Research Article

A Rademacher-Type Formula for $pod(n)$

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A Rademacher-type formula for the Fourier coefficients of the generating function for the partitions of n where no odd part is repeated is presented.

1. Partitions

A *partition* of a positive integer n is a representation of n as a sum of positive integers where order of summands (parts) does not matter. Let $p(n)$ represent the number of partitions of n . In 1937, Rademacher [1, 2] was able to express $p(n)$ as a convergent series:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \sqrt{k} \frac{d}{dn} \left(\frac{\sinh\left((\pi/k)\sqrt{(2/3)(n-1/24)}\right)}{\sqrt{n-1/24}} \right), \quad (1.1)$$

where

$$A_k(n) = \sum_{0 \leq h < k, \gcd(h,k)=1} e^{\pi i(s(h,k) - 2nh/k)} \quad (1.2)$$

is a Kloosterman sum and

$$s(h, k) = \sum_{r=1}^{k-1} \frac{r}{k} \left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2} \right) \quad (1.3)$$

is a Dedekind sum.

In 2011, Bruinier and Ono [3] announced a new formula that expresses $p(n)$ as a *finite* sum.

1.1. Formula for $\tilde{p}(n)$

Let

$$f(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m} \quad (1.4)$$

be Euler's generating function for $p(n)$. H. Rademacher used the classical circle method to find the coefficients of q^n . There are many other infinite products to which this method could be applied. We introduce one of these infinite products here and derive the formula for the coefficients of q^n . Define

$$G(q) := \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-q^{2m})^2} = \frac{(f(q^2))^2}{f(q)}. \quad (1.5)$$

Let $\tilde{p}(j)$ denote the coefficient of q^j in the expansion of $G(q)$, that is,

$$G(q) = \sum_{j=0}^{\infty} \tilde{p}(j)q^j. \quad (1.6)$$

We will find a closed expression for $\tilde{p}(j)$. Note that

$$G(-q) = \prod_{m=1}^{\infty} \frac{1+q^{2m-1}}{1-q^{2m}} = \sum_{n=0}^{\infty} pod(n)q^n, \quad (1.7)$$

where $pod(n)$ equals the number of partitions of n where no odd part is repeated. Thus

$$\begin{aligned} pod(n) &= (-1)^n \tilde{p}(n) \\ &= (-1)^n \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{(k,2)=2} \sqrt{k} \sum_{0 \leq h < k, (h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} \frac{d}{dn} \\ &\quad \times \left(\frac{\sinh\left(\pi \sqrt{8n-1}/2k\right)}{\sqrt{8n-1}} \right) \end{aligned} \quad (1.8)$$

which is simpler than the one given by Sills [4, page 4, Equation (1.13)] in 2010:

$$\begin{aligned} pod(n) = & \frac{2}{\pi} \sum_{k=1}^{\infty} \sqrt{k \left(1 - (-1)^k + \left\lfloor \frac{(4, k)}{4} \right\rfloor \right)} \sum_{0 \leq h < k(h, k)=1} \frac{\omega(h, k) \omega(4h/(k, 4), k/(k, 4))}{\omega(2h/(k, 2), k/(k, 2))} \\ & \times e^{-2\pi i n h/k} \frac{d}{dn} \left(\frac{\sinh \left(\pi \sqrt{(k, 4)(8n-1)/4k} \right)}{\sqrt{8n-1}} \right), \end{aligned} \quad (1.9)$$

where $\omega(h, k)$ is defined as

$$\omega(h, k) = \exp \left(\pi i \sum_{r=1}^{k-1} \frac{r}{k} \left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2} \right) \right). \quad (1.10)$$

2. Evaluation of the Path Integral

2.1. Convergence and Cauchy Residue Theorem

Considering q as a complex variable in

$$\prod_{m=1}^{\infty} \frac{(1 - q^m)}{(1 - q^{2m})^2} = \prod_{m=1}^{\infty} \frac{1}{(1 + q^m)(1 - q^{2m})} = \prod_{m=1}^{\infty} \frac{1}{(1 - (-q^m))(1 - q^{2m})}, \quad (2.1)$$

we see from the right-hand side that infinite product and thus also infinite series are convergent for $|q| < 1$ since

$$\sum_{n=0}^{\infty} (q^k)^n = \frac{1}{(1 - q^k)}, \quad (2.2)$$

is a geometric series which converges for $|q| < 1$ for any fixed $k \geq 1$.

