \rangle) - S_2(0) = \langle u, xZ_1(x) \rangle \neq 0. \quad (2.54)$$

Moreover, $\{Z_n\}_{n \geq 0}$ is orthogonal with respect to u , therefore it is strictly quasiorthogonal of order two with respect to xv , and then it satisfies (2.16) with $a_n \neq 0$, $n \geq 0$. This implies $\Delta_n \neq 0$, $n \geq 0$. Otherwise, if there exists an $n_0 \geq 1$ such that $\Delta_{n_0} = 0$, from (2.29), $\Delta_0 = 0$, which is a contradiction.

Sufficiency. Let

$$c_0 = -S_1(0) = \xi_0, \quad (2.55)$$

$$c_1 = -\langle u, (\theta_0 S_2) \rangle, \quad (2.56)$$

$$b_0 = \Delta_0 - \langle u, S_2 \rangle - c_1 \langle u, S_1 \rangle. \quad (2.57)$$

We get

$$\langle u, xZ_1(x) \rangle = \langle u, S_2 \rangle + c_1 \langle u, S_1 \rangle + b_0 = \Delta_0 \neq 0. \quad (2.58)$$

We have $\langle u, Z_1 \rangle = c_1 + \langle u, \theta_0 S_2 \rangle = 0$.

From (2.56) and (2.57) we get

$$S_2(0) + c_1 S_1(0) + b_0 = S_2(0) - \langle u, S_2 \rangle - \langle u, \theta_0 S_2 \rangle (S_1(0) - \langle u, S_1 \rangle) + \Delta_0. \quad (2.59)$$

On account of (2.54), we can deduce that $S_2(0) + c_1 S_1(0) + b_0 = 0$.

Then we had just proved that the initial conditions (2.27) are satisfied.

Furthermore, the system (2.26) is a Cramer system whose solution is given by (2.29), (2.30), and (2.31); with all these numbers a_n , b_n , and c_n ($n \geq 0$), define a sequence polynomials $\{Z_n\}_{n \geq 0}$ by (2.16). Then it follows from (2.26) and Lemma 2.2 that u is regular and $\{Z_n\}_{n \geq 0}$ is the corresponding MOPS.

Moreover, by (2.22) we get

$$\langle u, Z_{n+2}^2 \rangle = \lambda a_n \langle v, S_n^2 \rangle, \quad n \geq 0. \quad (2.60)$$

Making $n = 0$ in (2.60), it follows that

$$\langle u, Z_2^2 \rangle = \lambda a_0. \quad (2.61)$$

Based on relations (2.58), (2.60), (2.61), and (2.29), we, respectively, obtain

$$\begin{aligned}\gamma_1 &= \langle u, xZ_1(x) \rangle = \Delta_0; & \gamma_2 &= \frac{\langle u, Z_2^2 \rangle}{\langle u, xZ_1(x) \rangle} = -\lambda \Delta_1 \Delta_0^{-2}, \\ \gamma_{n+3} &= \frac{\langle u, Z_{n+3}^2 \rangle}{\langle u, Z_{n+2}^2 \rangle} = \frac{\Delta_n \Delta_{n+2}}{\Delta_{n+1}^2} \sigma_{n+1}, & n &\geq 0.\end{aligned}\tag{2.62}$$

We have proved (2.49) and (2.50).

When $\{Z_n\}_{n \geq 0}$ is orthogonal, we have

$$\beta_0 = (u)_1.\tag{2.63}$$

By (2.16) and the orthogonality of $\{Z_n\}_{n \geq 0}$, we get

$$\langle u, xZ_1^2(x) \rangle = c_1 \langle u, Z_1^2 \rangle + \langle u, S_2 Z_1 \rangle.\tag{2.64}$$

By virtue of (2.13) and the regularity of u we obtain

$$\begin{aligned}\langle u, S_2 Z_1 \rangle &= \langle x^2 u, Z_1 \rangle - (\xi_0 + \xi_1) \langle u, Z_1^2 \rangle = \lambda \langle v, xZ_1(x) \rangle - (\xi_0 + \xi_1) \langle u, Z_1^2 \rangle \\ &= \lambda b_0 - (\xi_0 + \xi_1) \langle u, Z_1^2 \rangle,\end{aligned}\tag{2.65}$$

and consequently, we get the second result in (2.51) from (2.58), and (2.64).

From (2.16), and the orthogonality of $\{Z_n\}_{n \geq 0}$, we have

$$\beta_{n+2} \langle u, Z_{n+2}^2 \rangle = c_{n+2} \langle u, Z_{n+2}^2 \rangle + \langle u, S_{n+3} Z_{n+2} \rangle, \quad n \geq 0.\tag{2.66}$$

Using (2.13), (2.16), and the the orthogonality of $\{S_n\}_{n \geq 0}$, we have

$$\langle u, S_{n+3} Z_{n+2} \rangle = \lambda b_{n+1} \langle v, S_{n+1}^2 \rangle - (\xi_{n+1} + \xi_{n+2}) \langle u, Z_{n+2}^2 \rangle, \quad n \geq 0.\tag{2.67}$$

Taking into account the previous relation, (2.66) becomes

$$\beta_{n+2} = c_{n+2} - \xi_{n+1} - \xi_{n+2} + \lambda b_{n+1} \frac{\langle v, S_{n+1}^2 \rangle}{\langle u, Z_{n+2}^2 \rangle}, \quad n \geq 0.\tag{2.68}$$

From (2.60) and (2.29), we have

$$\frac{\langle v, S_{n+1}^2 \rangle}{\langle u, Z_{n+2}^2 \rangle} = -\lambda^{-1} \Delta_n \Delta_{n+1}^{-1} \sigma_{n+1}, \quad n \geq 0.\tag{2.69}$$

Last equation and (2.68) give (2.52).

