Research Article

A New Hybrid Algorithm for a Pair of Quasi- ϕ -Asymptotically Nonexpansive Mappings and Generalized Mixed Equilibrium Problems in Banach Spaces

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The purpose of this paper is, by using a new hybrid method, to prove a strong convergence theorem for finding a common element of the set of solutions for a generalized mixed equilibrium problem, the set of solutions for a variational inequality problem, and the set of common fixed points for a pair of quasi- ϕ -asymptotically nonexpansive mappings. Under suitable conditions some strong convergence theorems are established in a uniformly smooth and strictly convex Banach space with Kadec-Klee property. The results presented in the paper improve and extend some recent results.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. We also assume that E is a real Banach space, E^* is the dual space of E , C is a nonempty closed convex subset of E , and $\langle \cdot, \cdot \rangle$ is the pairing between E and E^* .

Let $\varphi : C \rightarrow \mathbb{R}$ be a real-valued function, $\Theta : C \times C \rightarrow \mathbb{R}$ a bifunction, and $A : C \rightarrow E^*$ a nonlinear mapping. The “so-called” generalized mixed equilibrium problem is to find $u \in C$ such that

$$\Theta(u, y) + \langle Au, y - u \rangle + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C. \quad (1.1)$$

The set of solutions for (1.1) is denoted by Ω , that is,

$$\Omega = \{u \in C : \Theta(u, y) + \langle Au, y - u \rangle + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C\}. \quad (1.2)$$

Special examples are as follows.

(I) If $\varphi = 0$, the problem (1.1) is equivalent to finding $u \in C$ such that

$$\Theta(u, y) + \langle Au, y - u \rangle \geq 0, \quad \forall y \in C, \quad (1.3)$$

which is called the generalized equilibrium problem. The set of solutions for (1.3) is denoted by GEP.

(II) If $A = 0$, the problem (1.1) is equivalent to finding $u \in C$ such that

$$\Theta(u, y) + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C, \quad (1.4)$$

which is called the mixed equilibrium problem (MEP) [1]. The set of solutions for (1.4) is denoted by MEP.

(III) If $\Theta = 0$, the problem (1.1) is equivalent to finding $u \in C$ such that

$$\langle Au, y - u \rangle + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C, \quad (1.5)$$

which is called the mixed variational inequality of Browder type (VI) [2]. The set of solutions for (1.5) is denoted by $VI(C, A, \varphi)$.

(IV) If $\varphi = 0$ and $A = 0$, the problem (1.1) is equivalent to finding $u \in C$ such that

$$\Theta(u, y) \geq 0, \quad \forall y \in C, \quad (1.6)$$

which is called the equilibrium problem. The set of solutions for (1.6) is denoted by $EP(\Theta)$.

(V) If $\varphi = 0$ and $\Theta = 0$, the problem (1.1) is equivalent to finding $u \in C$ such that

$$\langle Au, y - u \rangle \geq 0, \quad \forall y \in C, \quad (1.7)$$

which is called the variational inequality of Browder type. The set of solutions for (1.7) is denoted by $VI(C, A)$.

The problem (1.1) is very general in the sense that numerous problems in physics, optimization and economics reduce to finding a solution for (1.1). Some methods have been proposed for solving the generalized equilibrium problem and the equilibrium problem in Hilbert space (see, e.g., [3–6]).

A mapping $S : C \rightarrow E$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (1.8)$$

We denote the fixed point set of S by $F(S)$.

In 2008, S. Takahashi and W. Takahashi [6] proved some strong convergence theorems for finding an element or a common element of EP , $EP(f) \cap F(S)$ or $VI(C, A) \cap F(S)$, respectively, in a Hilbert space.

Recently, Takahashi and Zembayashi [7, 8] proved some weak and strong convergence theorems for finding a common element of the set of solutions for equilibrium (1.6) and the set of fixed points of a relatively nonexpansive mapping in a Banach space.

In 2010, Chang et al. [9] proved a strong convergence theorem for finding a common element of the set of solutions for a generalized equilibrium problem (1.3) and the set of common fixed points of a pair of relatively nonexpansive mappings in a Banach space.

Motivated and inspired by [4–9], we intend in this paper, by using a new hybrid method, to prove a strong convergence theorem for finding a common element of the set of solutions for a generalized mixed equilibrium problem (1.1) and the set of common fixed points of a pair of quasi- ϕ -asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property.

2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions which will be needed in proving our main results.

The mapping $J : E \rightarrow 2^{E^*}$ defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| = \|x^*\|\}, \quad x \in E, \quad (2.1)$$

is called the normalized duality mapping. By the Hahn-Banach theorem, $J(x) \neq \emptyset$ for each $x \in E$.

In the sequel, we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively.

A Banach space E is said to be strictly convex if $\|x+y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be uniformly convex if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x+y\|/2 < 1 - \delta$ for all $x, y \in U$ with $\|x-y\| \geq \epsilon$. E is said to be smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t} \quad (2.2)$$

exists for all $x, y \in U$. E is said to be uniformly smooth if the above limit exists uniformly in $x, y \in U$.

Remark 2.1. The following basic properties can be found in Cioranescu [10].

- (i) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E .
- (ii) If E is a reflexive and strictly convex Banach space, then J^{-1} is hemicontinuous.
- (iii) If E is a smooth, strictly convex, and reflexive Banach space, then J is singlevalued, one-to-one and onto.
- (iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.
- (v) Each uniformly convex Banach space E has the Kadec-Klee property, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \rightarrow x$.

Next we assume that E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty closed convex subset of E . In the sequel, we always use $\phi : E \times E \rightarrow \mathbb{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (2.3)$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \quad (2.4)$$

Following Alber [11], the generalized projection $\Pi_C : E \rightarrow C$ is defined by

$$\Pi_C(x) = \arg \inf_{y \in C} \phi(y, x), \quad \forall x \in E. \quad (2.5)$$

Lemma 2.2 (see [11, 12]). *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E . Then, the following conclusions hold:*

- (a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in E$;
- (b) if $x \in E$ and $z \in C$, then

$$z = \Pi_C x \iff \langle z - y, Jx - Jz \rangle \geq 0, \quad \forall y \in C; \quad (2.6)$$

- (c) for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.

Remark 2.3. If E is a real Hilbert space H , then $\phi(x, y) = \|x - y\|^2$ and Π_C is the metric projection P_C of H onto C .

