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Research Article

Fuzzy Small Submodule and Jacobson L-Radical

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Using the notion of fuzzy small submodules of a module, we introduce the concept of fuzzy coessential extension of a fuzzy submodule of a module. We attempt to investigate various properties of fuzzy small submodules of a module. A necessary and sufficient condition for fuzzy small submodules is established. We investigate the nature of fuzzy small submodules of a module under fuzzy direct sum. Fuzzy small submodules of a module are characterized in terms of fuzzy quotient modules. This characterization gives rise to some results on fuzzy coessential extensions. Finally, a relation between small L-submodules and Jacobson L-radical is established.

1. Introduction

After the introduction of fuzzy sets by Zadeh [1], a number of applications of this fundamental concept have come up. Rosenfeld [2] was the first one to define the concept of fuzzy subgroups of a group. Since then many generalizations of this fundamental concept have been done in the last three decades. Naegoita and Ralescu [3] applied this concept to modules and defined fuzzy submodules of a module. Consequently, fuzzy finitely generated submodules, fuzzy quotient modules [4], radical of fuzzy submodules, and primary fuzzy submodules [5, 6] were investigated. Saikia and Kalita [7] defined fuzzy essential submodules and investigated various characteristics of such submodules. These modules play a prominent role in fuzzy Goldie dimension of modules.

In this paper we fuzzify various properties of small (or superfluous) submodules of a module. We define fuzzy small ephimorphism and fuzzy coessential extension of a fuzzy submodule. We investigate various characteristics of fuzzy small submodules. Necessary and sufficient conditions for fuzzy small submodules are established. We also investigate the nature of fuzzy small submodule under fuzzy direct sum. A relation regarding fuzzy small

submodule of a module and fuzzy quotient module is also established. We attempt to fuzzify the well-known relation between the Jacobson radical and the small submodules of a module. In [8] Basnet et. al. have shown that the relation between the Jacobson radical and the small submodules of a module does not hold in fuzzy setting whereas we have tried to achieve the relation. It is established that the Jacobson L-radical is the sum of all the small L-submodules of a module. In case of a finitely generated module, the Jacobson L-radical is also a small L-submodule of the module under the condition that L-$\{1\}$ possesses a maximal element.

2. Basic Definitions and Notations

By R we mean a commutative ring with unity 1 and M denotes a R-module. The zero elements of R and M are 0 and θ, respectively. A complete Heyting algebra L is a complete lattice such that for all $A \subseteq L$ and for all $b \in L$, $\vee\{a \wedge b \mid a \in A\} = (\vee\{a \mid a \in A\}) \wedge b$ and $\wedge\{a \vee b \mid a \in A\} = (\wedge\{a \mid a \in A\}) \vee b$.

Definition 2.1. A submodule S of a module M over a ring R is said to be a small submodule of M if for every submodule N of M with $N \neq M$ implies $S + N \neq M$.

The notation $S \ll M$ indicates that S is a small submodule of M.

Fuzzy set on a nonempty set was introduced by Zadeh [1] in 1965. It is defined as follows.

Definition 2.2. By a fuzzy set of a module M, we mean any mapping μ from M to $[0,1]$. By $[0,1]^M$ we will denote the set of all fuzzy subsets of M. If μ is a mapping from M to L, where L is a complete Heyting algebra then μ is called an L-subset of M. By L^M we will denote the set of all L-subsets of M.

For each fuzzy set μ in M and any $\alpha \in [0,1]$, we define two sets $U(\mu,\alpha) = \{x \in M \mid \mu(x) \geq \alpha\}$, $L(\mu,\alpha) = \{x \in M \mid \mu(x) \leq \alpha\}$, which are called an upper level cut and a lower level cut of μ, respectively. The complement of μ, denoted by μ^c, is the fuzzy set on M defined by $\mu^c(x) = 1 - \mu(x)$. The support of a fuzzy set μ, denoted by μ^*, is a subset of M defined by $\mu^* = \{x \in M \mid \mu(x) > 0\}$. The subset μ_* of M is defined as $\mu_* = \{x \in M \mid \mu(x) = \mu(\theta)\}$.