Next, we note that from

$$G(q) = \sum_{j=0}^{\infty} \tilde{p}(j) q^j, \quad (2.3)$$

we get that

$$\frac{G(q)}{q^{n+1}} = \sum_{j=0}^{\infty} \frac{\tilde{p}(j) q^j}{q^{n+1}} \quad \text{if } 0 < |q| < 1. \quad (2.4)$$

The series on the right side of (2.4) is a Laurent series of $G(q)/q^{(n+1)}$. It has a pole of order $n+1$ at $q=0$ with residue $\tilde{p}(n)$. Applying Cauchy's Residue Theorem we get that

$$\tilde{p}(n) = \frac{1}{2\pi i} \int_C \frac{G(q)}{q^{n+1}} dq = \frac{1}{2\pi i} \int_C \frac{(f(q^2))^2}{f(q)q^{n+1}} dq, \quad (2.5)$$

where C is any positively oriented simple closed contour lying inside the unit circle.

2.2. Change of the Variable

The change of the variable $q = e^{2\pi i \tau}$ maps the unit disk $|q| < 1$ into an infinite vertical strip of width 1 in the τ -plane. To see this we note that from $q = e^{2\pi i \tau}$ we get $\log q = 2\pi i \tau$, so $\tau = \log q / 2\pi i$. Choosing the branch cut to be $[0, 1]$, we get

$$\tau = \frac{\log|q|}{2\pi i} + \frac{\text{Arg}(q)}{2\pi}. \quad (2.6)$$

As q traverses a circle centered at $q=0$ of radius $e^{-2\pi}$ in the positive direction, the point τ varies from i to $i+1$ along a horizontal segment as could be easily deduced from (2.6).

Replacing the segment by the Rademacher path composed of upper arcs of the Ford circles formed by the Farey series $\mathcal{F}_N$, (2.5) becomes

$$\tilde{p}(n) = \frac{1}{2\pi i} \int_i^{i+1} \frac{(f(e^{4\pi i \tau}))^2 2\pi i e^{2\pi i \tau}}{f(e^{2\pi i \tau}) e^{2\pi i \tau (n+1)}} d\tau, \quad (2.7)$$

which simplifies to

$$\begin{aligned} \tilde{p}(n) &= \int_i^{i+1} \frac{(f(e^{4\pi i \tau}))^2}{f(e^{2\pi i \tau})} e^{-2\pi i \tau n} d\tau, \\ &= \int_{P(N)} \frac{(f(e^{4\pi i \tau}))^2}{f(e^{2\pi i \tau})} e^{-2\pi i \tau n} d\tau. \end{aligned} \quad (2.8)$$

The above can be written as

$$\int_{P(N)} \frac{(f(e^{4\pi i \tau}))^2}{f(e^{2\pi i \tau})} e^{-2\pi i \tau n} d\tau = \sum_{k=1}^N \sum_{0 \leq h < k(h,k)=1} \int_{\gamma(h,k)} \frac{(f(e^{4\pi i \tau}))^2}{f(e^{2\pi i \tau})} e^{-2\pi i \tau n} d\tau, \quad (2.9)$$

where $\gamma(h, k)$ is the upper arc of the Ford circle $C(h, k)$.

2.3. Another Change of the Variable

Consider another change of variable

$$\tau = \frac{h}{k} + \frac{iz}{k}, \quad (2.10)$$

so that

$$z = -ik \left(\tau - \frac{h}{k} \right) \quad (2.11)$$

$$dz = -ik \, d\tau. \quad (2.12)$$

Under this transformation the Ford circle $C(h, k)$ in the τ -plane with center at $h/k + i1/2k^2$ and radius $1/2k^2$ is mapped to a negatively oriented circle C_k in the z -plane with center at $1/2k$ and radius $1/2k$. This follows from the fact that any point on the Ford circle $C(h, k)$ is given by

$$\tau = \left(\frac{h}{k} + i \frac{1}{2k^2} \right) + \frac{1}{2k^2} e^{i\theta}, \quad 0 \leq \theta < 2\pi. \quad (2.13)$$

Substitution of (2.13) into (2.11) gives

$$z = \frac{1}{2k} + \frac{1}{2k} (-ie^{i\theta}), \quad (2.14)$$

which is a circle centered at $1/2k$ with radius $1/2k$. Now we make change of variable in (2.9). This gives

$$\tilde{p}(n) = i \sum_{k=1}^N k^{-1} \sum_{0 \leq h < k, (h,k)=1} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \frac{(f(e^{4\pi i h/k - 4\pi i z/k}))^2}{f(e^{2\pi i h/k - 2\pi z/k})} e^{2\pi n z/k} dz, \quad (2.15)$$

where

$$\begin{aligned} s_{h,k} &= \frac{k}{k^2 + k_p^2} + \frac{k_p}{k^2 + k_p^2} i, \\ t_{h,k} &= \frac{k}{k^2 + k_s^2} - \frac{k_s}{k^2 + k_s^2} i \end{aligned} \quad (2.16)$$

are initial and terminal points, respectively.