Moreover, if the form u is regular, for (2.29), (2.30), and (2.31), we get

$$a_n = -\frac{\Delta_{n+1}}{\Delta_n}, \quad n \geq 0, \quad (2.70)$$

$$b_{n+1} = (D_n \lambda^2 + H_n \lambda + I_n) \Delta_n^{-1} + \sigma_{n+2}, \quad n \geq 0, \quad (2.71)$$

$$c_{n+2} = -(J_n \lambda^2 + L_n \lambda + K_n) \Delta_n^{-1} + \xi_{n+2}, \quad n \geq 0, \quad (2.72)$$

where

$$\begin{aligned} D_n &= S_n(0) \left(\langle v, S_{n+1}^2 \rangle - \chi_{n+1}(0) \right) - \xi_{n+1} S_{n+2}(0) \left(\mu_n(0) + \frac{1}{2} \chi'_n(0) \right) \\ &\quad - \xi_{n+1} \left(S_{n+1}^{(1)}(0) - S'_{n+2}(0) \right) \left(\chi_n(0) - \langle v, S_n^2 \rangle \right), \quad n \geq 0, \\ H_n &= (u)_1 S_n(0) \left(2\chi_{n+1}(0) - \langle v, S_{n+1}^2 \rangle \right) + \xi_{n+1} S_{n+2}(0) (\chi_n(0)) \\ &\quad + (u)_1 (\chi'_n(0) + \mu_n(0)) + (u)_1 \xi_{n+1} \chi_n(0) \left(S_{n+1}^{(1)}(0) - S'_{n+2}(0) \right) \\ &\quad + \xi_{n+1} \left(\langle v, S_n^2 \rangle - \chi_n(0) \right) (S_{n+2}(0) + (u)_1 S'_{n+2}(0)), \quad n \geq 0, \\ I_n &= -(u)_1^2 \left\{ S_n(0) \chi_{n+1}(0) + \frac{1}{2} \xi_{n+1} (S_{n+2}(0) \chi'_n(0) - S'_{n+2}(0) \chi_n(0)) \right\}, \quad n \geq 0, \\ J_n &= S_{n+2}(0) \left(\mu_n(0) + \frac{1}{2} \chi'_n(0) \right) + (S_{n+1}^{(1)}(0) - S'_{n+2}(0)) \left(\chi_n(0) - \langle v, S_n^2 \rangle \right), \quad n \geq 0, \\ L_n &= (u)_1 \chi_n(0) \left(2S'_{n+2}(0) - S_{n+1}^{(1)}(0) \right) - (u)_1 S_{n+2}(0) (\mu_n(0) + \chi'_n(0)) \\ &\quad - \langle v, S_n^2 \rangle (S_{n+2}(0) + (u)_1 S'_{n+2}(0)), \quad n \geq 0, \\ K_n &= (u)_1^2 \left\{ \frac{1}{2} S_{n+2}(0) \chi'_n(0) - \chi_n(0) S'_{n+2}(0) \right\}, \quad n \geq 0. \end{aligned} \quad (2.73)$$

In the sequel, we will assume that v is a symmetric linear form.

We need the following lemmas.

Lemma 2.7. *If $\{y_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ are sequences of complex numbers fulfilling*

$$\begin{aligned} y_{n+1} + a_n y_n &= b_{n+1}, \quad n \geq 0, \quad a_n \neq 0, \quad n \geq 0, \\ y_0 &= b_0, \end{aligned} \quad (2.74)$$

then

$$y_n = (-1)^n a_n^{-1} \left(\prod_{\mu=0}^n a_\mu \right) \sum_{v=0}^n (-1)^v a_v \left(\prod_{\mu=0}^v a_\mu^{-1} \right) b_v, \quad n \geq 0. \quad (2.75)$$

Lemma 2.8. When $\{S_n\}_{n \geq 0}$ given by (2.13) is symmetric, one has

$$\begin{aligned}
 S_{2n}(0) &= \frac{(-1)^n}{\sigma_{2n+1}} \prod_{\mu=0}^n \sigma_{2\mu+1}, \quad n \geq 0, & S_{2n+1}(0) &= 0, \quad n \geq 0, \\
 S_{2n}^{(1)}(0) &= (-1)^n \prod_{\mu=0}^n \sigma_{2\mu}, \quad n \geq 0, & S_{2n+1}^{(1)}(0) &= 0, \quad n \geq 0, \\
 S'_{2n+1}(0) &= (-1)^n \left(\prod_{\mu=0}^n \sigma_{2\mu} \right) \Lambda_n, \quad n \geq 0, & S'_{2n}(0) &= 0, \quad n \geq 0, \\
 \left(S_{2n}^{(1)} \right)'(0) &= 0, \quad n \geq 0, & S''_{2n+1}(0) &= 0, \quad n \geq 0.
 \end{aligned} \tag{2.76}$$

Proof. As v is symmetric, then $\xi_n = 0$, $n \geq 0$, and therefore from (2.13) we have

$$\begin{aligned}
 S_0(0) &= 1, & S_1(0) &= 0, & S_0^{(1)}(0) &= 1, & S_1^{(1)}(0) &= 0, \\
 S_{n+2}(0) &= -\sigma_{n+1} S_n(0), \quad n \geq 0, & S_{n+2}^{(1)}(0) &= -\sigma_{n+2} S_n^{(1)}(0), \quad n \geq 0, \\
 S'_0(0) &= 0, & S'_1(0) &= 1, & S'_{n+2}(0) &= -\sigma_{n+1} S'_n(0) + S_{n+1}(0), \quad n \geq 0, \\
 \left(S_0^{(1)} \right)'(0) &= 0, & \left(S_{n+2}^{(1)} \right)'(0) &= -\sigma_{n+2} \left(S_n^{(1)} \right)'(0) + S_{n+1}^{(1)}(0), \quad n \geq 0, \\
 S''_0(0) &= 0, & S''_1(0) &= 0, & S''_{n+2}(0) &= -\sigma_{n+1} S''_n(0) + 2S'_{n+1}(0), \quad n \geq 0.
 \end{aligned} \tag{2.77}$$

Now, it is sufficient to use Lemma 2.7 in order to obtain the desired results. $\square$