Definition 2.3 (see [9]). If $N \subseteq M$ and $\alpha \in [0,1]^M$ then α_N is defined as,

$$\alpha_N(x) = \begin{cases} \alpha & \text{if } x \in N, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

If $N = \{x\}$ then $\alpha_{\{x\}}$ is often called a fuzzy point and is denoted by x_α. When $\alpha = 1$ then 1_N is known as the characteristic function of N. From now onwards, we will denote the characteristic function of N as χ_N.

If $\mu, \sigma \in [0,1]^M$ then

(1) $\mu \subseteq \sigma$, if and only if $\mu(x) \leq \sigma(x)$;

(2) $(\mu \cup \sigma)(x) = \max\{\mu(x), \sigma(x)\} = \mu(x) \vee \sigma(x)$;

(3) $(\mu \cap \sigma)(x) = \min\{\mu(x), \sigma(x)\} = \mu(x) \wedge \sigma(x)$.

For any family $\{\mu_i \mid i \in I\}$ of fuzzy subsets of M, where I is any nonempty index set,

(4) $(\bigcup_{i\in I}\mu_i)(x) = \sup_{i\in I}\mu_i(x) = \bigvee_{i\in I}\mu_i(x)$;

(5) $(\bigcap_{i\in I}\mu_i)(x) = \inf_{i\in I}\mu_i(x) = \bigwedge_{i\in I}\mu_i(x)$ for all $x \in M$;

(6) $(\mu+\sigma)(x) = \vee\{\mu(y)\wedge\sigma(z) \mid y,z\in M,\ y+z=x\}$.

Definition 2.4 (see [9]). Let X and Y be any two nonempty sets, and $f : X \to Y$ be a mapping. Let $\mu \in [0,1]^X$ and $\sigma \in [0,1]^Y$ then the *image* $f(\mu) \in [0,1]^Y$ and the *inverse image* $f^{-1}(\sigma) \in [0,1]^X$ are defined as follows: for all $y \in Y$

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) \mid x\in X, f(x)=y\}, & \text{if } f^{-1}(y)\neq\phi, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

and $f^{-1}(\sigma)(x) = \sigma(f(x))$ for all $x \in X$.

Definition 2.5 (see [9]). Let $\zeta \in [0,1]^R$ and $\mu \in [0,1]^M$. Then $\zeta\odot\mu$ is a fuzzy subset of M and it is defined by

$$(\zeta\odot\mu)(x) = \vee\left\{\bigwedge_{i=1}^{n}(\zeta(r_i)\wedge\mu(x_i)) \mid r_i\in R,\ x_i\in M,\ 1\le i\le n,\ n\in N,\ \sum_{i=1}^{n} r_i x_i = x\right\} \tag{2.3}$$

for all $x \in M$.

Definition 2.6 (see [9]). A fuzzy set μ of R is called a fuzzy ideal, if it satisfies the following properties:

(1) $\mu(x-y) \ge \mu(x)\wedge\mu(y)$,

(2) $\mu(xy) \ge \mu(x)\vee\mu(y)$, for all $x,y \in R$.

The following definition is given by Naegoita and Ralescu [3].

Definition 2.7 (see [3]). Let M be a module over a ring R and L be a Complete Heyting algebra. An L subset μ in M is called an L-submodule of M, if for every $x,y \in M$ and $r \in R$ the following conditions are satisfied:

(1) $\mu(\theta) = 1$,

(2) $\mu(x-y) \ge \mu(x)\wedge\mu(y)$,

(3) $\mu(rx) \ge \mu(x)$.

We denote the set of all L-submodules of M by $L(M)$. If $L = [0,1]$, then μ is called a fuzzy submodule of M. The set of all fuzzy submodules of M are denoted by $F(M)$.

Definition 2.8 (see [9]). Let $\mu,\nu \in F(M)$ be such that $\mu \subseteq \nu$. Then the quotient of ν with respect to μ, is a fuzzy submodule of M/μ^*, denoted by ν/μ, and is defined as follows:

$$\left(\frac{\nu}{\mu}\right)([x]) = \vee\{\nu(z) \mid z\in[x]\}, \quad \forall x\in\nu^*, \tag{2.4}$$

where $[x]$ denotes the coset $x+\mu^*$.