2.4. Modular Transformation

Next, we note that

$$f(q) = f(e^{2\pi i\tau}) = \frac{e^{\pi i\tau/12}}{\eta(\tau)}, \quad (2.17)$$

where $\eta(\tau)$ is the Dedekind eta function. Rewriting modular functional equation [5, page 96] for $\eta(\tau)$ in terms of $f(q) = f(e^{2\pi i\tau}) = f(e^{2\pi ih/k-2\pi z/k})$ we get

$$f(e^{2\pi ih/k-2\pi z/k}) = \omega(h, k) \exp\left(\frac{\pi(z^{-1}-z)}{12k}\right) \sqrt{z} f\left(\exp\left(2\pi i \frac{iz^{-1}+H}{k}\right)\right), \quad (2.18)$$

with $hH \equiv -1 \pmod{k}$, $(h, k) = 1$.

To evaluate (2.15) we would like to express

$$G(q) = G(e^{2\pi i\tau}) = G(e^{2\pi ih/k-2\pi z/k}) = \frac{(f(e^{4\pi ih/k-4\pi iz/k}))^2}{f(e^{2\pi ih/k-2\pi z/k})}, \quad (2.19)$$

in the same way we did for $f(q)$ above. Two cases have to be considered: $(k, 2) = 1$ and $(k, 2) = 2$. When $(k, 2) = 1$ we will replace h by $2h$ and z by $2z$, and when $(k, 2) = 2$, k will be replaced by $k/2$ in order to obtain $f(q^2)$ from $f(q)$. Hence, we have

$$G(e^{2\pi ih/k-2\pi z/k}) = \begin{cases} \frac{\omega^2(2h, k) e^{\pi((2z)^{-1}-2z)/6k} 2z f^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{\omega(h, k) e^{\pi(z^{-1}-z)/12k} \sqrt{z} f(e^{2\pi i(iz^{-1}+H_2)/k})}, & \text{if } (k, 2) = 1, \\ \frac{\omega^2(h, k/2) e^{\pi(z^{-1}-z)/3k} z f^2(e^{4\pi i(iz^{-1}+H_1)/k})}{\omega(h, k) e^{\pi(z^{-1}-z)/12k} \sqrt{z} f(e^{2\pi i(iz^{-1}+H_1)/k})}, & \text{if } (k, 2) = 2, \end{cases} \quad (2.20)$$

which simplifies to

$$G(e^{2\pi ih/k-2\pi z/k}) = \begin{cases} 2 \frac{\omega^2(2h, k)}{\omega(h, k)} e^{-\pi z/4k} \sqrt{z} \frac{f^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{f(e^{2\pi i(iz^{-1}+H_2)/k})}, & \text{if } (k, 2) = 1, \\ \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{\pi(z^{-1}-z)/4k} \sqrt{z} G(e^{2\pi i(iz^{-1}+H_1)/k}), & \text{if } (k, 2) = 2, \end{cases} \quad (2.21)$$

where $hH_j \equiv -1 \pmod{k}$ and $j \mid H_j$ for $j = 1, 2$.

We return to evaluation of (2.15). To proceed we note that

$$G(e^{2\pi i(iz^{-1}+H_1)/k}) = 1 + \left\{ G(e^{2\pi i(iz^{-1}+H_1)/k}) - 1 \right\}. \quad (2.22)$$

