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Research Article

An Intermediate Value Theorem for the Arboricities

Avapa Chantasartrassmee[1] and Narong Punnim[2]

[1] *Department of Mathematics, School of Science, University of the Thai Chamber of Commerce, Bangkok 10400, Thailand*
[2] *Department of Mathematics, Srinakharinwirot University, Sukhumvit 23, Bangkok 10110, Thailand*

Correspondence should be addressed to Narong Punnim, narongp@swu.ac.th

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Let G be a graph. The vertex (edge) arboricity of G denoted by $\mathtt{a}(G)(\mathtt{a}_1(G))$ is the minimum number of subsets into which the vertex (edge) set of G can be partitioned so that each subset induces an acyclic subgraph. Let $\mathbf{d}$ be a graphical sequence and let $\mathcal{R}(\mathbf{d})$ be the class of realizations of $\mathbf{d}$. We prove that if $\pi \in \{\mathtt{a}, \mathtt{a}_1\}$, then there exist integers $x(\pi)$ and $y(\pi)$ such that $\mathbf{d}$ has a realization G with $\pi(G) = z$ if and only if z is an integer satisfying $x(\pi) \leq z \leq y(\pi)$. Thus, for an arbitrary graphical sequence $\mathbf{d}$ and $\pi \in \{\mathtt{a}, \mathtt{a}_1\}$, the two invariants $x(\pi) = \min(\pi, \mathbf{d}) := \min\{\pi(G) : G \in \mathcal{R}(\mathbf{d})\}$ and $y(\pi) = \max(\pi, \mathbf{d}) := \max\{\pi(G) : G \in \mathcal{R}(\mathbf{d})\}$ naturally arise and hence $\pi(\mathbf{d}) := \{\pi(G) : G \in \mathcal{R}(\mathbf{d})\} = \{z \in \mathbb{Z} : x(\pi) \leq z \leq y(\pi)\}$. We write $\mathbf{d} = r^n := (r, r, \ldots, r)$ for the degree sequence of an r-regular graph of order n. We prove that $\mathtt{a}_1(r^n) = \{\lceil (r+1)/2 \rceil\}$. We consider the corresponding extremal problem on vertex arboricity and obtain $\min(\mathtt{a}, r^n)$ in all situations and $\max(\mathtt{a}, r^n)$ for all $n \geq 2r + 2$.

1. Introduction

We limit our discussion to graphs that are simple and finite. For the most part, our notation and terminology follows that of Chartrand and Lesniak [1]. A typical problem in graph theory deals with decomposition of a graph into various subgraphs possessing some prescribed property. There are ordinarily two problems of this type, one dealing with a decomposition of the vertex set and the other with a decomposition of the edge set. The vertex coloring problem is an example of vertex decomposition while the edge coloring problem is an example of edge decomposition with some additional property. For a graph G, it is always possible to partition $V(G)$ into subsets V_i, $1 \leq i \leq k$, such that each induced subgraph $\langle V_i \rangle$ contains no cycle. The *vertex arboricity* $\mathtt{a}(G)$ of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces an acyclic subgraph. Thus, $\mathtt{a}(G) = 1$ if and only if G is a forest. The vertex arboricity was first defined in [2] by

Chartrand et al. For a few classes of graphs, the vertex arboricity is easily determined. For example, $\mathtt{a}(C_n) = 2$. $\mathtt{a}(K_n) = \lceil n/2 \rceil$. Also $\mathtt{a}(K_{r,s}) = 1$ if $r = 1$ or $s = 1$ and $\mathtt{a}(K_{r,s}) = 2$ otherwise. It is clear that, for every graph G of order n, $\mathtt{a}(G) \leq \lceil n/2 \rceil$. We now turn to the second decomposition problem. The *edge arboricity* or simply the *arboricity* $\mathtt{a}_1(G)$ of a nonempty graph G is the minimum number of subsets into which $E(G)$ can be partitioned so that each subset induces an acyclic subgraph. The arboricity was first defined by Nash-Williams in [3]. As with vertex arboricity, a nonempty graph has arboricity 1 if and only if it is a forest. Also $\mathtt{a}_1(K_n) = \lceil n/2 \rceil$.