Definition 2.9 (see [9]). Let $\mu \in [0,1]^M$. Then $\cap\{\nu \mid \mu \subseteq \nu,\ \nu \in F(M)\}$ is a fuzzy submodule of M, and it is called the fuzzy submodule generated by the fuzzy subset μ. We denote this by $\langle\mu\rangle$, that is,

$$\langle\mu\rangle = \cap\{\nu \mid \mu \subseteq \nu,\ \nu \in F(M)\}. \tag{2.5}$$

Let $\xi \in F(M)$. If $\xi = \langle\mu\rangle$ for some $\mu \in [0,1]^M$, then μ is called a generating fuzzy subset of ξ.

Remark 2.10. (a) If A is a nonempty subset of M, then $\langle\chi_A\rangle = \chi_{\langle A\rangle}$, where $\langle A\rangle$ is the submodule of M generated by A.

(b) If $x \in M$, then $\chi_R \odot \chi_{\{x\}}$ is a fuzzy submodule of M generated by $\chi_{\{x\}}$, and in this case,

$$\chi_R \odot \chi_{\{x\}} = \langle\chi_{\{x\}}\rangle = \chi_{\langle\{x\}\rangle} = \chi_{Rx}. \tag{2.6}$$

3. Preliminaries

This section contains some preliminary results that are needed in the sequel.

Lemma 3.1 (see [10]). *Let M be a module and suppose that $K \leq N \leq M$ and $H \leq M$. Then*

(a) *$H + K \ll M$ if and only if $H \ll M$ and $K \ll M$;*

(b) *if $K \ll N$, then $K \ll M$;*

(c) *if N is a direct summand of M, then $K \ll M$ if and only if $K \ll N$;*

(d) *if $M = M_1 \oplus M_2$ and $K_i \leq M_i$ for $i = 1,2$, then $K_1 \oplus K_2 \ll M_1 \oplus M_2$ if and only if $K_1 \ll M_1$ and $K_1 \ll M_1$.*

Lemma 3.2 (see [9]). *Let $\mu, \nu \in F(M)$. Then $\mu + \nu \in F(M)$.*

Lemma 3.3 (see [9]). *Let $\mu_i \in F(M)$, for each $i \in I$, where $|I| > 1$. Then $\sum_{i\in I} \mu_i \in F(M)$ and $\langle\bigcup_{i\in I} \mu_i\rangle = \sum_{i\in I} \mu_i$.*

Lemma 3.4 (see [5]). *Let $\mu \in [0,1]^M$. Then the level subset $\mu_t = \{x \in M : \mu(x) \geq t\}$, $t \in \mathrm{Im}(\mu)$ is a submodule of M if and only if μ is a fuzzy submodule of M.*

Corollary 3.5. *μ_* is a submodule of M if and only if μ is a fuzzy submodule of M.*

In the next two sections we present our main results.

4. Fuzzy Small Submodule

Definition 4.1. Let M be a module over a ring R and let $\mu \in L(M)$. Then μ is said to be a Small L-Submodule of M, if for any $\nu \in L(M)$ satisfying $\nu \neq \chi_M$ implies $\mu + \nu \neq \chi_M$. The notation $\mu \ll_L M$ indicates that μ is a small L-submodule of M.

If $L = [0,1]$, then μ is called a fuzzy small submodule of M and it is indicated by the notation $\mu \ll_f M$. It is obvious that $\chi_{\{\theta\}}$ is always a fuzzy small submodule of M.

Definition 4.2. Let $\mu \in F(M)$. If $\mu \ll_f M$, then we say χ_M (or M) is a fuzzy small cover of χ_{M/μ^*} or M/μ.

Definition 4.3. Let M and L be any two modules over a ring R. Then an ephimorphism $f : M \to L$ is called a fuzzy small ephimorphism, if $f^{-1}(\chi_{\{\theta\}}) \ll_f M$.

It is obvious that M is a fuzzy small cover of M/μ if and only if the canonical projection $M \to M/\mu_*$ is a fuzzy small ephimorphism.

Definition 4.4. A fuzzy ideal μ of R with $\mu(0) = 1$ is called a fuzzy small ideal of R if it is a fuzzy small submodule of ${}_RR$.