Rewriting (2.15) in terms of (2.21) and (2.22) we obtain

$$\begin{aligned}
 \tilde{p}(n) &= i \sum_{k=1(k,2)=1}^N k^{-1} \sum_{0 \leq h < k(h,k)=1} 2 \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} \frac{f^2 \left(e^{2\pi i (i(2z)^{-1} + H_2)/k} \right)}{f \left(e^{2\pi i (iz^{-1} + H_2)/k} \right)} e^{\pi z/k(2n-1/4)} dz \\
 &+ i \sum_{k=1(k,2)=2}^N k^{-1} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} \left\{ 1 + \left(G \left(e^{2\pi i (iz^{-1} + H_1)/k} \right) - 1 \right) \right\} \\
 &\times e^{((\pi z/k)(2n-1/4) + \pi/4zk)} dz \\
 &= 2i \sum_{k=1(k,2)=1}^N k^{-1} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} \frac{f^2 \left(e^{2\pi i (i(2z)^{-1} + H_2)/k} \right)}{f \left(e^{2\pi i (iz^{-1} + H_2)/k} \right)} e^{\pi z/k(2n-1/4)} dz \\
 &+ i \sum_{k=1(k,2)=2}^N k^{-1} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} (J_1(h,k) + J_2(h,k)),
 \end{aligned} \tag{2.23}$$

where

$$\begin{aligned}
 J_1(h,k) &= \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} e^{(\pi z/k(2n-1/4) + \pi/4zk)} dz, \\
 J_2(h,k) &= \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} \left\{ G \left(e^{2\pi i (iz^{-1} + H_1)/k} \right) - 1 \right\} e^{(\pi z/k(2n-1/4) + \pi/4zk)} dz.
 \end{aligned} \tag{2.24}$$

2.5. Estimation of the First Term

We will estimate the first term in (2.23) and will show that it is small for large N . To do this we change variable again by letting $\xi = zk$. Then the first term in (2.23) becomes

$$2i \sum_{k=1(k,2)=1}^N k^{-5/2} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s_{h,k}^*}^{t_{h,k}^*} \sqrt{\xi} \frac{f^2 \left(e^{2\pi i (i(2\xi/k)^{-1} + H_2)/k} \right)}{f \left(e^{2\pi i (i(\xi/k)^{-1} + H_2)/k} \right)} e^{\pi \xi/k^2 (2n-1/4)} d\xi, \tag{2.25}$$

where

$$s_{h,k}^* = \frac{k^2}{k^2 + k_p^2} + \frac{kk_p}{k^2 + k_p^2} i, \tag{2.26}$$

$$t_{h,k}^* = \frac{k^2}{k^2 + k_s^2} - \frac{kk_s}{k^2 + k_s^2} i, \tag{2.27}$$

are initial and terminal points obtained from (2.16), respectively. Under this change of variable circle C_k in z -plane with center at $1/2k$ and radius $1/2k$ is mapped to a circle C_k^*

in ξ -plane centered at $1/2$ with radius $1/2$. Note also that the mapping $w = 1/\xi$ maps the circle C_k^* and its interior onto a half-plane $\Re(w) \geq 1$ (where $\Re(w)$ denotes the real part of complex variable w and $\Im(w)$ is the imaginary part). From elementary complex analysis we have that $\Re(w) = x/(x^2 + y^2)$ and $\Im(w) = -y/(x^2 + y^2)$, where $x + iy = \xi$. It is readily seen that the segment $0 < x \leq 1$ in the ξ -plane is mapped to an infinite strip $[1, \infty)$ in the w -plane. So, it follows that inside and on the circle C_k^* we have that $0 < \Re(\xi) \leq 1$ and $\Re(1/\xi) \geq 1$. We now show that $\Re(1/\xi) = 1$ on the circle C_k^* . To see this note that in the polar form $\xi = 1/2 + (1/2)e^{i\theta}$ on C_k^* , $0 \leq \theta \leq 2\pi$. From this we get that

$$\begin{aligned} \frac{1}{\xi} &= \frac{2}{1 + e^{i\theta}} = \frac{2}{(1 + \cos \theta) + i \sin \theta} \\ &= \frac{2[(1 + \cos \theta) - i \sin \theta]}{(1 + \cos \theta)^2 + \sin^2 \theta} \\ &= \frac{2(1 + \cos \theta)}{2 + 2 \cos \theta} - i \frac{2 \sin \theta}{2 + 2 \cos \theta} \\ &= 1 - i \frac{\sin \theta}{1 + \cos \theta}. \end{aligned} \quad (2.28)$$

So, $\Re(1/\xi) = 1$.

Furthermore, we may move path of integration from the arc joining $s_{h,k}^*$ and $t_{h,k}^*$ to a segment connecting these two points on the circle C_k^* . By [5, page 104], Theorem 5.9 the length of the path of integration is bounded by $2\sqrt{2}k/N$, and on the segment connecting $s_{h,k}^*$ and $t_{h,k}^*$, $|\xi| < \sqrt{2}k/N$.