A *linear forest* of a graph G is a forest of G in which each component is a path. The *linear arboricity* is the minimum number of subsets into which $E(G)$ can be partitioned so that each subset induces a linear forest. It was first introduced by Harary in [4] and is denoted by $\Xi(G)$. Note that the Greek letter, capital, Xi, looks like three paths! It was proved in [5] that $\Xi(G) = 2$ for all cubic graphs G and $\Xi(G) = 3$ for all 4-regular graphs G. It was conjectured that $\Xi(G) = \lceil (r+1)/2 \rceil$ for all r-regular graphs G.

A sequence $(d_1, d_2, \ldots, d_n)$ of nonnegative integers is called a *degree sequence* of a graph G if the vertices of G can be labeled $v_1, v_2, \ldots, v_n$ so that $\mathrm{d}eg\ \ v_i = d_i$ for all i. If a sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is a degree sequence of some graph G, then $\mathbf{d}$ is called *graphical sequence*. In this case, G is called a *realization* of $\mathbf{d}$. A graph G is *regular* of degree r if $\mathrm{d}eg\ \ v = r$ for each vertex v of G. Such graphs are called *r-regular*. We write $\mathbf{d} = r^n$ for the sequence $(r, r, \ldots, r)$ of length n, where r is a nonnegative integer and n is a positive integer. It is well known that an r-regular graph of order n exists if and only if $n \geq r + 1$ and $nr \equiv 0 (\mathrm{mod}\ 2)$. Furthermore, there exists a disconnected r-regular graph of order n if and only if $n \geq 2r + 2$.

Let G be a graph and $ab, cd \in E(G)$ be independent where $ac, bd \notin E(G)$. Put

$$G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) + \{ac, bd\}. \tag{1.1}$$

The operation $\sigma(a, b; c, d)$ is called a *switching operation*. It is easy to see that the graph obtained from G by a switching has the same degree sequence as G.

Havel [6] and Hakimi [7] independently obtained that if G_1 and G_2 are any two realizations of $\mathbf{d}$, then G_2 can be obtained from G_1 by a finite sequence of switchings. Let $\mathcal{R}(\mathbf{d})$ denote the set of all realizations of degree sequence $\mathbf{d}$. As a consequence of this result, Eggleton and Holton [8] defined, in 1978, *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ whose vertex set is the set $\mathcal{R}(\mathbf{d})$, two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. As a consequence of Havel and Hakimi, we have the following theorem.

Theorem 1.1. *Let* $\mathbf{d}$ *be a graphical sequence. Then, the graph* $\mathcal{R}(\mathbf{d})$ *is connected.*

Let $\pi \in \{\mathtt{a}, \mathtt{a}_1\}$ and $\mathbf{d}$ be a graphical sequence. Put

$$\pi(\mathbf{d}) := \{\pi(G) : G \in \mathcal{R}(\mathbf{d})\}. \tag{1.2}$$

A graph parameter π is said to satisfy an *intermediate value theorem over a class of graphs* $\mathcal{J}$ if $G, H \in \mathcal{J}$ with $\pi(G) < \pi(H)$, then, for every integer k with $\pi(G) \leq k \leq \pi(H)$, there is a graph $K \in \mathcal{J}$ such that $\pi(K) = k$. If a graph parameter π satisfies an intermediate value theorem over $\mathcal{J}$, then we write $(\pi, \mathcal{J}) \in \mathtt{IVT}$. The main purpose of this section is to prove that if $\pi \in \{\mathtt{a}, \mathtt{a}_1\}$, then $(\pi, \mathcal{R}(\mathbf{d})) \in \mathtt{IVT}$.

Theorem 1.2. *Let* $\mathbf{d}$ *be a graphical sequence and* $\pi \in \{\mathtt{a}, \mathtt{a}_1\}$. *Then, there exist integers* $x(\pi)$ *and* $y(\pi)$ *such that there exists a graph* $G \in \mathcal{R}(\mathbf{d})$ *with* $\pi(G) = z$ *if and only if* z *is an integer satisfying* $x(\pi) \le z \le y(\pi)$. *That is,* $\pi(\mathbf{d}) = \{z \in \mathbb{Z} : x(\pi) \le z \le y(\pi)\}$.

The proof of Theorem 1.2 follows from Theorem 1.1 and the following results.