Let E be a smooth, strictly, convex and reflexive Banach space, C a nonempty closed convex subset of E , $T : C \rightarrow C$ a mapping, and $F(T)$ the set of fixed points of T . A point $p \in C$ is said to be an asymptotic fixed point of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightarrow p$ and $\|x_n - Tx_n\| \rightarrow 0$. We denoted the set of all asymptotic fixed points of T by $\tilde{F}(T)$.

Definition 2.4 (see [13]). (1) A mapping $T : C \rightarrow C$ is said to be relatively nonexpansive if $F(T) \neq \emptyset$, $F(T) = \tilde{F}(T)$, and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, p \in F(T). \quad (2.7)$$

(2) A mapping $T : C \rightarrow C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ with $x_n \rightarrow x$ and $Tx_n \rightarrow y$, $Tx = y$.

Definition 2.5 (see [14]). (1) A mapping $T : C \rightarrow C$ is said to be quasi- ϕ -nonexpansive if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, p \in F(T). \quad (2.8)$$

(2) A mapping $T : C \rightarrow C$ is said to be quasi- ϕ -asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \quad \forall n \geq 1, x \in C, p \in F(T). \quad (2.9)$$

(3) A pair of mappings $T_1, T_2 : C \rightarrow C$ is said to be uniformly quasi- ϕ -asymptotically nonexpansive if $F(T_1) \cap F(T_2) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ such that for $i = 1, 2$

$$\phi(p, T_i^n x) \leq k_n \phi(p, x), \quad \forall n \geq 1, x \in C, p \in F(T_1) \cap F(T_2). \quad (2.10)$$

(4) A mapping $T : C \rightarrow C$ is said to be uniformly L -Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|, \quad \forall x, y \in C. \quad (2.11)$$

Remark 2.6. (1) From the definition, it is easy to know that each relatively nonexpansive mapping is closed.

(2) The class of quasi- ϕ -asymptotically nonexpansive mappings contains properly the class of quasi- ϕ -nonexpansive mappings as a subclass, and the class of quasi- ϕ -nonexpansive mappings contains properly the class of relatively nonexpansive mappings as a subclass, but the converse may be not true.

Lemma 2.7 (see [15]). *Let E be a uniformly convex Banach space, $r > 0$ a positive number, and $B_r(0)$ a closed ball of E . Then, for any given subset $\{x_1, x_2, \dots, x_N\} \subset B_r(0)$ and for any positive numbers $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ with $\sum_{i=1}^N \lambda_i = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \rightarrow [0, \infty)$ with $g(0) = 0$ such that, for any $i, j \in \{1, 2, \dots, N\}$ with $i < j$,*

$$\left\| \sum_{n=1}^N \lambda_n x_n \right\|^2 \leq \sum_{n=1}^N \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \quad (2.12)$$

Lemma 2.8 (see [15]). *Let E be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E . Let $T : C \rightarrow C$ be a closed and quasi- ϕ -asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \rightarrow 1$. Then $F(T)$ is a closed convex subset of C .*

For solving the generalized mixed equilibrium problem (1.1), let us assume that the function $\psi : C \rightarrow \mathbb{R}$ is convex and lower semicontinuous, the nonlinear mapping $A : C \rightarrow E^*$ is continuous and monotone, and the bifunction $\Theta : C \times C \rightarrow \mathbb{R}$ satisfies the following conditions:

- (A₁) $\Theta(x, x) = 0$, for all $x \in C$,
- (A₂) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$, $\forall x, y \in C$,
- (A₃) $\limsup_{t \downarrow 0} \Theta(x + t(z - x), y) \leq \Theta(x, y) \quad \forall x, z, y \in C$,
- (A₄) the function $y \mapsto \Theta(x, y)$ is convex and lower semicontinuous.

Lemma 2.9. Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E . Let $\Theta : C \times C \rightarrow \mathbb{R}$ a bifunction satisfying the conditions (A_1) – (A_4) . Let $r > 0$ and $x \in E$. Then, the followings hold.

(i) (Blum and Oettli [3]) there exists $z \in C$ such that

$$\Theta(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \quad (2.13)$$

(ii) (Takahashi and Zembayashi [8]) Define a mapping $T_r : E \rightarrow C$ by

$$T_r(x) = \left\{ z \in C : \Theta(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C \right\}, \quad x \in E. \quad (2.14)$$

Then, the following conclusions hold:

(a) T_r is single-valued,

(b) T_r is a firmly nonexpansive-type mapping, that is, $\forall z, y \in E$,

$$\langle T_r z - T_r y, JT_r z - JT_r y \rangle \leq \langle T_r z - T_r y, Jz - Jy \rangle, \quad (2.15)$$

(c) $F(T_r) = \text{EP}(\Theta) = \tilde{F}(T_r)$,

(d) $\text{EP}(\Theta)$ is closed and convex,

(e) $\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x)$, $\forall q \in F(T_r)$.

Lemma 2.10 (see [16]). Let E be a smooth, strictly convex, and reflexive Banach space, and C a nonempty closed convex subset of E . Let $A : C \rightarrow E^*$ be a continuous and monotone mapping, $\varphi : C \rightarrow \mathbb{R}$ a lower semicontinuous and convex function, and $\Theta : C \times C \rightarrow \mathbb{R}$ a bifunction satisfying conditions (A_1) – (A_4) . Let $r > 0$ be any given number and $x \in E$ any given point. Then, the following hold.

(i) There exists $u \in C$ such that

$$\Theta(u, y) + \langle Au, y - u \rangle + \varphi(y) - \varphi(u) + \frac{1}{r} \langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C. \quad (2.16)$$

(ii) If we define a mapping $K_r : C \rightarrow C$ by

$$K_r(x) = \left\{ u \in C : \Theta(u, y) + \langle Au, y - u \rangle + \varphi(y) - \varphi(u) + \frac{1}{r} \langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C \right\}, \quad \forall x \in C. \quad (2.17)$$

Then, the mapping K_r has the following properties:

- (a) K_r is single valued,
- (b) K_r is a firmly nonexpansive-type mapping, that is,

$$\langle K_r z - K_r y, JK_r z - JK_r y \rangle \leq \langle K_r z - K_r y, Jz - Jy \rangle, \quad \forall z, y \in E, \quad (2.18)$$

- (c) $F(K_r) = \Omega = \tilde{F}(K_r)$,
- (d) Ω is closed and convex,
- (e)

$$\phi(q, K_r z) + \phi(K_r z, z) \leq \phi(q, z), \quad \forall q \in F(K_r), z \in E. \quad (2.19)$$

Remark 2.11. It follows from Lemma 2.9 that the mapping K_r is a relatively nonexpansive mapping. Thus, it is quasi- ϕ -nonexpansive.