Next, let us define $\tilde{p}^*(m)$ by

$$\sum_{m=0}^{\infty} \tilde{p}^*(m) q^m = \frac{f^2 \left(e^{2\pi i (i(2\xi/k)^{-1} + H_2)/k} \right)}{f \left(e^{2\pi i (i(\xi/k)^{-1} + H_2)/k} \right)}, \quad (2.29)$$

which is a part of the integrand in (2.25). Then, estimating the integrand in (2.25) we get

$$\begin{aligned} & \left| \sqrt{\xi} e^{(\pi \xi / k^2) (2n-1/4)} \right| \times \left| -1 + \frac{f^2 \left(e^{2\pi i (i(2\xi/k)^{-1} + H_2)/k} \right)}{f \left(e^{2\pi i (i(\xi/k)^{-1} + H_2)/k} \right)} \right| \\ &= |\xi|^{1/2} \left| e^{\pi \xi / k^2 (2n-1/4)} \right| \times \left| \sum_{m=1}^{\infty} \tilde{p}^*(m) \exp \left(\frac{2\pi i m (i(\xi/k)^{-1} + H_2)}{k} \right) \right| \\ &= |\xi|^{1/2} \left| e^{\pi \xi / k^2 (2n-1/4)} \right| \times \left| \sum_{m=1}^{\infty} \tilde{p}^*(m) \exp \left(-\frac{2\pi m}{\xi} \right) \exp \left(\frac{2\pi i m H_2}{k} \right) \right| \end{aligned}$$

$$\begin{aligned}
&\leq |\xi|^{1/2} e^{\pi/k^2 (2n-1/4)\Re(\xi)} \left| \sum_{m=1}^{\infty} \tilde{p}^*(m) \exp\left(-\frac{2\pi m}{\xi}\right) \exp\left(\frac{2\pi i m H_2}{k}\right) \right| \\
&\leq |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^*(m)| \exp\left(-2\pi m \Re\left(\frac{1}{\xi}\right)\right) \\
&\leq |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^*(m)| e^{-2\pi m} \\
&= |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^*(m)| y^m, \quad (\text{where } y = e^{-2\pi}) \\
&= c |\xi|^{1/2},
\end{aligned} \tag{2.30}$$

where

$$c = e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^*(m)| y^m. \tag{2.31}$$

Note that c does not depend on ξ or N . It depends on n , but n remains fixed in the above analysis. So,

$$\begin{aligned}
&\left| \int_{s^*(h,k)}^{t^*(h,k)} \sqrt{\xi} \frac{f^2\left(e^{2\pi i(i(2\xi/k)^{-1}+H_2)/k}\right)}{f\left(e^{2\pi i(i(\xi/k)^{-1}+H_2)/k}\right)} e^{(\pi\xi/k^2)(2n-1/4)} d\xi \right| \leq c |\xi|^{1/2} \leq c \left(\frac{\sqrt{2}k}{N}\right)^{1/2} \frac{2\sqrt{2}N}{N} \\
&< \alpha k^{3/2} N^{-3/2},
\end{aligned} \tag{2.32}$$

for some constant α , and we have that

$$\begin{aligned}
&\left| 2i \sum_{k=1(k,2)=1}^N k^{-5/2} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s^*(h,k)}^{t^*(h,k)} \sqrt{\xi} \frac{f^2\left(e^{2\pi i(i(2\xi/k)^{-1}+H_2)/k}\right)}{f\left(e^{2\pi i(i(\xi/k)^{-1}+H_2)/k}\right)} e^{\pi\xi/k^2(2n-1/4)} (d\xi) \right| \\
&\leq 2 \sum_{k=1(k,2)=1}^N \sum_{0 \leq h < k(h,k)=1} \alpha k^{-1} N^{-3/2} \\
&\leq 2\alpha N^{-3/2} \sum_{k=1(k,2)=1}^N 1 = 2\alpha N^{-1/2}.
\end{aligned} \tag{2.33}$$

This completes the estimation of the first term in (2.23). We proceed to the second term.