Lemma 1.3. *Let* $\pi \in \{\mathtt{a}, \mathtt{a}_1\}$ *and* σ *be a switching on a graph* G. *Then,* $\pi(G^\sigma) \le \pi(G) + 1$.

Proof. Let G be a graph and $\sigma = \sigma(a, b; c, d)$ be a switching on G. Then, $\mathtt{a}(G^\sigma - \{a, b\}) \le \mathtt{a}(G)$ and hence $\mathtt{a}(G^\sigma) \le \mathtt{a}(G^\sigma - \{a, b\}) + 1 \le \mathtt{a}(G) + 1$. Since $ac, bd \in E(G^\sigma)$, we have that $\mathtt{a}_1(G^\sigma - \{ac, bd\}) \le \mathtt{a}_1(G)$ and hence $\mathtt{a}_1(G^\sigma) \le \mathtt{a}_1(G^\sigma - \{ac, bd\}) + 1 \le \mathtt{a}_1(G) + 1$. □

Let $\pi \in \{a, a_1\}$. By Lemma 1.3, if $\pi(G) \le \pi(G^\sigma)$, then we have that $\pi(G) \le \pi(G^\sigma) \le \pi(G) + 1$. If $\pi(G) \ge \pi(G^\sigma)$, then, by Lemma 1.3, we see that $\pi(G) = \pi((G^\sigma)^{\sigma^{-1}}) \le \pi(G^\sigma) + 1 \le \pi(G) + 1$, where $\sigma^{-1} = \sigma(a, c; b, d)$.

We have the following corollary.

Corollary 1.4. *Let* G *be a graph and let* σ *be a switching on* G. *If* $\pi \in \{\mathtt{a}, \mathtt{a}_1\}$, *then* $|\pi(G) - \pi(G^\sigma)| \le 1$.

2. Extremal Results

Let $\pi \in \{\mathtt{a}, \mathtt{a}_1\}$. By Theorem 1.2, $\pi(\mathbf{d})$ is uniquely determined by $x(\pi)$ and $y(\pi)$, thus it is reasonable to denote $x(\pi) = \min(\pi, \mathbf{d}) := \min\{\pi(G) : G \in \mathcal{R}(\mathbf{d})\}$ and $y(\pi) = \max(\pi, \mathbf{d}) := \max\{\pi(G) : G \in \mathcal{R}(\mathbf{d})\}$. We first state a famous result on edge arboricity of Nash-Williams [3].

Theorem 2.1. *For every nonempty graph* G, $\mathtt{a}_1(G) = \max\lceil |E(H)|/(|V(H)| - 1)\rceil$, *where the maximum is taken over all nontrivial induced subgraphs* H *of* G.

As a consequence of the theorem, it follows that $\mathtt{a}_1(K_n) = \lceil n/2 \rceil$ and $\mathtt{a}_1(K_{r,s}) = \lceil rs/(r+s-1)\rceil$.

If $\pi \in \{\mathtt{a}, \mathtt{a}_1\}$ and H is a subgraph of G, then $\pi(H) \le \pi(G)$. It is well known that if G is a graph with a degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ $(d_1 \ge d_2 \ge \cdots \ge d_n)$, then G can be embedded as an induced subgraph of a d_1-regular graph H and $\pi(G) \le \pi(H)$. Thus, it is reasonable to determine $\min(\pi, \mathbf{d})$ and $\max(\pi, \mathbf{d})$ in the case where $\mathbf{d} = r^n$. It should be noted that if $r \in \{0, 1\}$, then $\mathtt{a}_1(G) = 1$ for all $G \in \mathcal{R}(r^n)$. Therefore, we may assume from now on that $r \ge 2$.

Theorem 2.2. $\min(\mathtt{a}_1, r^n) = \max(\mathtt{a}_1, r^n) = \lceil (r+1)/2 \rceil$.