Let

$$\omega = \lambda^{-1}(u)_1. \tag{2.78}$$

Corollary 2.9. If v is a symmetric form, one has

$$\begin{aligned}
 \Delta_{2n} &= \lambda^2 \frac{(-1)^{n+1}}{\sigma_{2n+1}} \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right) \left(\prod_{\mu=0}^n \sigma_{2\mu} \right)^2 \{(\omega - 1)\Lambda_n + 1\}^2, \quad n \geq 0, \\
 \Delta_{2n+1} &= \lambda(-1)^n \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right)^2 \left(\prod_{\mu=0}^n \sigma_{2\mu} \right), \quad n \geq 0,
 \end{aligned} \tag{2.79}$$

where

$$\Lambda_n = \sum_{v=0}^n \frac{1}{\sigma_{2v+1}} \prod_{\mu=0}^v \frac{\sigma_{2\mu+1}}{\sigma_{2\mu}}, \quad n \geq 0, \quad \sigma_0 = 1. \tag{2.80}$$

Proof. Following Lemma 2.8, for (2.43) we have

$$\begin{aligned}
 E_{2n} &= \frac{(-1)^{n+1}}{\sigma_{2n+1}} \left(\prod_{\mu=0}^n \sigma_{2\mu} \right)^2 \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right) (1 - \Lambda_n), \quad n \geq 0; \quad E_{2n+1} = 0, \quad n \geq 0, \\
 F_{2n} &= 2\omega\lambda \frac{(-1)^{n+1}}{\sigma_{2n+1}} \left(\prod_{\mu=0}^n \sigma_{2\mu} \right)^2 \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right) (1 - \Lambda_n) \Lambda_{n+1}, \quad n \geq 0, \\
 F_{2n+1} &= (-1)^n \left(\prod_{\mu=0}^n \sigma_{2\mu} \right) \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right)^2, \quad n \geq 0, \\
 G_{2n} &= \omega^2 \lambda^2 \frac{(-1)^{n+1}}{\sigma_{2n+1}} \left(\prod_{\mu=0}^n \sigma_{2\mu} \right)^2 \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right) \Lambda_n^2, \quad n \geq 0; \quad G_{2n+1} = 0, \quad n \geq 0.
 \end{aligned} \tag{2.81}$$

As a consequence, relations (2.81) and (2.42) yield (2.79). $\square$

Theorem 2.10. *The form u is regular if and only if $(\omega - 1)\Lambda_n + 1 \neq 0$, $n \geq 0$, where Λ_n is defined in (2.80).*

In this case one has

$$a_{2n} = \frac{\sigma_{2n+1}}{\lambda \Theta_n ((\omega - 1)\Lambda_n + 1)^2}, \quad a_{2n+1} = -\lambda \sigma_{2n+2}^2 \Theta_n ((\omega - 1)\Lambda_n + 1)^2, \quad n \geq 0, \tag{2.82}$$

$$b_{2n} = \sigma_{2n+1}, \quad n \geq 0, \quad b_{2n+1} = \sigma_{2n+2} \frac{(\omega - 1)\Lambda_{n+1} + 1}{(\omega - 1)\Lambda_n + 1}, \quad n \geq 0, \tag{2.83}$$

$$c_0 = 0, \quad c_1 = -\omega\lambda, \quad c_{2n+2} = \frac{1}{\lambda \Theta_n ((\omega - 1)\Lambda_n + 1)^2}, \quad n \geq 0,$$

$$c_{2n+3} = -\lambda \sigma_{2n+2} \Theta_n ((\omega - 1)\Lambda_{n+1} + 1) ((\omega - 1)\Lambda_n + 1), \quad n \geq 0,$$

$$\gamma_1 = -\lambda^2 \omega^2, \quad \gamma_2 = -\frac{\sigma_1^2}{\lambda^2 \omega^4}, \quad \gamma_{2n+4} = \frac{1}{\lambda^2 \Theta_{n+1}^2} ((\omega - 1)\Lambda_{n+1} + 1)^2, \quad n \geq 0, \tag{2.84}$$

$$\gamma_{2n+3} = \lambda^2 \sigma_{2n+2}^2 \Theta_n^2 ((\omega - 1)\Lambda_n + 1)^2 ((\omega - 1)\Lambda_{n+1} + 1)^2, \quad n \geq 0,$$

$$\beta_0 = \lambda\omega, \quad \beta_1 = -\lambda\omega - \frac{\sigma_1}{\lambda\omega^2},$$

$$\beta_{2n+2} = \frac{1}{\lambda \Theta_n ((\omega - 1)\Lambda_n + 1)^2} + \lambda \sigma_{2n+2} \Theta_n ((\omega - 1)\Lambda_n + 1) ((\omega - 1)\Lambda_{n+1} + 1),$$

$$\beta_{2n+3} = \frac{1}{\lambda \Theta_{n+1} ((\omega - 1)\Lambda_{n+1} + 1)^2} - \lambda \sigma_{2n+2} \Theta_n ((\omega - 1)\Lambda_n + 1) ((\omega - 1)\Lambda_{n+1} + 1), \quad n \geq 0, \tag{2.85}$$

where $\Theta_n = \prod_{\mu=0}^n \sigma_{2\mu} / \sigma_{2\mu+1}$, $n \geq 0$.

Proof. From Proposition 2.6 and Corollary 2.9, we can deduce that u is regular if and only if $(\omega - 1)\Lambda_n + 1 \neq 0$, $n \geq 0$.

Moreover, from (2.70) we can deduce (2.82).