2.6. Estimation of the Second Term

First, we will show that

$$J_2(h, k) = \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} \left\{ G\left(e^{2\pi i(iz^{-1}+H_1)/k}\right) - 1 \right\} e^{(\pi z/k(2n-1/4)+\pi/4zk)} dz \quad (2.34)$$

is small for large N . Making change of variable $\xi = zk$ as before, we get that

$$J_2(h, k) = k^{-3/2} \int_{s_{h,k}^*}^{t_{h,k}^*} \sqrt{\xi} \left\{ G\left(e^{2\pi i(i(\xi/k)^{-1}+H_1)/k}\right) - 1 \right\} e^{(\pi \xi/k^2(2n-1/4)+\pi/4\xi)} d\xi, \quad (2.35)$$

where $s_{h,k}^*$ and $t_{h,k}^*$ are as in (2.26), respectively. As before, we define $\tilde{p}^{**}(m)$ by

$$\sum_{m=0}^{\infty} \tilde{p}^{**}(m) q^m = G\left(e^{2\pi i(i(\xi/k^{-1})+H_1)/k}\right) - 1. \quad (2.36)$$

Then, estimating the integrand, we see that

$$\begin{aligned} & \left| \sqrt{\xi} e^{(\pi \xi/k^2)(2n-1/4)+\pi/4\xi} \right| \times \left| G\left(e^{2\pi i(i(\xi/k)^{-1}+H_1)/k}\right) - 1 \right| \\ &= |\xi|^{1/2} \left| e^{\pi \xi/k^2(2n-1/4)} e^{\pi/4\xi} \right| \times \left| \sum_{m=0}^{\infty} \tilde{p}^{**}(m) \exp\left(\frac{2\pi i m(ik/\xi + H_1)}{k}\right) - 1 \right| \\ &\leq |\xi|^{1/2} e^{(\pi/k^2)(2n-1/4)\Re(\xi)} e^{\pi/4\Re(\xi)} \left| \sum_{m=1}^{\infty} \tilde{p}^{**}(m) \exp\left(\frac{-2\pi m}{\xi}\right) \exp\left(\frac{2\pi i m H_1}{k}\right) \right| \\ &\leq |\xi|^{1/2} e^{2\pi n} e^{\pi/4\Re(1/\xi)} \sum_{m=1}^{\infty} |\tilde{p}^{**}(m)| \exp\left(-2\pi m \Re\left(\frac{1}{\xi}\right)\right) \\ &= |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^{**}(m)| \exp\left(\left(-2\pi m + \frac{\pi}{4}\right) \Re\left(\frac{1}{\xi}\right)\right) \\ &\leq |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^{**}(m)| \exp\left(-2\pi m + \frac{\pi}{4}\right) \\ &= |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^{**}(m)| e^{-\pi/4(8m-1)} \\ &\leq |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^{**}(8m-1)| e^{-\pi/4(8m-1)} \end{aligned}$$

$$\begin{aligned}
&= |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^{**}(8m-1)| x^{8m-1}, \quad \text{where } x = e^{-\pi/4} \\
&= b |\xi|^{1/2},
\end{aligned} \tag{2.37}$$

where

$$b = e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^{**}(8m-1)| x^{8m-1}. \tag{2.38}$$

Note that b does not depend on ξ or N . It depends on n , but n is fixed. It follows, therefore, that

$$|J_2(h, k)| \leq b \left(\frac{\sqrt{2}k}{N} \right)^{1/2} \frac{2\sqrt{2}N}{N} < \beta k^{3/2} N^{-3/2}, \tag{2.39}$$

for some constant β . Then we have that

$$\begin{aligned}
\left| i \sum_{k=1(k,2)=2}^N k^{-5/2} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-\frac{2\pi i n h}{k}} J_2(h, k) \right| &< \sum_{k=1(k,2)=2}^N \sum_{0 \leq h < k(h,k)=1} \beta k^{-1} N^{-3/2} \\
&\leq \beta N^{-3/2} \sum_{k=1(k,2)=2}^N 1 = \beta N^{-1/2}.
\end{aligned} \tag{2.40}$$

Combining the results from (2.33) and (2.40) we have that

$$\begin{aligned}
\tilde{p}(n) &= i \sum_{k=1(k,2)=2}^N k^{-5/2} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} J_1(h, k) + O\left(\beta N^{-1/2} + 2\alpha N^{-1/2}\right) \\
&= i \sum_{k=1(k,2)=2}^N k^{-5/2} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} J_1(h, k) + O\left(N^{-1/2}\right).
\end{aligned} \tag{2.41}$$

Finally, we turn our attention to

$$J_1(h, k) = k^{-3/2} \int_{S_{h,k}^*}^{t_{h,k}^*} \sqrt{\xi} e^{((\pi \xi/k^2)(2n-1/4) + \pi/4)\xi} d\xi. \tag{2.42}$$

