Research Article

Almost $\alpha$-Hyponormal Operators with Weyl Spectrum of Area Zero

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We define the class of almost $\alpha$-hyponormal operators and prove that for an operator $T$ in this class, $(TT^*)^\alpha - (TT^*)^\alpha$ is trace-class and its trace is zero when $\alpha \in (0,1]$ and the area of the Weyl spectrum is zero.

This note is dedicated to Professor Carl M. Pearcy with the occasion of his 75th birthday.

Let $\mathcal{A}$ be a complex, separable, infinite-dimensional Hilbert space, and let $L(\mathcal{A})$ denote the algebra of all linear bounded operators on $\mathcal{A}$, and for $1 \leq p < \infty$, let $C_p(\mathcal{A})$ denote the $p$-Schatten class on $\mathcal{A}$. For $K \in C_p(\mathcal{A})$, the expression $\|K\|_p := \left(\sum_{n=1}^{\infty} \mu_n(K)^p\right)^{1/p}$, where $\mu_1(K) \geq \mu_2(K) \geq \cdots$ are the singular values of $K$, is a norm for $p \geq 1$, and is only a quasinorm for $0 < p < 1$ (it does not satisfy the triangle inequality). Nevertheless, the latter case will be used in what follows.

For $T \in L(\mathcal{A})$, $\sigma(T)$ and $\sigma_w(T)$ will denote the spectrum and the Weyl spectrum, respectively. Recall that Weyl spectrum is the union of the essential spectrum, $\sigma_e(T)$, and all bounded components of $\mathbb{C} \setminus \sigma_e(T)$ associated with nonzero Fredholm index. An operator $T \in L(\mathcal{A})$ is called $(C_p,\alpha)$-normal (notation: $T \in N_p^\alpha(\mathcal{A})$) if $C_p^\alpha := (TT^*)^\alpha - (TT^*)^\alpha$ belongs to $C_p(\mathcal{A})$, and $T$ is called $(C_p,\alpha)$-hyponormal (notation: $T \in H_p^\alpha(\mathcal{A})$) if $C_p^\alpha$ is the sum of a positive definite operator and an operator in $C_p(\mathcal{A})$, or equivalently, $(C_p^\alpha)_-$ (the negative part of $C_p^\alpha$) belongs to $C_p(\mathcal{A})$, where $\alpha$ is a positive number. This note will be concerned with the particular class $H_p^\alpha(\mathcal{A})$, which by some parallelism with some terminology used in [1], would be appropriate to be referred as almost $\alpha$-hyponormal operators.

Voiculescu’s [1] generalization of Berger-Shaw inequality gives an estimate for the trace of $C_p^\alpha$. The result was extended in [2]. The combination of these results will be stated after recalling some terminology and notation. The rational cyclic multiplicity of an operator
Let \( T \in L(H^1(\mathfrak{H})) \), denoted by \( m(T) \), is the smallest cardinal number \( m \) with the property that there are \( m \) vectors \( x_1, \ldots, x_m \) in \( H^1(\mathfrak{H}) \) such that

\[
\forall \{ f(T)x_j \mid 1 \leq j \leq m, f \in \text{Rat}(\sigma(T)) \} = \mathfrak{H},
\]

where \( \text{Rat}(\sigma(T)) \) is the algebra of complex-valued rational functions with poles off \( \sigma(T) \).

For a Borel subset \( E \subseteq \mathbb{C} \) and \( \alpha > 0 \), denote \( \mu_{\alpha}(E) = (\alpha/2) \int_E \rho^{\alpha-1} \, d\rho \, d\theta \). In particular, \( \mu_2 \) is the planar Lebesgue measure.