By (2.49), (2.51), (2.78), and (2.79), for (2.55), (2.56), and (2.57) we get

$$\begin{aligned} c_0 &= 0, & c_1 &= -(u)_1 = -\omega\lambda, \\ b_0 &= \sigma_1. \end{aligned} \tag{2.86}$$

When $n \geq 0$ by Lemma 2.8, for (2.73) we get

$$\begin{aligned} D_{2n} &= \frac{(-1)^n}{\sigma_{2n+1}} \left(\prod_{\mu=0}^n \sigma_{2\mu} \right) \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right)^2 (1 - \Lambda_n); & D_{2n+1} &= 0, \\ H_{2n} &= \omega\lambda \frac{(-1)^n}{\sigma_{2n+1}} \left(\prod_{\mu=0}^n \sigma_{2\mu} \right) \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right)^2 (2\Lambda_n - 1); & H_{2n+1} &= 0, \\ I_{2n} &= \omega^2 \lambda^2 \frac{(-1)^{n+1}}{\sigma_{2n+1}} \left(\prod_{\mu=0}^n \sigma_{2\mu} \right) \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right)^2 \Lambda_n; & I_{2n+1} &= 0, \\ J_{2n} &= 0; & J_{2n+1} &= (-1)^n \sigma_{2n+2} \left(\prod_{\mu=0}^n \sigma_{2\mu} \right)^2 \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right) (1 - \Lambda_n)(1 - \Lambda_{n+1}), \\ L_{2n} &= \frac{(-1)^{n+1}}{\sigma_{2n+1}} \left(\prod_{\mu=0}^n \sigma_{2\mu} \right) \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right)^2, \\ L_{2n+1} &= \omega\lambda (-1)^n \sigma_{2n+2} \left(\prod_{\mu=0}^n \sigma_{2\mu} \right)^2 \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right) (\Lambda_{n+1} + (1 - 2\Lambda_{n+1})\Lambda_n), \\ K_{2n} &= 0, \quad n \geq 0; & K_{2n+1} &= \omega^2 \lambda^2 (-1)^n \sigma_{2n+2} \left(\prod_{\mu=0}^n \sigma_{2\mu} \right)^2 \left(\prod_{\mu=0}^n \sigma_{2\mu+1} \right) \Lambda_n \Lambda_{n+1}. \end{aligned} \tag{2.87}$$

Taking into account (2.79), (2.80), and (2.86)–(2.87), relations (2.70), (2.71) and (2.72) give (2.82)–(2.84).

As a result of relations (2.82)–(2.84) and Proposition 2.6 we get (2.85). $\square$

Corollary 2.11. (1) *If v is a symmetric positive definite form, then the form u is regular when $\omega \in \mathbb{C} -] - \infty, 1[$.*

(2) When u is regular, it is positive definite form if and only if

$$\lambda\omega^2 < 0, \quad \frac{\sigma_1^2}{\omega^2} > 0, \quad \frac{1}{\lambda^2\Theta_{n+1}^2}((\omega-1)\Lambda_{n+1}+1)^2, \quad n > 0, \quad (2.88)$$

$$\lambda^2\sigma_{2n+2}^2\Theta_n^2((\omega-1)\Lambda_n+1)^2((\omega-1)\Lambda_{n+1}+1)^2, \quad n > 0.$$

Proof. (1) If v is positive definite, then $\sigma_{n+1} > 0$, $n \geq 0$, therefore $\Lambda_n > 0$, $n \geq 0$ and so $(\omega-1)\Lambda_n+1 \neq 0$, $n \geq 0$ under the hypothesis of the corollary.

(2) If u is regular, it is positive definite if and only if $\gamma_{n+1} > 0$, $n \geq 0$. By Theorem 2.10, we conclude the desired results. $\square$

3. Some Results on the Semiclassical Case

Let us recall that a form v is called semiclassical when it is regular and its formal Stieltjes function $S(\cdot; v)$ satisfies [15]

$$\phi(z)S'(z; v) = C(z)S(z; v) + D(z), \quad (3.1)$$

where ϕ monic, C , and D are polynomials with

$$D(z) = -(v\theta_0\phi)'(z) + (v\theta_0C)(z),$$

$$S(z; v) = -\sum_{n \geq 0} \frac{(v)_n}{z^{n+1}}. \quad (3.2)$$

The class of the semi-classical form v is $s = \max(\deg \phi - 2, \deg C - 1)$ if and only if the following condition is satisfied [22]:

$$\prod_c (|C(c)| + |D(c)|) > 0, \quad (3.3)$$

where $c \in \{x : \phi(x) = 0\}$, that is, ϕ , C , and D are coprime.

In the sequel, we will suppose that the form v is semi-classical of class s satisfying (3.1).

Proposition 3.1. When u is regular, it is also semi-classical and satisfies

$$\tilde{\phi}(z)S'(z; u) = \tilde{C}(z)S(z; u) + \tilde{D}(z), \quad (3.4)$$

where

$$\tilde{\phi}(z) = z^3\phi(z), \quad \tilde{C}(z) = z^3C(z) - z^2\phi(z),$$

$$\tilde{D}(z) = z(z + (u)_1 - \lambda)C(z) + \lambda z^2D(z) + ((u)_1 - \lambda)\phi(z). \quad (3.5)$$

Moreover, the class of u depends on the zero $x = 0$ of ϕ .

Proof. We need the following formula:

$$S(z; fw) = fS(z; w) + (w\theta_0 f)(z), \quad w \in \mathcal{P}', \quad f \in \mathcal{P}. \quad (3.6)$$

From (2.7), we have $S(z; x^2u) = \lambda S(z; xv)$. Using (3.6), we get

$$z^2 S(z; u) + z + (u)_1 = \lambda z S(z; v) + \lambda. \quad (3.7)$$

Differentiating the previous equation, we obtain

$$z^2 S'(z; u) + 2z S(z; u) + 1 = \lambda z S'(z; v) + \lambda S(z; v). \quad (3.8)$$

By (3.1) we can deduce (3.4) and (3.5).

Since v is a semi-classical, $S(z; v)$ satisfies (3.1) where ϕ , C and D are coprime.

Let c be a zero of $\tilde{\phi}$ different from 0, which implies that $\phi(c) = 0$. We know that $|C(c)| + |D(c)| \neq 0$.