We note that

$$J_1(h, k) = \int_{C_k^*} - \int_{S_{h,k}^*}^0 - \int_0^{t_{h,k}^*} = \int_{C_k^*} -S_1 - S_2, \tag{2.43}$$

where C_k^* is a circle in the ξ -plane centered at $1/2$ with radius $1/2$, as before. It is easily seen that the length of the arc connecting 0 and $s_{h,k}^*$ is less than

$$2\pi \frac{|s_{h,k}^*|}{2} \leq \pi |s_{h,k}^*| \leq \pi \sqrt{2} \frac{k}{N}. \quad (2.44)$$

From the discussion above we know that $\Re(1/\xi) = 1$ and $0 < \Re(\xi) \leq 1$ on C_k^* . So, the integrand in S_1 could be estimated as

$$\begin{aligned} \left| \sqrt{\xi} e^{(\pi\xi/k^2)(2n-1/4)+\pi/4\xi} \right| &= |\xi|^{1/2} \left| e^{\pi\xi/k^2(2n-1/4)} \right| \left| e^{\pi/4\xi} \right| \\ &= |\xi|^{1/2} e^{\pi/k^2(2n-1/4)\Re(\xi)} e^{\pi/4\Re(1/\xi)} \\ &\leq 2^{1/4} \frac{k^{1/2}}{N^{1/2}} e^{2\pi n} e^{\pi/4}. \end{aligned} \quad (2.45)$$

2.7. Combining the Results

We combine the results in (2.44) and (2.45) to get

$$|S_1| < \gamma k^{3/2} N^{-3/2}, \quad (2.46)$$

where γ is a constant. We can obtain similar estimate for S_2 and, as before, we get an error term $O(N^{-1/2})$ in the formula for $\tilde{p}(n)$. Therefore, we can write

$$\begin{aligned} \tilde{p}(n) &= i \sum_{k=1}^N \sum_{(k,2)=2} k^{-5/2} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} \\ &\quad \times \int_{C_k^*} \sqrt{\xi} e^{((\pi\xi/k^2)(2n-1/4)+\pi/4\xi)} d\xi + O(N^{-1/2}). \end{aligned} \quad (2.47)$$

Letting $N \rightarrow \infty$ we have that

$$\begin{aligned} \tilde{p}(n) &= i \sum_{k=1}^{\infty} \sum_{(k,2)=2} k^{-5/2} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} \\ &\quad \times \int_{C_k^*} \sqrt{\xi} e^{((\pi\xi/k^2)(2n-1/4)+\pi/4\xi)} d\xi. \end{aligned} \quad (2.48)$$

We introduce another change of variable

$$\xi = \frac{1}{w}, \quad d\xi = -\frac{1}{w^2}. \quad (2.49)$$

Then (2.48) becomes

$$\begin{aligned}\tilde{p}(n) &= \frac{1}{i} \sum_{k=1(k,2)=2}^{\infty} k^{-5/2} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} \\ &\quad \times \int_{1-\infty i}^{1+\infty i} w^{-5/2} e^{((\pi/k^2)(2n-1/4)1/w+\pi w/4)} dw.\end{aligned}\quad (2.50)$$

Let $t = \pi w/4$ in (2.50), then the above becomes

$$\begin{aligned}\tilde{p}(n) &= 2\pi \left(\frac{\pi^{3/2}}{8} \right) \sum_{k=1(k,2)=2}^{\infty} k^{-5/2} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} \frac{1}{2\pi i} \\ &\quad \times \int_{\pi/4-\infty i}^{\pi/4+\infty i} t^{-5/2} e^{(t+(\pi^2/4k^2)(2n-1/4)(1/t))} dt.\end{aligned}\quad (2.51)$$

2.8. Bessel Function

In Watson's Treatise on Bessel functions [6, page 181], we find a formula equivalent to the following:

$$I_{\nu}(z) = \frac{(1/2z)^{\nu}}{2\pi i} \int_{c-\infty i}^{c+\infty i} t^{-\nu-1} e^{t+(z^2/4t)} dt, \quad (\text{if } c > 0, \quad \Re(\nu) > 0). \quad (2.52)$$