3. Main Results

In this section, we will prove a strong convergence theorem for finding a common element of the set of solutions for the generalized mixed equilibrium problem (1.1) and the set of common fixed points for a pair of quasi- ϕ -asymptotically nonexpansive mappings in Banach spaces.

Theorem 3.1. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E . Let $A : C \rightarrow E^*$ be a continuous and monotone mapping, $\psi : C \rightarrow \mathbb{R}$ a lower semicontinuous and convex function, and $\Theta : C \times C \rightarrow \mathbb{R}$ a bifunction satisfying conditions (A_1) – (A_4) . Let $S, T : C \rightarrow C$ be two closed and uniformly quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$. Suppose that S and T are uniformly L -Lipschitz continuous and that $G = F(T) \cap F(S) \cap \Omega$ is a nonempty and bounded subset in C . Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} x_0 &\in C, \quad C_0 = C, \quad Q_0 = C, \\ z_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n), \\ y_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n)JS^n z_n), \\ u_n &\in C \text{ such that, } \quad \forall y \in C, \\ \Theta(u_n, y) + \langle Au_n, y - u_n \rangle + \psi(y) - \psi(u_n) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle &\geq 0, \\ C_n &= \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n, \phi(v, u_n) \leq \phi(v, x_n) + (1 + k_n)(1 - \beta_n)\xi_n\}, \\ Q_n &= \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \quad (3.1)$$

where $J : E \rightarrow E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\xi_n = \sup_{u \in G} (k_n - 1)\phi(u, x_n)$. Suppose that the following conditions are satisfied:

$$(i) \liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0,$$

$$(ii) \liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0.$$

Then $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap F(T) \cap \Omega} x_0$, where $\Pi_{F(S) \cap F(T) \cap \Omega}$ is the generalized projection of E onto $F(S) \cap F(T) \cap \Omega$.

Proof. Firstly, we define two functions $H : C \times C \rightarrow \mathbb{R}$ and $K_r : C \rightarrow C$ by

$$\begin{aligned} H(x, y) &= \Theta(x, y) + \langle Ax, y - x \rangle + \psi(y) - \psi(x), \quad \forall x, y \in C, \\ K_r(x) &= \left\{ u \in C : H(u, y) + \frac{1}{r} \langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C \right\}, \quad x \in C. \end{aligned} \quad (3.2)$$

By Lemma 2.10, we know that the function H satisfies conditions (A_1) – (A_4) and K_r has properties (a)–(e). Therefore, (3.1) is equivalent to

$$\begin{aligned} x_0 &\in C, \quad C_0 = C, \quad Q_0 = C, \\ z_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n), \\ y_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n)JS^n z_n), \\ u_n &\in C \text{ such that, } \quad \forall y \in C, \\ H(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle &\geq 0, \\ C_n &= \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n, \phi(v, u_n) \leq \phi(v, x_n) + (1 + k_n)(1 - \beta_n)\xi_n\}, \\ Q_n &= \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0. \end{aligned} \quad (3.3)$$

We divide the proof of Theorem 3.1 into five steps.

(I) First we prove that C_n and Q_n are both closed and convex subsets of C for all $n \geq 0$.

In fact, it is obvious that Q_n is closed and convex for all $n \geq 0$. Again we have that

$$\begin{aligned} \phi(v, z_n) \leq \phi(v, x_n) + \xi_n &\iff 2\langle v, Jx_n - Jz_n \rangle \leq \|x_n\|^2 - \|z_n\|^2 + \xi_n, \\ \phi(v, u_n) \leq \phi(v, x_n) + (1 + k_n)(1 - \beta_n)\xi_n &\iff 2\langle v, Jx_n - Ju_n \rangle \\ &\leq \|x_n\|^2 - \|u_n\|^2 + (1 + k_n)(1 - \beta_n)\xi_n. \end{aligned} \quad (3.4)$$

Hence $C_n, \forall n \geq 0$, is closed and convex, and so $C_n \cap Q_n$ is closed and convex for all $n \geq 0$.

(II) Next we prove that $F(T) \cap F(S) \cap \Omega \subset C_n \cap Q_n, \forall n \geq 0$.

Putting $u_n = K_{r_n} y_n, \forall n \geq 0$, by Lemma 2.10 and Remark 2.11, K_{r_n} is relatively nonexpansive. Again since S and T are quasi- ϕ -asymptotically nonexpansive, for any given $u \in F(S) \cap F(T) \cap \Omega$, we have that

$$\begin{aligned}
 \phi(u, z_n) &= \phi\left(u, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n)\right) \\
 &= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1 - \alpha_n)JT^n x_n \rangle + \|\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n\|^2 \\
 &\leq \|u\|^2 - 2\alpha_n \langle u, Jx_n \rangle - 2(1 - \alpha_n) \langle u, JT^n x_n \rangle + \alpha_n \|x_n\|^2 \\
 &\quad + (1 - \alpha_n) \|JT^n x_n\|^2 - \alpha_n(1 - \alpha_n)g(\|Jx_n - JT^n x_n\|) \\
 &= \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, JT^n x_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JT^n x_n\|) \\
 &\leq \alpha_n \phi(u, x_n) + (1 - \alpha_n)k_n \phi(u, x_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JT^n x_n\|) \\
 &\leq k_n \phi(u, x_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JT^n x_n\|) \\
 &\leq \phi(u, x_n) + \sup_{p \in G} (k_n - 1) \phi(p, x_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JT^n x_n\|) \\
 &= \phi(u, x_n) + \xi_n - \alpha_n(1 - \alpha_n)g(\|Jx_n - JT^n x_n\|) \\
 &\leq \phi(u, x_n) + \xi_n.
 \end{aligned} \tag{3.5}$$