If $C(c) \neq 0$, then $\tilde{C}(c) \neq 0$. if $C(c) = 0$, then $\tilde{D}(c) = \lambda c^2 D(c) \neq 0$. Hence $|\tilde{C}(c)| + |\tilde{D}(c)| \neq 0$. $\square$

Corollary 3.2. *Introducing*

$$\begin{aligned} \vartheta_1 &:= ((u)_1 - \lambda)\phi(0), & \vartheta_2 &:= ((u)_1 - \lambda)(C(0) + \phi'(0)), \\ \vartheta_3 &:= C(0) + ((u)_1 - \lambda)(C'(0) + \phi''(0)) + \lambda D(0), \end{aligned} \quad (3.9)$$

- (1) if $\vartheta_1 \neq 0$, then $\tilde{s} = s + 3$;
- (2) if $\vartheta_1 = 0$ and $\vartheta_2 \neq 0$, then $\tilde{s} = s + 2$;
- (3) if $\vartheta_1 = \vartheta_2 = 0$ and $\phi(0) \neq 0$ or $\vartheta_3 \neq 0$, then $\tilde{s} = s + 1$.

Proof. (1) From (3.9) and (3.5), we obtain $\tilde{C}(0) = 0$, $\tilde{D}(0) = \vartheta_1 \neq 0$. Therefore, it is not possible to simplify, which means that the class of u is $s + 3$.

(2) If $\vartheta_1 = 0$, then from (3.5) we have $\tilde{C}(0) = \tilde{D}(0) = 0$. Consequently, (3.4)–(3.6) is divisible by z . Thus, u fulfils (3.4) with

$$\begin{aligned} \tilde{\phi}(z) &= z^2 \phi(z), & \tilde{C}(z) &= z^2 C(z) - z\phi(z), \\ \tilde{D}(z) &= (z + (u)_1 - \lambda)C(z) + \lambda z D(z) + ((u)_1 - \lambda)\theta_0 \phi(z). \end{aligned} \quad (3.10)$$

If $\tilde{D}(0) = \vartheta_2 \neq 0$, it is not possible to simplify, which means that the class of u is $s + 2$.

(3) When $\vartheta_1 = \vartheta_2 = 0$, then it is possible to simplify (3.4)–(3.10) by z . Thus, u fulfils (3.4) with

$$\begin{aligned} \tilde{\phi}(z) &= z\phi(z), & \tilde{C}(z) &= zC(z) - \phi(z), \\ \tilde{D}(z) &= ((u)_1 - \lambda)(\theta_0 C(z) + \theta_0^2 \phi(z)) + \lambda D(z) + C(z). \end{aligned} \quad (3.11)$$

Since we have $\tilde{C}(0) = -\phi(0)$, $\tilde{D}(0) = \vartheta_3$, then we can deduce that if $\phi(0) \neq 0$ or $\vartheta_3 \neq 0$, it is not possible to simplify, which means that the class of u is $s + 1$. $\square$

4. Some Examples

In the sequel the examples treated generalize some of the cases studied in [13].

4.1. v the Generalized Hermite Form

Let us describe the case $v := \mathcal{H}(\tau)$, where $\mathcal{H}(\tau)$ is the generalized Hermite form. Here is [1]

$$\xi_n = 0, \quad n \geq 0, \quad \sigma_{n+1} = \frac{1}{2}(n+1 + \tau(1 + (-1)^n)), \quad n \geq 0. \quad (4.1)$$

From (4.1), we get

$$\prod_{\mu=0}^n \sigma_{2\mu+1} = \frac{\Gamma(n + \tau + 3/2)}{\Gamma(\tau + 1/2)}, \quad n \geq 0, \quad \prod_{\mu=0}^n \sigma_{2\mu} = \Gamma(n + 1), \quad n \geq 0. \quad (4.2)$$

We want $\Lambda_n = \sum_{v=0}^n 1/\sigma_{2v+1} \prod_{\mu=0}^v \sigma_{2\mu+1} / \sigma_{2\mu}$, $n \geq 0$.

But from (4.1) and (4.2), we have $1/\sigma_{2v+1} \prod_{\mu=0}^v \sigma_{2\mu+1} / \sigma_{2\mu} = (1/\Gamma(\tau + 1/2))h_v$, with

$$h_n = \frac{\Gamma(n + \tau + 1/2)}{\Gamma(n + 1)}, \quad n \geq 0, \quad (4.3)$$

fulfilling

$$(n+1)h_{n+1} - nh_n = \left(\tau + \frac{1}{2}\right)h_n, \quad n \geq 0, \quad (4.4)$$

and so

$$\Lambda_n = \frac{1}{(\tau + 1/2)\Gamma(\tau + 1/2)} \sum_{v=0}^n (v+1)h_{v+1} - v h_v = \frac{1}{\Gamma(\tau + 3/2)} \frac{\Gamma(n + \tau + 3/2)}{\Gamma(n + 1)}, \quad n \geq 0. \quad (4.5)$$

Then we get Table 1.

Proposition 4.1. *If $v = \mathcal{H}(\tau)$ is the generalized Hermite form, then the form $u(\tau, \omega, \lambda)$ given by (2.9) has the following integral representation:*

$$\langle u(\tau, \omega, \lambda), f \rangle = f(0) + \lambda(\omega - 1)f'(0) + \frac{\lambda}{\Gamma(\tau + 1/2)} P \int_{-\infty}^{+\infty} \frac{|x|^{2\tau}}{x} e^{-x^2} f(x) dx, \quad \forall f \in \mathcal{P}. \quad (4.6)$$