Let

$$\frac{z}{2} = \left\{ \frac{\pi^2}{4k^2} \left(2n - \frac{1}{4} \right) \right\}^{1/2} \quad (2.53)$$

and $\nu = 3/2$. Then we have

$$\begin{aligned}\tilde{p}(n) &= 2\pi \left(\frac{\pi^{3/2}}{8} \right) \sum_{k=1(k,2)=2}^{\infty} k^{-5/2} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} \\ &\quad \times \frac{\pi^{-3/2}(2n-1/4)^{-3/4}}{4^{-3/4}k^{-3/2}} I_{3/2} \left(\frac{\pi}{k} \sqrt{2n - \frac{1}{4}} \right) \\ &= \frac{2\pi(2n-1/4)^{-3/4}}{\sqrt{8}} \sum_{k=1(k,2)=2}^{\infty} k^{-1} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} I_{3/2} \left(\frac{\pi}{k} \sqrt{2n - \frac{1}{4}} \right).\end{aligned}\quad (2.54)$$

Note that Bessel functions of this order can be expressed as

$$I_{3/2}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left(\frac{\sinh z}{z} \right). \quad (2.55)$$

Expanding (2.55) we have that

$$I_{3/2}(z) = \sqrt{\frac{2z}{\pi}} \left(\frac{\cosh z}{z} - \frac{\sinh z}{z^2} \right). \quad (2.56)$$

Substituting (2.53) into (2.56), we get

$$\begin{aligned} I_{3/2}(z) &= I_{3/2} \left(\frac{\pi}{k} \left(2n - \frac{1}{4} \right)^{1/2} \right) = I_{3/2} \left(\frac{\pi \sqrt{8n-1}}{2k} \right) \\ &= \sqrt{\frac{2 \left(\pi \sqrt{8n-1}/2k \right)}{\pi}} \left(\frac{\cosh \left(\pi \sqrt{8n-1}/2k \right)}{\left(\pi \sqrt{8n-1}/2k \right)} - \frac{\sinh \left(\pi \sqrt{8n-1}/2k \right)}{\left(\pi \sqrt{8n-1}/2k \right)^2} \right) \\ &= \frac{(8n-1)^{1/4}}{\sqrt{k}} \left(\frac{2 \cosh \left(\pi \sqrt{8n-1}/2k \right)}{\frac{\pi}{k} \sqrt{8n-1}} - \frac{4k/\pi \sinh \left(\pi \sqrt{8n-1}/2k \right)}{\frac{\pi}{k} (8n-1)} \right) \\ &= \frac{1}{\pi \sqrt{\sqrt{8n-1}/k}} \left(2 \cosh \left(\frac{\pi \sqrt{8n-1}}{2k} \right) - \frac{4k \sinh \left(\pi \sqrt{8n-1}/2k \right)}{\pi \sqrt{8n-1}} \right). \end{aligned} \quad (2.57)$$

Multiplying (2.57) by

$$\frac{2\pi(2n-1/4)^{-3/4}}{\sqrt{8}} = \frac{2\pi}{(8n-1)^{3/4}}, \quad (2.58)$$

we get

$$\frac{2 \left(2 \cosh \left(\pi \sqrt{8n-1}/2k \right) - 4k \sinh \left(\pi \sqrt{8n-1}/2k \right) / \pi \sqrt{8n-1} \right)}{(8n-1)^{3/4} \sqrt{\sqrt{8n-1}/k}}. \quad (2.59)$$

2.9. Final Form

Finally, we rewrite (2.54) in terms of (2.59) to get

$$\begin{aligned} \tilde{p}(n) = & \sum_{k=1(k,2)=2}^{\infty} k^{-1} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h / k} \\ & \times \frac{2 \left(2 \cosh \left(\pi \sqrt{8n-1} / 2k \right) - 4k \sinh \left(\pi \sqrt{8n-1} / 2k \right) / \pi \sqrt{8n-1} \right)}{(8n-1)^{3/4} \sqrt{\sqrt{8n-1}/k}}. \end{aligned} \quad (2.60)$$

Thus,

$$\begin{aligned} pod(n) = & (-1)^n \sum_{k=1(k,2)=2}^{\infty} k^{-1} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h / k} \\ & \times \frac{2 \left(2 \cosh \left(\pi \sqrt{8n-1} / 2k \right) - 4k \sinh \left(\pi \sqrt{8n-1} / 2k \right) / \pi \sqrt{8n-1} \right)}{(8n-1)^{3/4} \sqrt{\sqrt{8n-1}/k}}, \end{aligned} \quad (2.61)$$

or equivalently

$$pod(n) = (-1)^n \frac{2}{\pi} \sum_{k=1(k,2)=2}^{\infty} \sqrt{k} \sum_{0 \leq h < k(h,k)=1} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h / k} \frac{d}{dn} \left(\frac{\sinh \left(\pi \sqrt{8n-1} / 2k \right)}{\sqrt{8n-1}} \right). \quad (2.62)$$

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