From (3.5) we have that

$$\begin{aligned}
 \phi(u, u_n) &= \phi(u, K_{r_n} y_n) \leq \phi(u, y_n) \\
 &\leq \phi\left(u, J^{-1}(\beta_n Jx_n + (1 - \beta_n)JS^n z_n)\right) \\
 &= \|u\|^2 - 2\langle u, \beta_n Jx_n + (1 - \beta_n)JS^n z_n \rangle + \|\beta_n Jx_n + (1 - \beta_n)JS^n z_n\|^2 \\
 &\leq \|u\|^2 - 2\beta_n \langle u, Jx_n \rangle - 2(1 - \beta_n) \langle u, JS^n z_n \rangle + \beta_n \|x_n\|^2 \\
 &\quad + (1 - \beta) \|JS^n z_n\|^2 - \beta_n(1 - \beta_n)g(\|Jx_n - JS^n z_n\|) \\
 &= \beta_n \phi(u, x_n) + (1 - \beta_n) \phi(u, JS^n z_n) - \beta_n(1 - \beta_n)g(\|Jx_n - JS^n z_n\|) \\
 &\leq \beta_n \phi(u, x_n) + (1 - \beta_n)k_n \phi(u, z_n) - \beta_n(1 - \beta_n)g(\|Jx_n - JS^n z_n\|) \\
 &\leq \beta_n \phi(u, x_n) + (1 - \beta_n)k_n (\phi(u, x_n) + \xi_n) - \beta_n(1 - \beta_n)g(\|Jx_n - JS^n z_n\|) \\
 &\leq \beta_n \phi(u, x_n) + (1 - \beta_n)(\phi(u, x_n) + \xi_n) + (1 - \beta_n)k_n \xi_n - \beta_n(1 - \beta_n)g(\|Jx_n - JS^n z_n\|) \\
 &\leq \phi(u, x_n) + (1 - \beta_n)\xi_n + (1 - \beta_n)k_n \xi_n - \beta_n(1 - \beta_n)g(\|Jx_n - JS^n z_n\|) \\
 &\leq \phi(u, x_n) + (1 + k_n)(1 - \beta_n)\xi_n - \beta_n(1 - \beta_n)g(\|Jx_n - JS^n z_n\|) \\
 &\leq \phi(u, x_n) + (1 + k_n)(1 - \beta_n)\xi_n \quad \forall n \geq 0.
 \end{aligned} \tag{3.6}$$

This implies that $u \in C_n, \forall n \geq 0$, and so $F(T) \cap F(S) \cap \Omega \subset C_n, \forall n \geq 0$.

Now we prove that $F(T) \cap F(S) \cap \Omega \subset C_n \cap Q_n, \forall n \geq 0$.

In fact, from $Q_0 = C$, we have that $F(T) \cap F(S) \cap \Omega \subset C_0 \cap Q_0$. Suppose that $F(T) \cap F(S) \cap \Omega \subset C_k \cap Q_k$, for some $k \geq 0$. Now we prove that $F(T) \cap F(S) \cap \Omega \subset C_{k+1} \cap Q_{k+1}$. In fact, since $x_{k+1} = \Pi_{C_k \cap Q_k} x_0$, we have that

$$\langle x_{k+1} - z, Jx_0 - Jx_{k+1} \rangle \geq 0, \quad \forall z \in C_k \cap Q_k. \quad (3.7)$$

Since $F(T) \cap F(S) \cap \Omega \subset C_k \cap Q_k$, for any $z \in F(T) \cap F(S) \cap \Omega$, we have that

$$\langle x_{k+1} - z, Jx_0 - Jx_{k+1} \rangle \geq 0. \quad (3.8)$$

This shows that $z \in Q_{k+1}$, and so $F(T) \cap F(S) \cap \Omega \subset Q_{k+1}$. The conclusion is proved.

(III) Now we prove that $\{x_n\}$ is bounded.

From the definition of Q_n , we have that $x_n = \Pi_{Q_n} x_0, \forall n \geq 0$. Hence, from Lemma 2.2(1),

$$\begin{aligned} \phi(x_n, x_0) &= \phi(\Pi_{Q_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{Q_n} x_0) \\ &\leq \phi(u, x_0), \quad \forall u \in F(T) \cap F(S) \cap \Omega \subset Q_n, \forall n \geq 0. \end{aligned} \quad (3.9)$$

This implies that $\{\phi(x_n, x_0)\}$ is bounded. By virtue of (2.4), $\{x_n\}$ is bounded. Denote

$$M = \sup_{n \geq 0} \{\|x_n\|\} < \infty. \quad (3.10)$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n \cap Q_n \subset Q_n$ and $x_n = \Pi_{Q_n} x_0$, from the definition of Π_{Q_n} , we have that

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \leq (M + \|x_0\|)^2, \quad \forall n \geq 0. \quad (3.11)$$

This implies that $\{\phi(x_n, x_0)\}$ is nondecreasing, and so the limit $\lim_{n \rightarrow \infty} \phi(x_n, x_0)$ exists. Without loss of generality, we can assume that

$$\lim_{n \rightarrow \infty} \phi(x_n, x_0) = r \geq 0. \quad (3.12)$$

By the way, from the definition of $\{\xi_n\}$, (2.4), and (3.10), it is easy to see that

$$\xi_n = \sup_{u \in G} (k_n - 1) \phi(u, x_n) \leq \sup_{u \in G} (k_n - 1) (\|u\| + M)^2 \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \quad (3.13)$$

(IV) Now, we prove that $\{x_n\}$ converges strongly to some point $p \in G = F(T) \cap F(S) \cap \Omega$.

In fact, since $\{x_n\}$ is bounded in C and E is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p$. Again since Q_n is closed and convex for each $n \geq 0$, it is weakly

closed, and so $p \in Q_n$ for each $n \geq 0$. Since $x_n = \Pi_{Q_n} x_0$, from the definition of Π_{Q_n} , we have that

$$\phi(x_{n_i}, x_0) \leq \phi(p, x_0), \quad n \geq 0. \quad (3.14)$$

Since

$$\begin{aligned} \liminf_{n_i \rightarrow \infty} \phi(x_{n_i}, x_0) &= \liminf_{n_i \rightarrow \infty} \left\{ \|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0 \rangle + \|x_0\|^2 \right\} \\ &\geq \|p\|^2 - 2\langle p, Jx_0 \rangle + \|x_0\|^2 = \phi(p, x_0), \end{aligned} \quad (3.15)$$

we have that

$$\phi(p, x_0) \leq \liminf_{n_i \rightarrow \infty} \phi(x_{n_i}, x_0) \leq \limsup_{n_i \rightarrow \infty} \phi(x_{n_i}, x_0) \leq \phi(p, x_0). \quad (3.16)$$

This implies that $\lim_{n_i \rightarrow \infty} \phi(x_{n_i}, x_0) = \phi(p, x_0)$, that is, $\|x_{n_i}\| \rightarrow \|p\|$. In view of the Kadec-Klee property of E , we obtain that $\lim_{n \rightarrow \infty} x_{n_i} = p$.