Table 1

Δ_n	$\Delta_{2n} = (-1)^{n+1} \frac{\lambda^2}{\Gamma(\tau+1/2)} \Gamma(n+\tau+1/2) \Gamma^2(n+1) ((\omega-1)\Lambda_n+1)^2, \quad n \geq 0,$ $\Delta_{2n+1} = (-1)^n \frac{\lambda}{\Gamma^2(\tau+1/2)} \Gamma^2(n+\tau+3/2) \Gamma(n+1), \quad n \geq 0.$
a_n	$a_{2n} = \frac{(n+\tau+1/2)^2}{\lambda \Gamma(\tau+1/2)} \frac{h_n}{((\omega-1)\Lambda_n+1)^2}, \quad n \geq 0, \quad a_{2n+1} = -\lambda \Gamma(\tau+1/2) \frac{n+1}{h_{n+1}} ((\omega-1)\Lambda_n+1)^2, \quad n \geq 0.$
b_n	$b_{2n} = n+\tau+1/2, \quad n \geq 0, \quad b_{2n+1} = (n+1) \frac{(\omega-1)\Lambda_{n+1}+1}{(\omega-1)\Lambda_n+1}, \quad n \geq 0.$
c_n	$c_0 = 0, \quad c_1 = -\omega\lambda, \quad c_{2n+2} = \frac{1}{\lambda} \frac{n+\tau+1/2}{\Gamma(\tau+1/2)} \frac{h_n}{((\omega-1)\Lambda_n+1)^2}, \quad n \geq 0,$ $c_{2n+3} = -\lambda \frac{(n+1)\Gamma(\tau+1/2)}{(n+\tau+1/2)h_n} ((\omega-1)\Lambda_{n+1}+1)((\omega-1)\Lambda_n+1), \quad n \geq 0.$
γ_1	$\gamma_1 = -\lambda^2\omega^2, \quad \gamma_2 = -\frac{(\tau+1/2)^2}{\lambda^2\omega^4},$
γ_{n+1}	$\gamma_{2n+3} = -\frac{\lambda^2\Gamma^2(\tau+1/2)}{h_{n+1}^2} ((\omega-1)\Lambda_{n+1}+1)^2 ((\omega-1)\Lambda_n+1)^2, \quad n \geq 0,$ $\gamma_{2n+4} = -\frac{1}{\lambda^2\Gamma^2(\tau+1/2)} \frac{(n+\tau+3/2)^2 h_{n+1}^2}{((\omega-1)\Lambda_{n+1}+1)^4}, \quad n \geq 0.$
β_0	$\beta_0 = \omega\lambda, \quad \beta_1 = -\omega\lambda - \frac{\tau+1/2}{\lambda\omega^2},$
β_n	$\beta_{2n+3} = -\frac{\lambda(n+1)\Gamma(\tau+1/2)}{(n+\tau+1/2)h_n} ((\omega-1)\Lambda_{n+1}+1)((\omega-1)\Lambda_n+1) - \frac{n+\tau+3/2}{\lambda\Gamma(\tau+1/2)} \frac{h_{n+1}}{((\omega-1)\Lambda_{n+1}+1)^2}, \quad n \geq 0,$ $\beta_{2n+2} = \frac{1}{\lambda} \frac{1}{\Gamma(\tau+1/2)} \frac{(n+\tau+1/2)h_n}{((\omega-1)\Lambda_n+1)^2} + \frac{\lambda\Gamma(\tau+1/2)}{h_{n+1}} ((\omega-1)\Lambda_{n+1}+1)((\omega-1)\Lambda_n+1), \quad n \geq 0.$

It is a quasi-antisymmetric $((u(\tau, \omega, \lambda))_{2n+2} = 0, n \geq 0)$ and semi-classical form of class s satisfying the following functional equation:

$$\tau = 0, \quad \omega \neq 1, \quad z^3 S'(z; u(0, \omega, \lambda)) = -z^2 (2z^2 + 1) S(z; u(0, \omega, \lambda)) - 2z^3 - 2\lambda\omega z^2 + \lambda(\omega - 1), \quad s = 3, \quad (4.7)$$

$$\tau = 0, \quad \omega = 1, \quad z S'(z; u(0, 1, \lambda)) = -(2z^2 + 1) S(z; u(0, 1, \lambda)) - 2z - 2\lambda, \quad s = 1,$$

$$\tau \neq 0, \quad \omega \neq 1, \quad z^3 S'(z; u(\tau, \omega, \lambda)) = -z^2 (2z^2 - 2\tau + 1) S(z; u(\tau, \omega, \lambda)) - 2z^3 - 2\lambda\omega z^2 + 2\tau z + 2\tau\lambda(\omega - 1) + \lambda(\omega - 1), \quad s = 3,$$

$$\tau \neq 0, \quad \omega = 1, \quad z^2 S'(z; u(\tau, 1, \lambda)) = z(-2z^2 + 2\tau - 1) S(z; u(\tau, 1, \lambda)) - 2z^2 - 2\lambda z + 2\tau, \quad s = 2. \quad (4.8)$$

Proof. It is well known that the generalized Hermite form possesses the following integral representation [1]:

$$\langle v, f \rangle = \int_{-\infty}^{+\infty} \frac{1}{\Gamma(\tau + 1/2)} |x|^{2\tau} e^{-x^2} f(x) dx, \quad \Re(\tau) > -\frac{1}{2}, \quad \forall f \in \mathcal{P}. \quad (4.9)$$

Following (2.11), we obtain (4.6). Also the form u is quasi-antisymmetric because it satisfies

$$\langle u, x^{2n+2} \rangle = \lambda \langle v, x^{2n+1} \rangle = 0, \quad n \geq 0, \quad (4.10)$$

since v is symmetric by hypothesis.

When $\tau = 0$, v is classical and satisfies (3.4) with [22]

$$\phi(x) = 1, \quad C(z) = -2z, \quad D(z) = -2. \quad (4.11)$$

Then, $\vartheta_1 = \lambda(\omega - 1)$, $\vartheta_2 = 0$.

Now, it is sufficient to use Corollary 3.2 and Proposition 3.1 in order to obtain (4.7).

If $\tau \neq 0$, the form v is semi-classical of class one and satisfies (3.4) with [23]

$$\phi(x) = x, \quad C(z) = -2z^2 + 2\tau, \quad D(z) = -2z. \quad (4.12)$$

Therefore $\vartheta_1 = 0$, $\vartheta_2 = \lambda(\omega - 1)(2\tau + 1)$, $\vartheta_3 = 2\tau$.