**Theorem A** (see [1, 2]). Suppose \( T \in H^1_1(\mathfrak{H}) \). If there exists \( K \in C_{2, \alpha}(\mathfrak{H}) \) such that either \( m(T + K) < \infty \) or \( \mu_2(\sigma(T + K)) = 0 \), then \( T \in N^1_{1, \alpha}(\mathfrak{H}) \). Moreover, when \( m(T + K) < \infty \),

\[
\text{tr}(C^1_\tau) \leq \frac{m(T + K)}{\pi} \cdot \mu_2(\sigma(T + K)),
\]

and when \( \mu_2(\sigma(T + K)) = 0 \), \( \text{tr}(C^1_\tau) \leq 0 \), and consequently, \( \text{tr}(C^1_\tau) = 0 \).

In fact, it was observed in [2] that the inequality can be improved by replacing \( m(T + K) \) with \( \tau(T + K) \), where

\[
\tau(S) := \liminf \, [\text{rank}(I - P)SP],
\]

and the liminf is taken over all sequences of finite-rank orthogonal projections such that \( P \to I \) in the strong operator topology.

**Corollary B** (see [2]). Let \( T \in H^1_1(\mathfrak{H}) \) such that \( \mu_2(\sigma_{\omega}(T)) = 0 \). Then \( T \in N^1_{1, \alpha}(\mathfrak{H}) \) and \( \text{tr}(C^1_\tau) = 0 \).

On the other hand, Berger-Shaw inequality was extended to operators in \( H^1_1(\mathfrak{H}) \) using similar circle of ideas used in [1]. This was done in [3] for the case \( \alpha \in [(1/2), 1] \) and later on in [4] for the case \( \alpha \in (0, (1/2)] \).

**Theorem C** (see [3, 4]). Let \( 0 < \alpha \leq 1 \), and let \( T \in H^1_1(\mathfrak{H}) \) and \( K \in C_{2, \alpha}(\mathfrak{H}) \) with \( m(T + K) < \infty \). Then \( T \in N^1_{1, \alpha}(\mathfrak{H}) \) and

\[
\text{tr}(C^1_\alpha) \leq \frac{m(T + K)}{\pi} \cdot \mu_{2\alpha}(\sigma(T + K)).
\]

The case in which \( m(T + K) = \infty \) and \( \mu_{2\alpha}(\sigma(T + K)) = 0 \) was not discussed in [4] or [3]. It is the goal of this note to make some progress towards this case. We have the following.

**Theorem 1.** Let \( \alpha \in (0, 1) \) and let \( T \in H^1_1(\mathfrak{H}) \) and \( K \in C_{\alpha}(\mathfrak{H}) \) with \( \mu_{2\alpha}(\sigma(T + K)) = 0 \). Then \( T \in N^1_{1, \alpha}(\mathfrak{H}) \) and \( \text{tr}(C^1_\alpha) = 0 \).

**Remark.** It would have been desirable that Theorem 1 be proved with the hypothesis that \( K \in C_{2\alpha}(\mathfrak{H}) \).

Before we prove Theorem 1, we extract a similar consequence to Corollary B.
Corollary 2. Let $\alpha \in (0,1]$ and let $T \in \mathcal{H}_1^\alpha$ such that $\mu_2(\sigma_\alpha(T)) = 0$. Then $T \in N_1^\alpha(\mathcal{K})$ and $\text{tr} \ (C_\alpha^T) = 0$.

Proof. If $\alpha = 1$, then conclusion holds according to Corollary B. Let $\alpha \in (0,1)$. First, a careful inspection of the proof of a result of Stampfli [5] leads to the following. For $T \in L(\mathcal{K})$ and $\alpha > 0$, there exists $K_\alpha \in C_\alpha(\mathcal{K})$ such that $\sigma(T + K_\alpha) \setminus \sigma_\alpha(T)$ consists of a countable set which clusters only on $\sigma_\alpha(T)$. Therefore $\mu_2(\sigma(T + K_\alpha)) = 0$ and thus Theorem 1 applies.

The proof of Theorem 1 makes use of the following three inequalities.

Proposition D (Hansen’s inequality [6]). If $A, B \in L(\mathcal{K})$, $A \geq 0$, $||B|| \leq 1$, and $\alpha \in (0,1]$, then $B^\alpha A^\beta B \leq (B^\alpha AB)^\alpha$.