Now we first prove that $x_n \rightarrow p$ ($n \rightarrow \infty$). In fact, if there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightarrow q$, then we have that

$$\begin{aligned} \phi(p, q) &= \lim_{n_i \rightarrow \infty, n_j \rightarrow \infty} \phi(x_{n_i}, x_{n_j}) \leq \lim_{n_i \rightarrow \infty, n_j \rightarrow \infty} \phi(x_{n_i}, x_0) - \phi(\Pi_{Q_{n_j}} x_0, x_0) \\ &= \lim_{n_i \rightarrow \infty, n_j \rightarrow \infty} \phi(x_{n_i}, x_0) - \phi(x_{n_j}, x_0) = 0 \quad (\text{by (3.12)}). \end{aligned} \quad (3.17)$$

Therefore we have that $p = q$. This implies that

$$\lim_{n \rightarrow \infty} x_n = p. \quad (3.18)$$

Now we first prove that $p \in F(T) \cap F(S)$. In fact, by the construction of Q_n , we have that $x_n = \Pi_{Q_n} x_0$. Therefore, by Lemma 2.2(a) we have that

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{Q_n} x_0) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \end{aligned} \quad (3.19)$$

In view of $x_{n+1} \in C_n \cap Q_n \subset C_n$ and noting the construction of C_n we obtain

$$\begin{aligned} \phi(x_{n+1}, z_n) &\leq \phi(x_{n+1}, x_n) + \xi_n, \\ \phi(x_{n+1}, u_n) &\leq \phi(x_{n+1}, x_n) + (1 + k_n)(1 - \beta_n)\xi_n. \end{aligned} \quad (3.20)$$

From (3.13) and (3.19), we have that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, u_n) = 0, \quad \lim_{n \rightarrow \infty} \phi(x_{n+1}, z_n) = 0. \quad (3.21)$$

From (2.4) it yields that $(\|x_{n+1}\| - \|u_n\|)^2 \rightarrow 0$ and $(\|x_{n+1}\| - \|z_n\|)^2 \rightarrow 0$. Since $\|x_{n+1}\| \rightarrow \|p\|$, we have that

$$\|u_n\| \rightarrow \|p\|, \quad \|z_n\| \rightarrow \|p\| \quad (\text{as } n \rightarrow \infty). \quad (3.22)$$

Hence, we have that

$$\|Ju_n\| \rightarrow \|Jp\|, \quad \|Jz_n\| \rightarrow \|Jp\| \quad (\text{as } n \rightarrow \infty). \quad (3.23)$$

This implies that $\{Jz_n\}$ is bounded in E^* . Since E is reflexive, and so E^* is reflexive, there exists a subsequence $\{Jz_{n_i}\} \subset \{Jz_n\}$ such that $Jz_{n_i} \rightharpoonup p_0 \in E^*$. In view of the reflexivity of E , we see that $J(E) = E^*$. Hence, there exists $x \in E$ such that $Jx = p_0$. Since

$$\phi(x_{n_i+1}, z_{n_i}) = \|x_{n_i+1}\|^2 - 2\langle x_{n_i+1}, Jz_{n_i} \rangle + \|z_{n_i}\|^2 = \|x_{n_i+1}\|^2 - 2\langle x_{n_i+1}, Jz_{n_i} \rangle + \|Jz_{n_i}\|^2, \quad (3.24)$$

taking $\liminf_{n \rightarrow \infty}$ on both sides of the equality above and in view of the weak lower semi-continuity of norm $\|\cdot\|$, it yields that

$$\begin{aligned} 0 &\geq \|p\|^2 - 2\langle p, p_0 \rangle + \|p_0\|^2 = \|p\|^2 - 2\langle p, Jx \rangle + \|Jx\|^2 \\ &= \|p\|^2 - 2\langle p, Jx \rangle + \|x\|^2 = \phi(p, x), \end{aligned} \quad (3.25)$$

that is, $p = x$. This implies that $p_0 = Jp$, and so $Jz_n \rightharpoonup Jp$. It follows from (3.23) and the Kadec-Klee property of E^* that $Jz_{n_i} \rightarrow Jp$ (as $n \rightarrow \infty$). Noting that $J^{-1} : E^* \rightarrow E$ is hemicon-
tinuous, it yields that $z_{n_i} \rightarrow p$. It follows from (3.22) and the Kadec-Klee property of E that $\lim_{n_i \rightarrow \infty} z_{n_i} = p$.

By the same way as given in the proof of (3.18), we can also prove that

$$\lim_{n \rightarrow \infty} z_n = p. \quad (3.26)$$

From (3.18) and (3.26), we have that

$$\|x_n - z_n\| \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (3.27)$$

Since J is uniformly continuous on any bounded subset of E , we have that

$$\|Jx_n - Jz_n\| \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (3.28)$$

For any $u \in F(T) \cap F(S) \cap \Omega$, it follows from (3.5) that

$$\alpha_n(1 - \alpha_n)g(\|Jx_n - T^n x_n\|) \leq \phi(u, x_n) - \phi(u, z_n) + \xi_n. \quad (3.29)$$

Since

$$\begin{aligned}\phi(u, x_n) - \phi(u, z_n) &= \|x_n\|^2 - \|z_n\|^2 - 2\langle u, Jx_n - Jz_n \rangle \\ &\leq \left| \|x_n\|^2 - \|z_n\|^2 \right| + 2\|u\| \cdot \|Jx_n - Jz_n\| \\ &\leq \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|u\| \cdot \|Jx_n - Jz_n\|,\end{aligned}\tag{3.30}$$

From (3.27) and (3.28), it follows that

$$\phi(u, x_n) - \phi(u, z_n) \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty).\tag{3.31}$$

In view of condition (i) and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$, we see that

$$g(\|Jx_n - JT^n x_n\|) \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty).\tag{3.32}$$

It follows from the property of g that

$$\|Jx_n - JT^n x_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty).\tag{3.33}$$

Since $x_n \rightarrow p$ and J is uniformly continuous, it yields that $Jx_n \rightarrow Jp$. Hence from (3.33) we have that

$$JT^n x_n \longrightarrow Jp \quad (\text{as } n \longrightarrow \infty).\tag{3.34}$$