By Proposition 3.1 and Corollary 3.2 we can deduce (4.8). $\square$

4.2. v the Corecursive of the Second Kind Chebychev Form

Let us describe the case $v := \mathcal{J}_{(-1/2, 1/2)}$; it is the corecursive of the second kind Chebychev functional. Here is [1]

$$\xi_0 = -\frac{1}{2}, \quad \xi_{n+1} = 0, \quad n \geq 0, \quad \sigma_{n+1} = \frac{1}{4}, \quad n \geq 0. \quad (4.13)$$

In this case we have the following result.

Lemma 4.2. For $n \geq 0$, one has

$$\begin{aligned} S_{2n}(0) &= \frac{(-1)^n}{2^{2n}}, & S_{2n+1}(0) &= \frac{(-1)^n}{2^{2n+1}}, & S_{2n}^{(1)}(0) &= \frac{(-1)^n}{2^{2n}}, & S_{2n+1}^{(1)}(0) &= 0, \\ S_{2n}'(0) &= n \frac{(-1)^{n+1}}{2^{2n-1}}, & S_{2n+1}'(0) &= (n+1) \frac{(-1)^n}{2^{2n}}, & (S_{2n}^{(1)})' &= 0, \\ (S_{2n+1}^{(1)})' &= (n+1) \frac{(-1)^n}{2^{2n}}, & S_{2n}''(0) &= n(n+1) \frac{(-1)^{n+1}}{2^{2n-2}}, & S_{2n+1}''(0) &= n(n+1) \frac{(-1)^{n+1}}{2^{2n-1}}. \end{aligned} \quad (4.14)$$

Proof. The proof is analogous for the demonstration of Lemma 2.8. $\square$

Following Lemma 4.2, for (2.44) we have

$$\begin{aligned}\chi_{2n}(0) &= \frac{2n+1}{2^{4n}}, \quad n \geq 0; & \chi_{2n+1}(0) &= \frac{n+1}{2^{4n+1}}, \quad n \geq 0; & \chi'_{2n}(0) &= 0, \quad n \geq 0; \\ \chi'_{2n+1}(0) &= \frac{n+1}{2^{4n}}, \quad n \geq 0; & \mu_{2n}(0) &= -\frac{n}{2^{4n-1}}, \quad n \geq 0; & \mu_{2n+1}(0) &= -\frac{n+1}{2^{4n+1}}, \quad n \geq 0.\end{aligned}\tag{4.15}$$

Therefore, we get for (2.42)

$$\begin{aligned}\Delta_{2n} &= n(2n+1) \frac{(-1)^{n+1}}{2^{6n}} \lambda^2 + (8n(n+1)(u)_1 - 1) \frac{(-1)^n}{2^{6n+1}} \lambda \\ &\quad + (n+1)(2n+1)(u)_1^2 \frac{(-1)^{n+1}}{2^{6n}}, \quad n \geq 0, \\ \Delta_{2n+1} &= (n+1)(2n+1) \frac{(-1)^{n+1}}{2^{6n+3}} \lambda^2 \\ &\quad + \left(8(n+1)^2(u)_1 + 1\right) \frac{(-1)^n}{2^{6n+4}} \lambda (n+1)(2n+3)(u)_1^2 \frac{(-1)^{n+1}}{2^{6n+3}}, \quad n \geq 0.\end{aligned}\tag{4.16}$$

Then we obtain

$$\begin{aligned}\Delta_{2n} &= 4 \frac{(-1)^{n+1}}{2^{6n+1}} (tn - x_1)(tn - x_2), \quad n \geq 0, \\ \Delta_{2n+1} &= 4 \frac{(-1)^{n+1}}{2^{6n+4}} (tn - x_3)(tn - x_4), \quad n \geq 0,\end{aligned}\tag{4.17}$$

where

$$\begin{aligned}x_1 &= \frac{1}{4} \left\{ -3t - 2\lambda + \left(t^2 - 4\lambda t - 4\lambda^2 - 4\lambda \right)^{1/2} \right\}, & x_2 &= \frac{1}{4} \left\{ -3t - 2\lambda - \left(t^2 - 4\lambda t - 4\lambda^2 - 4\lambda \right)^{1/2} \right\}, \\ x_3 &= \frac{1}{4} \left\{ -5t - 2\lambda + \left((t+2\lambda)^2 + 4\lambda \right)^{1/2} \right\}, & x_4 &= \frac{1}{4} \left\{ -5t - 2\lambda - \left((t+2\lambda)^2 + 4\lambda \right)^{1/2} \right\}, \\ (u)_1 &= t + \lambda.\end{aligned}\tag{4.18}$$

On account of Proposition 2.6, we can deduce that the form u given by (2.9) is regular if and only if $tn - x_i \neq 0$, $n \geq 0$, $1 \leq i \leq 4$.

In the sequel, we suppose that the last condition is satisfied.

By virtue of (4.17) and Lemma 4.2, relations (2.49)–(2.52), and (2.55)–(2.57), (2.70)–(2.72) give Table 2.