Proposition E (Lowner’s inequality [7]). If $A, B \in L(\mathcal{K})$, $A \geq B \geq 0$, and $\alpha \in (0,1]$, then $A^\alpha \geq B^\alpha$.

The following is a consequence of Theorem 3.4 of [8].

Proposition F (Jocic’s inequality [8]). Let $A, B \in L(\mathcal{K})$, $A, B \geq 0$, $\alpha \in (0,1]$, and $1 \leq p < \infty$. If $A - B \in C_{ap}(\mathcal{K})$, then $A^\alpha - B^\alpha \in C_p(\mathcal{K})$ and $||B^\alpha - A^\alpha||_p \leq \||B - A||^\alpha\|_p$.

Proof of Theorem 1. Let $\alpha \in (0,1)$, $T \in \mathcal{H}_1^\alpha$, and $K \in C_\alpha(\mathcal{K})$ with $\mu_2(\sigma(T + K)) = 0$, and assume $m(T + K) = \infty$, otherwise Theorem C implies $T \in N_1^\alpha(\mathcal{K})$.

Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{K}$ and let

$$\mathcal{K}_n = \bigvee \{r(T + K)e_j \mid j = 1, \ldots, n, r \in \text{Rat}(\sigma(T + K))\}. \tag{5}$$

Assume that with respect to the decomposition $\mathcal{K} = \mathcal{K}_n \oplus \mathcal{K}_n^\perp$, operators $T$ and $K$ are written as

$$T = \begin{pmatrix} T_{1n} & T_{2n} \\ T_{3n} & T_{4n} \end{pmatrix}, \quad K = \begin{pmatrix} K_{1n} & K_{2n} \\ K_{3n} & K_{4n} \end{pmatrix}. \tag{6}$$

Since $\mathcal{K}_n$ is a rationally invariant subspace for $T + K$, we have $T_{3n} + K_{3n} = 0$, and thus $T_{3n} = -K_{3n} \in C_\alpha(\mathcal{K}_n) \subseteq C_{2\alpha}(\mathcal{K}_n)$, and $\sigma(T_{1n} + K_{1n}) \subseteq \sigma(T + K)$, which implies $\mu_2(\sigma(T_{1n} + K_{1n})) = 0$.

Let $P_n$ be the orthogonal projection onto $\mathcal{K}_n$, and thus $P_n \uparrow I$ strongly. We will prove next that $T_{1n} \in \mathcal{H}_1^\alpha$ by first establishing that

$$P_n C_1^T P_n - C_1^\alpha T_{1n} = -Q_n' + K_n', \tag{7a}$$

where $Q_n' \in L(\mathcal{K}_n)$ is positive semidefinite and $K_n' \in C_1(\mathcal{K}_n)$.

Assuming that equality (7a) was already proved and writing $C_1^T = Q + K$ with $Q \geq 0$ and $K \in C_1(\mathcal{K})$, then we have

$$C_1^T_{1n} = P_n Q P_n + P_n K P_n + Q_n' - K_n', \tag{7b}$$

that is, $C_1^T_{1n}$ is the sum of $P_n Q P_n + Q_n'$, which is a positive semidefinite operator, and of $P_n K P_n - K_n'$, which is a trace-class operator.
Indeed, the expression $P_nC^a_nP_n - C^a_{T_n}$ can be written as $D_1 - D_2$, where

$$D_1 = P_n(T^*T)^a P_n - (T^*_n T_n)^a,$$

$$D_2 = P_n(TT^*)^a P_n - (T^*_n T_n)^a.$$  

(8)

We can write $D_1 = -Q''_n + K''_n$, where

$$Q''_n = [(P_n T^* T P_n)^a - P_n (T^* T)^a P_n],$$

(9)

which according to Hansen’s inequality is a positive semidefinite operator, and

$$K''_n = [(P_n T^* T P_n)^a - (P_n T P_n T P_n)^a] P_n,$$

(10)

which according to Jocic’s inequality is a trace-class operator that satisfies

$$\|K''_n\|_1 \leq \|(P_n T^* T P_n - P_n T P_n T P_n)^a\|_1 = \|(P_n T^*_n T_n)^a\|_1$$

$$= \|(T^*_n T_n)^a\|_1 \leq \|T^*_n\|_a \cdot \|T_n\|_a \leq \|T\|_a \cdot \|T_n\|_a.$$  