Since $J^{-1} : E^* \rightarrow E$ is hemicontinuous, it follows that

$$T^n x_n \rightharpoonup p.\tag{3.35}$$

On the other hand, we have that

$$\left| \|T^n x_n\| - \|p\| \right| = \left| \|J(T^n x_n)\| - \|Jp\| \right| \leq \|JT^n x_n - Jp\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty).\tag{3.36}$$

This together with (3.35) shows that

$$T^n x_n \longrightarrow p.\tag{3.37}$$

Furthermore, by the assumption that T is uniformly L -Lipschitz continuous, we have that

$$\begin{aligned}\|T^{n+1} x_n - T^n x_n\| &\leq \|T^{n+1} x_n - T^{n+1} x_{n+1}\| + \|T^{n+1} x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - T^n x_n\| \\ &\leq (L + 1)\|x_{n+1} - x_n\| + \|T^{n+1} x_{n+1} - x_{n+1}\| + \|x_n - T^n x_n\|.\end{aligned}\tag{3.38}$$

This together with (3.18) and (3.37), yields $\|T^{n+1}x_n - T^n x_n\| \rightarrow 0$ (as $n \rightarrow \infty$). Hence from (3.37) we have that $T^{n+1}x_n \rightarrow p$, that is, $TT^n x_n \rightarrow p$. In view of (3.37) and the closeness of T , it yields that $Tp = p$. This implies that $p \in F(T)$.

By the same way as given in the proof of (3.23) to (3.31), we can also prove that

$$\lim_{n \rightarrow \infty} u_n = p, \quad \phi(u, x_n) - \phi(u, u_n) \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (3.39)$$

Since $u_n = K_{r_n} y_n$, from (2.19), (3.6), (3.13), and (3.39), we have that

$$\begin{aligned} \phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \leq \phi(u, y_n) - \phi(u, u_n) \\ &\leq \phi(u, x_n) - \phi(u, u_n) + (1 + k_n)(1 - \beta_n)\xi_n \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \end{aligned} \quad (3.40)$$

From (2.4) it yields that $(\|u_n\| - \|y_n\|)^2 \rightarrow 0$. Since $\|u_n\| \rightarrow \|p\|$, we have that

$$\|y_n\| \rightarrow \|p\| \quad (\text{as } n \rightarrow \infty). \quad (3.41)$$

Hence we have that

$$\|Jy_n\| \rightarrow \|Jp\| \quad (\text{as } n \rightarrow \infty). \quad (3.42)$$

By the same way as given in the proof of (3.26), we can also prove that

$$\lim_{n \rightarrow \infty} y_n = p. \quad (3.43)$$

From (3.39) and (3.43) we have that

$$\|u_n - y_n\| \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (3.44)$$

Since J is uniformly continuous on any bounded subset of E , we have that

$$\|Ju_n - Jy_n\| \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (3.45)$$

For any $u \in F(T) \cap F(S) \cap \Omega$, it follows from (3.6), (3.13), and (3.39) that

$$\beta_n(1 - \beta_n)g(\|Jx_n - S^n z_n\|) \leq \phi(u, x_n) - \phi(u, u_n) + (1 + k_n)(1 - \beta_n)\xi_n \rightarrow 0. \quad (3.46)$$

In view of condition (ii) and $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$, we see that

$$g(\|Jx_n - JS^n z_n\|) \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (3.47)$$

It follows from the property of g that

$$\|Jx_n - JS^n z_n\| \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (3.48)$$

Since $x_n \rightarrow p$ and J is uniformly continuous, it yields, $Jx_n \rightarrow Jp$. Hence from (3.48) we have that

$$JS^n z_n \rightarrow Jp \quad (\text{as } n \rightarrow \infty). \quad (3.49)$$

Since $J^{-1} : E^* \rightarrow E$ is hemicontinuous, it follows that

$$S^n z_n \rightarrow p. \quad (3.50)$$

On the other hand, we have that

$$|\|S^n z_n\| - \|p\|| = |\|J(S^n z_n)\| - \|Jp\|| \leq \|JS^n z_n - Jp\| \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (3.51)$$

This together with (3.50) shows that

$$S^n z_n \rightarrow p. \quad (3.52)$$

Furthermore, by the assumption that S is uniformly L -Lipschitz continuous, we have that

$$\begin{aligned} \|S^{n+1} z_n - S^n z_n\| &\leq \|S^{n+1} z_n - S^{n+1} z_{n+1}\| + \|S^{n+1} z_{n+1} - z_{n+1}\| + \|z_{n+1} - z_n\| + \|z_n - S^n z_n\| \\ &\leq (L+1)\|z_{n+1} - z_n\| + \|S^{n+1} z_{n+1} - z_{n+1}\| + \|z_n - S^n z_n\|. \end{aligned} \quad (3.53)$$

This together with (3.26) and (3.52), yields that $\|S^{n+1} z_n - S^n z_n\| \rightarrow 0$ (as $n \rightarrow \infty$). Hence from (3.52) we have that $S^{n+1} z_n \rightarrow p$, that is, $SS^n z_n \rightarrow p$. In view of (3.52) and the closeness of T , it yields that $Sp = p$. This implies that $p \in F(S)$.

Next we prove that $p \in \Omega$. From (3.45) and the assumption that $r_n \geq a$, $\forall n \geq 0$, we have that

$$\lim_{n \rightarrow \infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \quad (3.54)$$

Since $u_n = K_{r_n} y_n$, we have that

$$H(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C. \quad (3.55)$$

Replacing n by n_k in (3.55), from condition (A_2) , we have that

$$\frac{1}{r_{n_k}} \langle y - u_{n_k}, Ju_{n_k} - Jy_{n_k} \rangle \geq -H(u_{n_k}, y) \geq H(y, u_{n_k}), \quad \forall y \in C. \quad (3.56)$$

By the assumption that $y \mapsto H(x, y)$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \rightarrow \infty$ in (3.55), from (3.54) and condition (A_4) , we have that $H(y, p) \leq 0, \forall y \in C$.

For $t \in (0, 1]$ and $y \in C$, letting $y_t = ty + (1 - t)p$, there are $y_t \in C$ and $H(y_t, p) \leq 0$. By conditions (A_1) and (A_4) , we have that

$$0 = H(y_t, y_t) \leq tH(y_t, y) + (1 - t)H(y_t, p) \leq tH(y_t, y). \quad (3.57)$$

Dividing both sides of the above equation by t , we have that $H(y_t, y) \geq 0, \forall y \in C$. Letting $t \downarrow 0$, from condition (A_3) , we have that $H(p, y) \geq 0, \forall y \in C$, that is, $\Theta(p, y) + \langle Ap, y - p \rangle + \psi(y) - \psi(p) \geq 0, \forall y \in C$. Therefore $p \in \Omega$, and so $p \in F(T) \cap F(S) \cap \Omega$.