Proof. Let G be an r-regular graph of order n. By Theorem 2.1, we see that $\mathtt{a}_1(G) \ge \lceil nr/(2(n-1))\rceil$ for any r-regular graph G of order n. Since $nr/(2(n-1)) = r/2 + r/(2(n-1))$, it follows that $\mathtt{a}_1(G) \ge \lceil (r+1)/2 \rceil$. Let H be an induced subgraph of G of order m. If $m \ge r+1$, then $|E(H)|/(|V(H)|-1) \le mr/(2(m-1))$. It follows that $|E(H)|/(|V(H)|-1) \le (r+1)/2$. If $m \le r$, then $|E(H)| \le \binom{m}{2}$. It is clear that $|E(H)|/(|V(H)| - 1) \le (r+1)/2$. Thus, $\mathtt{a}_1(G) \le (r+1)/2$. Therefore, $\min(\mathtt{a}_1, r^n) = \max(\mathtt{a}_1, r^n) = \lceil (r+1)/2 \rceil$. □

In order to obtain the corresponding extremal results on vertex arboricity, we need to introduce some notation on graph construction.

Let X and Y be finite nonempty sets. We denote by $K(X,Y)$ the complete bipartite graph with partite sets X and Y. Let G and H be any two graphs. Then, $G * H$ is the graph with $V(G*H) = V(G) \cup V(H)$ and $E(G*H) = E(G) \cup E(H)$. It is clear that the binary operation "$*$" is associative. If G_1 and G_2 are disjoint (i.e., $V(G_1) \cap V(G_2) = \emptyset$), then $G_1 * G_2 = G_1 \cup G_2$. We also use pG for the union of p disjoint copies of G.

It should be noted once again that an r-regular graph G of order n have $\mathtt{a}(G) = 1$ if and only if $r \in \{0,1\}$. Thus, $\min(\mathtt{a}, r^n) = \max(\mathtt{a}, r^n) = 1$ if and only if $r \in \{0,1\}$. Moreover, $\min(\mathtt{a}, 2^n) = \max(\mathtt{a}, 2^n) = 2$.

Let r and n be integers with $r \geq 3$ and $\mathcal{R}(r^n) \neq \emptyset$. Then,

(1) if n is even and $n \geq 2r$, then there exists an r-regular bipartite graph G of order n. Therefore, $\mathtt{a}(G) = 2$;

(2) if n is odd and $n \geq 2r+1$, then r is even and there exists an r-regular bipartite graph H of order $n-1$. Let F be a set of $r/2$ independent edges of H. Then, $H - F$ is a bipartite graph of order n having r vertices of degree $r-1$ and $n-1-r$ vertices of degree r. Let G be a graph obtained from $H - F$ by joining the r vertices of $H - F$ to a new vertex. Therefore, G is an r-regular graph of order n having $\mathtt{a}(G) = 2$;

(3) if n is even and $n = 2(r-1)$, then $K_{r-1,r-1}$ is the complete $(r-1)$-regular bipartite graph of order n. Let $X = \{x_1, x_2, \ldots, x_{r-1}\}$ and $Y = \{y_1, y_2, \ldots, y_{r-1}\}$ be the corresponding partite sets of $K_{r-1,r-1}$. Let G be defined by $G = (K_{r-1,r-1} - \{x_2y_2, x_3y_3, \ldots, x_{r-2}y_{r-2}\}) + \{x_1x_2, x_2x_3, \ldots, x_{r-2}x_{r-1}, y_1y_2, y_2y_3, \ldots, y_{r-2}y_{r-1}\}$. Then G is an r-regular graph of order n having $\mathtt{a}(G) = 2$;

(4) if $n = 2r-1$, then r is even. Let X and Y be as described in the previous case and u be a new vertex. Then, $H = K(X,Y) + E_1$, where $E_1 = \{ux_1, uy_1, ux_2, uy_2, \ldots, ux_{r/2}, uy_{r/2}\}$, is a graph of order $2r-1$ having $r+1$ vertices of degree r and $r-2$ vertices of degree $r-1$. Put $X_1 = \{x_i \in X : r/2 < i \leq r-1\}$ and $Y_1 = \{y_i \in Y : r/2 < i \leq r-1\}$. Thus, $|X_1| = |Y_1| = (r-2)/2$.

Case 1. If $(r-2)/2$ is even, then we can construct an r-regular graph G from H by adding $(r-2)/4$ independent edges to each of X_1 and Y_1. This construction yields $\mathtt{a}(G) = 2$.