(11)

Concerning operator $D_2$, we can write $D_2 = Q'''_n + K'''_n$, where

$$Q'''_n = P_n (T T^*)^a P_n - P_n (T P_n T^*)^a P_n,$$

(12)

which according to Lowner’s inequality is a positive semidefinite operator, and

$$K'''_n = P_n (T P_n T^*)^a P_n - (P_n T P_n T^* P_n)^a = P_n [(T P_n T^*)^a - (P_n T P_n T^* P_n)^a] P_n,$$

(13)

which is also a trace-class operator since

$$T P_n T^* - P_n T P_n T^* P_n = (T P_n T^* - T P_n T^* P_n) + (T P_n T^* P_n - P_n T P_n T^* P_n)$$

$$= T P_n T^* (I - P_n) + (I - P_n) T P_n T^* P_n$$

$$= TT^*_n T_n T^* P_n \in \mathcal{C}_a(\mathcal{L}),$$

and according to Jocic’s inequality

$$\|K'''_n\|_1 \leq \|(T P_n T^*)^a - (P_n T P_n T^* P_n)^a\|_1 \leq \|T T^*_n + T_n T^* P_n\|_a^a$$

$$= \|T T^*_n + T_n T^* P_n\|_a^a \leq C (\|T T^*_n\|_a^a + \|T_n T^* P_n\|_a^a)$$

$$\leq C \|T\|_a^a (\|T T^*_n\|_a^a + \|T_n T^* P_n\|_a^a) = 2C \|T\|_a^a \|T_n\|_a^a.$$  

(15)
Therefore,

\[ D_2 = Q_n'' + K_n''' \quad \text{with} \quad Q_n'' \geq 0, \quad K_n''' \in C_1(H), \]  

(16)
and consequently, \( D_1 - D_2 = (Q_n'' + K_n''') - (Q_n'' + K_n''') = -(Q_n'' + Q_n''') + (K_n'' - K_n''') \), where \( Q_n'' + Q_n''' = Q_n \) is positive semidefinite and \( K_n - K_n''' =: K_n' \) is trace-class, which establishes equality (7a).

According to (7b), \( T_{1n} \in H^2(\mathcal{H}_n) \), and since \( m(T_{1n} + K_{1n}) \leq n \) and \( \sigma(T_{1n} + K_{1n}) \subseteq \sigma(T + K) \), Theorem C implies that \( \text{tr}(C_{T_{1n}}') \leq 0 \), and furthermore, by replacing \( T_{1n} \) with \( T_{1n}' \), \( \text{tr}(C_{T_{1n}}') = 0 \). Furthermore, equality (7a) implies

\[ P_n C_n^* P_n \leq C_{T_{1n}}' + K_n', \]  

(17)
which further implies

\[ \text{tr}(P_n C_n^* P_n) \leq \text{tr}(K_n'). \]  

(18)

Similar utilization of Löwner’s and Hansen’s inequalities implies that \( K_n'' \) and \( -K_n''' \) are positive semidefinite, and thus so is \( K_n' = K_n'' - K_n''' \). Therefore

\[ \text{tr}(K_n') \leq \| (K_n') \|_1 + \| (K_n'') \|_1 \leq (1 + 2C) \| T \| \| T_{3n} \|_a^2. \]  

(19)

Since \( T_{3n} = -K_{3n} \in C_p(\mathcal{H}_n) \) and \( K_{3n} \rightarrow 0 \) weakly and both \( |T_{3n}| \) and \( |T_{3n}|_a \leq \| T \|_a \), we have \( \| T_{3n} \|_a \rightarrow 0 \), and thus \( \text{tr}(C_n^a) \leq 0 \). Replacing \( T \) with \( T^* \) we conclude that \( \text{tr}(C_n^a) = 0 \).

References

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