Table 2

a_n	$a_{2n} = -\frac{1}{8} \frac{(tn - x_3)(tn - x_4)}{(tn - x_1)(tn - x_2)}, \quad n \geq 0,$	$a_{2n+1} = \frac{1}{8} \frac{(tn - x_1)(tn - x_2)}{(tn - x_3)(tn - x_4)}, \quad n \geq 0.$
b_n	$b_0 = -2x_1x_2 + \frac{1}{4} - \frac{t}{2} + (t + \lambda + \frac{1}{2})^2,$	$b_{2n+1} = \frac{1}{4} + \frac{t}{8} \frac{2(n+1)t - \lambda}{(tn - x_1)(tn - x_2)}, \quad n \geq 0,$
	$b_{2n+2} = \frac{1}{4} + \frac{t}{8} \frac{(2n+3)t + \lambda}{(tn - x_3)(tn - x_4)}, \quad n \geq 0.$	
c_n	$c_0 = -\frac{1}{2}, \quad c_1 = -\frac{1}{2} - t - \lambda,$	$c_{2n+3} = \frac{1}{8} \frac{2t(2n+1)((n+1)t - \lambda) - \lambda}{(tn - x_3)(tn - x_4)}, \quad n \geq 0,$
	$c_{2n+2} = -\frac{1}{8} \frac{2t(2n+1)((n+1)t + \lambda) - \lambda}{(tn - x_1)(tn - x_2)}, \quad n \geq 0.$	
	$\gamma_1 = -2x_1x_2, \quad \gamma_2 = \frac{\lambda}{16} \frac{x_3x_4}{x_1^2x_2^2},$	
γ_{n+1}	$\gamma_{2n+3} = -\frac{1}{4} \frac{(tn - x_1)(t(n+1) - x_1)(tn - x_2)(t(n+1) - x_2)}{(tn - x_3)^2(tn - x_4)^2}, \quad n \geq 0,$	
	$\gamma_{2n+4} = -\frac{1}{4} \frac{(tn - x_3)(t(n+1) - x_3)(tn - x_4)(t(n+1) - x_4)}{(t(n+1) - x_1)^2(t(n+1) - x_2)^2}, \quad n \geq 0.$	
	$\beta_0 = t + \lambda, \quad \beta_1 = -t - \lambda - \frac{\lambda}{2x_1x_2} \{-2x_1x_2 + \frac{1}{4} - \frac{t}{2} + (t + \lambda + \frac{1}{2})^2\},$	
	$\beta_{2n+3} = \frac{1}{8} \frac{2t(2n+1)((n+1)t - \lambda) - \lambda}{(tn - x_3)(tn - x_4)} + \frac{1}{2} \frac{(tn - x_3)(tn - x_4)}{(t(n+1) - x_1)(t(n+1) - x_2)}$	
β_n	$+ \frac{t}{4} \frac{(2n+3)t + \lambda}{(t(n+1) - x_1)(t(n+1) - x_2)}, \quad n \geq 0,$	
	$\beta_{2n+2} = -\frac{1}{8} \frac{2t(2n+1)((n+1)t + \lambda) - \lambda}{(tn - x_1)(tn - x_2)} - \frac{1}{2} \frac{(tn - x_1)(tn - x_2)}{(tn - x_3)(tn - x_4)} - \frac{t}{4} \frac{2(n+1)t - \lambda}{(tn - x_3)(tn - x_4)}, \quad n \geq 0.$	

Proposition 4.3. If $v = \mathcal{J}_{(-1/2, 1/2)}$ is the corecursive of the second kind Chebychev form, then the form $u(t, \lambda)$ given by (2.9) has the following integral representation:

$$\langle u(t, \lambda), f \rangle = (1 - \lambda)f(0) + tf'(0) + \frac{\lambda}{\pi} P \int_{-1}^1 \frac{1}{x} \sqrt{\frac{1-x}{1+x}} f(x) dx, \quad \forall f \in \mathcal{P}. \quad (4.19)$$

It is a semi-classical form of class s satisfying the following functional equation:

$$t \neq 0, \quad z^3(z^2 - 1)S'(z; u(t, \lambda)) = -z^2(z^2 - z - 1)S(z; u(t, \lambda)) + (t - 2\lambda + 1)z^2 + tz - t, \quad s = 3$$

$$t = 0, \quad z(z^2 - 1)S'(z; u(0, \lambda)) = (-z^2 + z + 1)S(z; u(0, \lambda)) - 2\lambda + 1, \quad s = 1. \quad (4.20)$$

Proof. It is well known that $v = \mathcal{J}_{(-1/2, 1/2)}$ possesses the following integral representation [1]:

$$\langle v, f \rangle = \int_{-1}^1 \frac{1}{\pi} \sqrt{\frac{1-x}{1+x}} f(x) dx, \quad f \in \mathcal{P}. \quad (4.21)$$

From (2.11) we easily obtain (4.19).

The form v satisfies (3.4) with [15]

$$\phi(x) = x^2 - 1, \quad C(z) = 1, \quad D(z) = -2. \quad (4.22)$$

Therefore, $\vartheta_1 = -t$, $\vartheta_2 = t$, $\phi(0) \neq 0$.

Now, we can simply use Proposition 3.1 and Corollary 3.2 in order to obtain (4.20). $\square$

Corollary 4.4. *When $t = 0$ and $\lambda = -1$, one has*

$$\begin{aligned} \beta_n &= (-1)^{n+1}, \quad n \geq 0, & \gamma_1 &= -\frac{1}{2}, & \gamma_{n+2} &= -\frac{1}{4}, \quad n \geq 0, \\ z(z^2 - 1)S'(z; u(0, -1)) &= (-z^2 + z + 1)S(z; u(0, -1)) + 3, \quad s = 1. \end{aligned} \quad (4.23)$$

Proof. From Table 2, we reach the desired results. $\square$

Remarks 4.5. (1) One has the form $h_{-1}u(0, -1) = \mathcal{L}(-3/2, 1/2)$, where $\mathcal{L}(\alpha, \beta)$ is studied in [24].

(2) Let $\{Z_n^{(1)}\}_{n \geq 0}$ [15, 19] be the first associated sequence of $\{Z_n\}_{n \geq 0}$ orthogonal with respect to $u(0, -1)$ and $\beta_n^{(1)}, \gamma_{n+1}^{(1)}$ the coefficients of the three-term recurrence relations; we have

$$\beta_n^{(1)} = \beta_{n+1} = (-1)^n, \quad n \geq 0; \quad \gamma_{n+1}^{(1)} = \gamma_{n+2} = -\frac{1}{4}, \quad n \geq 0. \quad (4.24)$$

The sequence $\{Z_n^{(1)}\}_{n \geq 0}$ is a second-order self-associated sequence; that is, $\{Z_n^{(1)}\}_{n \geq 0}$ is identical to its associated orthogonal sequence of second kind (see [25]).

Acknowledgments

Sincere thanks are due to the referee for his valuable comments and useful suggestions and his careful reading of the manuscript. The author is indebted to the proofreader the English teacher Hajer Rebai who checked the language of this work.

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