Case 2. If $(r-2)/2$ is odd, then $H - x_{r/2}y_{r/2}$ is a graph having $r-1$ vertices of degree r and r vertices of degree $r-1$. Thus, it has $r/2$ vertices of degree $r-1$ in X and also $r/2$ vertices in Y. Since $r/2$ is even, we can easily construct an r-regular graph G from $H - x_{r/2}y_{r/2}$ by adding appropriate independent edges so that $\mathtt{a}(G) = 2$.

With these observations, we easily obtain the following result.

Lemma 2.3. $\min(\mathtt{a}, r^n) = 2$ *for all* $n \geq 2r-2$.

In order to obtain $\min(\mathtt{a}, r^n)$ in all other cases, we first state some known result concerning the forest number of regular graphs. Let G be a graph and $F \subseteq V(G)$. F is called an *induced forest* of G if the subgraph $\langle F \rangle$ of G contains no cycle. The maximum cardinality of an induced forest of a graph G is called the *forest number* of G and is denoted by $\mathtt{f}(G)$. That is,

$$\mathtt{f}(G) := \max\{|F| : F \text{ is an induced forest of } G\}. \tag{2.1}$$

The second author proved in [9] the following result: Let r and s be integers with $1 \le s \le r$. Then,

(1) $\mathtt{f}(G) \le s+1$ for all $G \in \mathcal{R}(r^{r+s})$,

(2) there exists a graph $H \in \mathcal{R}(r^{r+s})$ such that $\mathtt{f}(H) = s+1$.

With these observations, we have the following result.

Lemma 2.4. *If* $1 \le s \le r-3$, *then* $\min(\mathtt{a}, r^{r+s}) \ge \lceil (r+s)/(s+1) \rceil$.

It is well known that $\mathtt{a}(K_{r+1}) = \lceil (r+1)/2 \rceil$. Thus, $\min(\mathtt{a}, r^{r+1}) = \max(\mathtt{a}, r^{r+1}) = \lceil (r+1)/2 \rceil$. It is also well known that $G = K_{r+2} - F$, where F is a 1-factor of K_{r+2}, is the unique r-regular graph of order $r+2$. Since $E(\overline{G}) = F$, it follows that a subgraph of $\overline{G}$ induced by any four vertices of $V(\overline{G})$ contains at most two edges. Equivalently, a subgraph of G induced by any four vertices must contain a cycle. Thus, $\mathtt{a}(G) \ge \lceil (r+2)/3 \rceil$. It is easy to show that $\mathtt{a}(G) = \lceil (r+2)/3 \rceil$. Thus, $\min(\mathtt{a}, r^{r+2}) = \max(\mathtt{a}, r^{r+2}) = \lceil (r+2)/3 \rceil$.

Let $T_{n,q}$ be the Turán graph of order n containing no clique of order $q+1$. Thus, there exists a unique pair of integers x and y such that $n = qx + y$, $0 \le y < q$. Note that $T_{n,q}$ is a complete q-partite graph of order n consisting of y partite sets of cardinality $x+1$ and $q-y$ partite sets of cardinality x. Therefore, $T_{n,q}$ is an $(n-x)$-regular graph if and only if $y = 0$. If $y > 0$, then $T_{n,q}$ contains $y(x+1)$ vertices of degree $n-x-1$ and $(q-y)x$ vertices of degree $n-x$. A little modification of $T_{n,q}$ yields a regular graph with prescribed arboricity number as in the following theorem.

Theorem 2.5. *Let* n, x, q, *and* y *be positive integers satisfying* $n = qx + y$, $0 < y < q$. *Then,*

(1) *there exists an* $(n-x)$*-regular graph* G *of order* n *such that* $\mathtt{a}(G) \le q$ *if* x *is odd,*

(2) *there exists an* $(n-x)$*-regular graph* G *of order* n *such that* $\mathtt{a}(G) \le q$ *if* x *and* y *are even,*

(3) *there exists an* $(n-x-1)$*-regular graph* H *of order* n *such that* $\mathtt{a}(G) \le q$ *if* x *is even and* y *is odd.*