(V) Finally, we prove that $x_n \rightarrow \Pi_{F(T) \cap F(S) \cap \Omega} x_0$.

Let $w = \Pi_{F(T) \cap F(S) \cap \Omega} x_0$. From $w \in F(T) \cap F(S) \cap \Omega \subset C_n \cap Q_n$, and $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$, we have that

$$\phi(x_{n+1}, x_0) \leq \phi(w, x_0), \quad \forall n \geq 0. \quad (3.58)$$

Since the norm is weakly lower semicontinuous, this implies that

$$\begin{aligned} \phi(p, x_0) &= \|p\|^2 - 2\langle p, Jx_0 \rangle + \|x_0\|^2 \leq \liminf_{n_k \rightarrow \infty} \left\{ \|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0 \rangle + \|x_0\|^2 \right\} \\ &\leq \liminf_{n_k \rightarrow \infty} \phi(x_{n_k}, x_0) \leq \limsup_{n_k \rightarrow \infty} \phi(x_{n_k}, x_0) \leq \phi(w, x_0). \end{aligned} \quad (3.59)$$

It follows from the definition of $\Pi_{F(T) \cap F(S) \cap \Omega} x_0$ and (3.59) that we have $p = w$. Therefore, $x_n \rightarrow \Pi_{F(T) \cap F(S) \cap \Omega} x_0$. This completes the proof of Theorem 3.1. $\square$

Remark 3.2. Theorem 3.1 improves and extends the corresponding results in [7–9].

- (a) For the framework of spaces, we extend the space from a uniformly smooth and uniformly convex Banach space to a uniformly smooth and strictly convex Banach space with the Kadec-Klee property (note that each uniformly convex Banach space must have the Kadec-Klee property).
- (b) For the mappings, we extend the mappings from nonexpansive mappings, relatively nonexpansive mappings, or weak relatively nonexpansive mappings to a pair of quasi- ϕ -asymptotically nonexpansive mappings.
- (c) For the equilibrium problem, we extend the generalized equilibrium problem to the generalized mixed equilibrium problem.

The following theorems can be obtained from Theorem 3.1 immediately.

Theorem 3.3. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E . Let $A : C \rightarrow E^*$ be a continuous and monotone mapping and $\Theta : C \times C \rightarrow \mathbb{R}$ a bifunction satisfying conditions (A_1) – (A_4) . Let $S, T : C \rightarrow C$ be two closed and uniformly quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$. Suppose that S and T are uniformly L -Lipschitz continuous and that*

$G = F(T) \cap F(S) \cap \text{GEP}$ is a nonempty and bounded subset in C . Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned}
 x_0 &\in C, \quad C_0 = C, \quad Q_0 = C, \\
 z_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n), \\
 y_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n)JS^n z_n), \\
 u_n &\in C \text{ such that, } \forall y \in C, \\
 \Theta(u_n, y) + \langle Au_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle &\geq 0, \\
 C_n &= \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n, \phi(v, u_n) \leq \phi(v, x_n) + (1 + k_n)(1 - \beta_n)\xi_n\}, \\
 Q_n &= \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
 x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
 \end{aligned} \tag{3.60}$$

where $J : E \rightarrow E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\xi_n = \sup_{u \in G} (k_n - 1)\phi(u, x_n)$. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions (i)-(ii) in Theorem 3.1, then $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap F(T) \cap \text{GEP}} x_0$, where GEP is the set for the solutions of generalized equilibrium problem (1.3).

Proof. Putting $\varphi = 0$ in Theorem 3.1, the conclusion of Theorem 3.3 can be obtained from Theorem 3.1. $\square$

Theorem 3.4. Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E . Let $\varphi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex function and $\Theta : C \times C \rightarrow \mathbb{R}$ a bifunction satisfying conditions (A_1) – (A_4) . Let $S, T : C \rightarrow C$ be two closed and uniformly quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$. Suppose that S and T are uniformly L -Lipschitz continuous and that $G = F(T) \cap F(S) \cap \text{MEP}$ is a nonempty and bounded subset in C . Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned}
 x_0 &\in C, \quad C_0 = C, \quad Q_0 = C, \\
 z_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n), \\
 y_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n)JS^n z_n), \\
 u_n &\in C \text{ such that, } \forall y \in C, \\
 \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle &\geq 0, \\
 C_n &= \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n, \phi(v, u_n) \leq \phi(v, x_n) + (1 + k_n)(1 - \beta_n)\xi_n\}, \\
 Q_n &= \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
 x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
 \end{aligned} \tag{3.61}$$

where $J : E \rightarrow E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\xi_n = \sup_{u \in G} (k_n - 1)\phi(u, x_n)$. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions

(i)-(ii) in Theorem 3.1, then $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap F(T) \cap \text{MEP}} x_0$, where MEP is the set of solutions for mixed equilibrium problem (1.4).

Proof. Putting $A = 0$ in Theorem 3.1, the conclusion of Theorem 3.4 can be obtained from Theorem 3.1. $\square$

Theorem 3.5. Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E . Let $A : C \rightarrow E^*$ be a continuous and monotone mapping and $\varphi : C \rightarrow \mathbb{R}$ a lower semicontinuous and convex function. Let $S, T : C \rightarrow C$ be two closed and uniformly quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$. Suppose that S and T are uniformly L -Lipschitz continuous and that $G = F(T) \cap F(S) \cap \text{VI}(C, A, \varphi)$ is a nonempty and bounded subset in C . Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &\in C, \quad C_0 = C, \quad Q_0 = C, \\ z_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n), \\ y_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n)JS^n z_n), \\ u_n &\in C \text{ such that, } \forall y \in C, \\ \langle Au_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle &\geq 0, \\ C_n &= \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n, \phi(v, u_n) \leq \phi(v, x_n) + (1 + k_n)(1 - \beta_n)\xi_n\}, \\ Q_n &= \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \tag{3.62}$$

where $J : E \rightarrow E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\xi_n = \sup_{u \in G} (k_n - 1)\phi(u, x_n)$. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions (i)-(ii) in Theorem 3.1, then $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap F(T) \cap \text{VI}(C, A, \varphi)} x_0$, where $\text{VI}(C, A, \varphi)$ is the set of solutions for the mixed variational inequality (1.5).