Proof. Note that $T_{n,q}$ is a complete q-partite graph of order n consisting of y partite sets of cardinality $x+1$ and $q-y$ partite sets of cardinality x. Let $V_1, V_2, \ldots, V_y$ be partite sets of $T_{n,q}$ of cardinality $x+1$ and $U_1, U_2, \ldots, U_{q-y}$ be partite sets of $T_{n,q}$ of cardinality x. It is clear that $\deg_{T_{n,q}} v = n-x-1$ if $v \in V_1 \cup V_2 \cup \cdots \cup V_y$ and $\deg_{T_{n,q}} u = n-x$ if $u \in U_1 \cup U_2 \cup \cdots \cup U_{q-y}$. For each $i = 1, 2, \ldots y$, let $V_i = \{v_{i1}, v_{i2}, \ldots, v_{i(x+1)}\}$; similarly, for each $j = 1, 2, \ldots, q-y$, let $U_j = \{u_{j1}, u_{j2}, \ldots, u_{jx}\}$.

(1) Suppose that x is odd. Since $|V_i| = x+1$ for each $i = 1, 2, \ldots, y$, an $(n-x)$-regular graph G can be obtained from $T_{n,q}$ by adding $(x+1)/2$ independent edges to each V_i, $1 \le i \le y$. It is clear that $\mathtt{a}(G) \le q$.

(2) Suppose that x and y are even. Let $H = \bigcup_{i=1}^{y} P(V_i)$ where $P(V_i)$ is a path with V_i as its vertex set and $E_i = \{v_{ik}v_{i(k+1)} : 1 \le k \le x\}$ as its edge set. Let $F_1, F_2, \ldots, F_{y/2}$ be defined by $F_k = \{v_{kt}v_{(k+y/2)t} : 2 \le t \le x\}$ and put $F = \bigcup_{k=1}^{y/2} F_k$. Finally, we obtain a graph $G = (T_{n,q} - F) * H$ which is an $(n-x)$-regular graph of order n and $\mathtt{a}(G) \le q$.

(3) Suppose that x is even and y is odd. Then, $n = qx + y$ is odd. If $q - y \ge 2$, then, by Dirac [10], a subgraph of $T_{n,q}$ induced by $U_1 \cup U_2 \cup \cdots \cup U_{q-y}$ contains a Hamiltonian cycle C. Since x is even and C is of order $x(q-y)$, it follows that C is of even order. Let F be a set of $x(q-y)/2$ independent edges of C. Then, $H = T_{n,q} - F$ is an $(n-x-1)$-regular graph of order n and $\mathtt{a}(H) \le q$.

Suppose that $q - y = 1$. Let $F_1 = \{v_{11}u_{11}, v_{12}u_{12}, \ldots, v_{1x}u_{1x}\}$ and $F_2 = \{v_{11}v_{12}, v_{13}v_{14}, \ldots, v_{1(x-1)}v_{1x}\}$. Then, $H = (T_{n,q} - F_1) + F_2$ is an $(n - x - 1)$-regular graph of order n and $\mathtt{a}(H) \leq q$. □

Note that if G is an r-regular graph on $r+s$ vertices where $1 \leq s \leq r$, then $r \geq (r+s)/2$. It follows, by Dirac [10], that G is Hamiltonian.

Theorem 2.6. *Let r and s be integers with $r \geq 3$ and $1 \leq s \leq r-3$. Then,*

$$\min(\mathtt{a}, r^{r+s}) = \left\lceil \frac{r+s}{s+1} \right\rceil. \tag{2.2}$$

Proof. By Lemma 2.4, we have that $\min(\mathtt{a}, r^{r+s}) \geq \lceil (r+s)/(s+1) \rceil$ if $1 \leq s \leq r-3$. We have already shown that $\min(\mathtt{a}, r^{r+s}) = \lceil (r+s)/(s+1) \rceil$ if $s \in \{1, 2\}$. Thus, in order to prove this theorem, it suffices to construct an r-regular graph G of order $r+s$ such that $\mathtt{a}(G) \leq \lceil (r+s)/(s+1) \rceil$ for $r \geq 3$ and $3 \leq s \leq r-3$. Suppose that $3 \leq s \leq r-3$. Let $q = \lceil (r+s)/(s+1) \rceil$ and put $r+s = qx+y$, $0 \leq y \leq q-1$. Then, $x \leq s+1$ and $x = s+1$ if and only if $y = 0$ and $r+s$ is a multiple of $s+1$.

Note that $T_{n,q}$ is a complete q-partite graph of order n consisting of q partite sets of cardinality $s+1$. Suppose that s is odd. Let G be a graph obtained from $T_{n,q}$ by adding $(s+1)/2$ independent edges to each partite set. Then, G is an r-regular graph of order n having $\mathtt{a}(G) = q$.

If s is even, then q is even. By the same argument with 2 in Theorem 2.5, there exists an r-regular graph G obtained from $T_{n,q}$ with $\mathtt{a}(G) = q$.

Now suppose that $y > 0$ and therefore $x \leq s$. Thus, $T_{r+s,q}$ contains y partite sets of cardinality $x+1$ and $q-y$ partite sets of cardinality x. By Theorem 2.5, there exists an $(r+s-x)$-regular graph G of order $r+s$ with $\mathtt{a}(G) \leq q$ if x is odd or both x and y are even and there exists an $(r+s-x-1)$-regular graph H of order $r+s$ with $\mathtt{a}(H) \leq q$ if x is even and y is odd. Since $x \leq s$, it follows that $r+s-x \geq r$. We will consider two cases according to the parity of x and y as in Theorem 2.5.

Case 1. If x is odd or both x and y are even, then, by Theorem 2.5, there exists an $(r+s-x)$-regular graph G of order $r+s$ with $\mathtt{a}(G) \leq q$. Since $x \leq s$, it follows that $r+s-x \geq r$. If $s-x = 0$, then $G \in \mathcal{R}(r^{r+s})$ with $\mathtt{a}(G) \leq q$, as required. Suppose that $s-x$ is even and $s-x \geq 2$. By Dirac [10], G contains enough edge-disjoint Hamiltonian cycles whose removal produces an r-regular graph G' of order $r+s$ and $\mathtt{a}(G') \leq \mathtt{a}(G) \leq q$, as required.

Suppose $s-x$ is odd. Then, either $r+s-x$ or r is odd and therefore $r+s$ is even. Also by Dirac [10], G contains a 1-factor F where $G-F$ is an $(r+s-x-1)$-regular graph and $\mathtt{a}(G-F) \leq \mathtt{a}(G) \leq q$. Since $s-x-1$ is even, the result follows by the same argument as above.

Case 2. If x is even and y is odd, then, by Theorem 2.5, there exists an $(r+s-x-1)$-regular graph H of order $r+s$ and $\mathtt{a}(H) \leq q$. Since $r+s = qx+y$ is odd, it follows that both $r+s-x-1$ and r are even. Since $x \leq s$, it follows that $r+s-x-1 \geq r-1$. Therefore, it follows by the parity that $r+s-x-1 \geq r$. The result follows easily by the same argument as in Case 1.

This completes the proof. □

Chartrand and Kronk [11] obtained a good upper bound for the vertex arboricity. In particular, they proved the following theorem.

Theorem 2.7. *For each graph G, $\mathtt{a}(G) \leq 1 + \lfloor \max \delta(G')/2 \rfloor$, where the maximum is taking over all induced subgraphs G' of G. In particular, $\mathtt{a}(G) \leq 1 + \lfloor r/2 \rfloor$ if G is an r-regular graph.*

In general, the bound given in Theorem 2.7 is not sharp but it is sharp in the class of r-regular graphs of order $n \geq 2r + 2$ as we will prove in the next theorem.

Theorem 2.8. *For $n \geq 2r + 2$, $\max(\mathtt{a}, r^n) = 1 + \lfloor r/2 \rfloor$.*

Proof. The result follows easily if $r \in \{0, 1, 2\}$. Suppose that $r \geq 3$. We write $n = (r+1)q+t$, $0 \leq t \leq r$. Thus, $q \geq 2$. Note that an r-regular graph of order n exists implying rn must be even. Thus, if r is odd, then n must be even and hence t must be even. Put $G = (q-1)K_{r+1} \cup H$, where H is an r-regular graph of order $r + t + 1$. Since $\mathtt{a}(K_{r+1}) = \lceil (r+1)/2 \rceil = 1 + \lfloor r/2 \rfloor$ and $\mathtt{a}(H) \leq 1 + \lfloor r/2 \rfloor$, it follows that $\mathtt{a}(G) = 1 + \lfloor r/2 \rfloor$. □

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