Proof. Putting $\Theta = 0$ in Theorem 3.1, the conclusion of Theorem 3.5 can be obtained from Theorem 3.1. $\square$

Theorem 3.6. Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E . Let $\Theta : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying conditions (A_1) – (A_4) . Let $S, T : C \rightarrow C$ be two closed and uniformly quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$. Suppose that S and T are uniformly L -Lipschitz continuous and that $G = F(T) \cap F(S) \cap \text{EP}(\Theta)$ is a nonempty and bounded subset in C . Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &\in C, \quad C_0 = C, \quad Q_0 = C, \\ z_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n), \\ y_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n)JS^n z_n), \\ u_n &\in C \text{ such that, } \forall y \in C, \end{aligned}$$

$$\begin{aligned}
& \Theta(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \\
& C_n = \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n, \phi(v, u_n) \leq \phi(v, x_n) + (1 + k_n)(1 - \beta_n)\xi_n\}, \\
& Q_n = \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
& x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{3.63}$$

where $J : E \rightarrow E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\xi_n = \sup_{u \in G} (k_n - 1)\phi(u, x_n)$. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions (i)-(ii) in Theorem 3.1, then $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap F(T) \cap \text{EP}(\Theta)} x_0$, where $\text{EP}(\Theta)$ is the set of solutions for the equilibrium problem (1.6).

Proof. Putting $\varphi = 0$ and $A = 0$ in Theorem 3.1, the conclusion of Theorem 3.6 can be obtained from Theorem 3.1. $\square$

Theorem 3.7. Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E . Let $A : C \rightarrow E^*$ be a continuous and monotone mapping and $S, T : C \rightarrow C$ two closed and uniformly quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$. Suppose that S and T are uniformly L -Lipschitz continuous and that $G = F(T) \cap F(S) \cap \text{VI}(C, A)$ is a nonempty and bounded subset in C . Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned}
& x_0 \in C, \quad C_0 = C, \quad Q_0 = C, \\
& z_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n), \\
& y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JS^n z_n), \\
& u_n \in C \text{ such that, } \quad \forall y \in C, \\
& \langle Au_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \\
& C_n = \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n, \phi(v, u_n) \leq \phi(v, x_n) + (1 + k_n)(1 - \beta_n)\xi_n\}, \\
& Q_n = \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
& x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{3.64}$$

where $J : E \rightarrow E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\xi_n = \sup_{u \in G} (k_n - 1)\phi(u, x_n)$. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions (i)-(ii) in Theorem 3.1, then $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap F(T) \cap \text{VI}(C, A)} x_0$, where $\text{VI}(C, A)$ is the set of solutions for the variational inequality (1.7)

Proof. Putting $\varphi = 0$ and $\Theta = 0$ in Theorem 3.1, the conclusion of Theorem 3.7 can be obtained from Theorem 3.1. $\square$

Theorem 3.8. Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E . Let $A : C \rightarrow E^*$ be a continuous and monotone

mapping, $\varphi : C \rightarrow \mathbb{R}$ a lower semicontinuous and convex function, and $\Theta : C \times C \rightarrow \mathbb{R}$ a bifunction satisfying conditions (A_1) – (A_4) . Let $S : C \rightarrow C$ be a closed and quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$. Suppose that S is uniformly L -Lipschitz continuous and that $F(S) \cap \Omega$ is a nonempty and bounded subset in C . Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &\in C, \quad C_0 = C, \quad Q_0 = C, \\ y_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n)JS^n x_n), \\ u_n &\in C \text{ such that, } \quad \forall y \in C, \\ \Theta(u_n, y) + \langle Au_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle &\geq 0, \quad (3.65) \\ C_n &= \{v \in C_{n-1} : \phi(v, u_n) \leq \phi(v, x_n) + \xi_n\}, \\ Q_n &= \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned}$$

where $J : E \rightarrow E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\xi_n = \sup_{u \in F(S) \cap \Omega} (k_n - 1)\phi(u, x_n)$. If $\{\beta_n\}$ satisfy condition (ii) in Theorem 3.1, then $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap \Omega} x_0$.

Proof. Taking $T = I$ in Theorem 3.1, we have that $z_n = x_n, \forall n \geq 0$. Hence, the conclusion of Theorem 3.8 is obtained. $\square$

Theorem 3.9. Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E . Let $A : C \rightarrow E^*$ be a continuous and monotone mapping, $\varphi : C \rightarrow \mathbb{R}$ a lower semicontinuous and convex function, and $\Theta : C \times C \rightarrow \mathbb{R}$ a bifunction satisfying conditions (A_1) – (A_4) . Suppose that Ω is a nonempty subset in C . Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &\in C, \quad C_0 = C, \quad Q_0 = C, \\ u_n &\in C \text{ such that, } \quad \forall y \in C, \\ \Theta(u_n, y) + \langle Au_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jx_n \rangle &\geq 0, \quad (3.66) \\ C_n &= \{v \in C_{n-1} : \phi(v, u_n) \leq \phi(v, x_n)\}, \\ Q_n &= \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned}$$

where $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{\Omega} x_0$.

Proof. Taking $T = S = I$ in Theorem 3.1, the conclusion is obtained. $\square$

Theorem 3.10. Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E . Let $S, T : C \rightarrow C$ be two closed and uniformly quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$. Suppose that S and T are uniformly L -Lipschitz continuous and that $F(T) \cap F(S)$ is a nonempty and bounded subset in C . Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned}x_0 &\in C, \quad C_0 = C, \quad Q_0 = C, \\z_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n), \\y_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n)JS^n z_n), \\u_n &= \Pi_C y_n, \\C_n &= \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n, \phi(v, u_n) \leq \phi(v, x_n) + (1 + k_n)(1 - \beta_n)\xi_n\}, \\Q_n &= \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,\end{aligned}\tag{3.67}$$

where $J : E \rightarrow E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, and $\xi_n = \sup_{u \in F(S) \cap F(T)} (k_n - 1)\phi(u, x_n)$. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions (i)-(ii) in Theorem 3.1, then $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap F(T)} x_0$.

Proof. Taking $A = \Theta = 0$ and $r_n = 1, \forall n \geq 0$ in Theorem 3.1, the conclusion of Theorem 3.10 is obtained. $\square$

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