Let μ and σ be any two fuzzy submodules of M such that $\mu \subseteq \sigma$, then μ is called a fuzzy submodule of σ. And μ is called a fuzzy small submodule in σ, denoted by $\mu \ll_f \sigma$, if $\mu \ll_f \sigma^*$ in the sense that for every submodule γ in M satisfying $\gamma_{|\sigma^*} \neq \chi_{\sigma^*}$ implies $\mu_{|\sigma^*} + \gamma_{|\sigma^*} \neq \chi_{\sigma^*}$ (by $\mu_{|\sigma^*}$, $\gamma_{|\sigma^*}$ we mean the restriction mapping of μ, γ on σ^* resp.).

Definition 4.5. Let M be a module over a ring R and suppose that $\mu, \nu \in F(M)$ with $\mu \subseteq \nu$. Then we say ν lies above μ in M or μ is a coessential extension of ν if M/μ is a fuzzy small cover of M/ν, that is, $\nu/\mu \ll_f M/\mu (=\chi_{M/\mu^*})$.

Example 4.6. Consider $M = Z_8 = \{0,1,2,3,4,5,6,7\}$ under addition modulo 8. Then M is a module over the ring Z. Let $S = \{0,2,4,6\}$. Define $\mu \in [0,1]^M$ as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x \in S, \\ \alpha & \text{otherwise.} \end{cases} \tag{4.1}$$

where $0 \leq \alpha < 1$. Then μ is a fuzzy small submodule of M.

Remark 4.7. Let $K = \{0,4\}$. Clearly S, K are the only proper submodules of M and $S + K = \{0,2,4,6\} + \{0,4\} \neq M$. Therefore $S \ll M$. Also $\mu_* = \{0,2,4,6\} = S$. Moreover, if we take $\alpha = 0$ then μ becomes the characteristic function of S.

The above remark inspires us to state the following two theorems.

Theorem 4.8. *Let M be a module and $N \leq M$. Then $N \ll M$ if and only if $\chi_N \ll_f M$.*

Proof. Let $N \ll M$. We assume χ_N is not a fuzzy small submodule of M. Thus there exists, $\nu \in F(M)$, $\nu \neq \chi_M$ such that $\chi_N + \nu = \chi_M$. Let $x \in M$. Then

$$1 = (\chi_N + \nu)(x) = \vee\{\chi_N(y) \wedge \nu(z) \mid y,\ z \in M,\ y + z = x\}. \tag{4.2}$$

So, there exist $y_0, z_0 \in M$ with $y_0 + z_0 = x$ such that $\chi_N(y_0) \wedge \nu(z_0) = 1$. Thus we have $\chi_N(y_0) = 1$ and $\nu(z_0) = 1$, and so $y_0 \in N$, $z_0 \in \nu_*$. This implies that $x = y_0 + z_0 \in N + \nu_*$. Since $x \in M$ is arbitrary, so this implies $M = N + \nu_*$. But $N \ll M$. So, we must have $M = \nu_*$ and this implies $\nu = \chi_M$, a contradiction. Therefore, $\chi_N \ll_f M$.

Conversely, we assume $\chi_N \ll_f M$. If possible let N be not a small submodule of M. Thus there exists $T \leq M$, $T \neq M$, but $N + T = M$. Thus $\chi_N, \chi_T \in F(M)$ and $\chi_N \neq \chi_M$,

$\chi_T \neq \chi_M$. Let $x \in M$. Since $N + T = M$, so there exist, $n \in N$, $l \in T$ such that $x = n + l$. Now,

$$(\chi_N + \chi_T)(x) = \vee\{\chi_N(y) \wedge \chi_T(z) \mid y,\ z \in M,\ y + z = x\} \geq \chi_N(n) \wedge \chi_T(l) = 1. \tag{4.3}$$

This implies $\chi_N + \chi_T = \chi_M$ and it contradicts the fact that $\chi_N \ll_f M$. Hence $N \ll M$. □

Here we present an alternative proof of the following theorem.

Theorem 4.9 (see [8]). *Let $\mu \in F(M)$. Then $\mu \ll_f M$ if and only if $\mu_* \ll M$.*

Proof. Suppose, $\mu \ll_f M$. Let $N \leq M$ and $N \neq M$. We claim $\mu_* + N \neq M$. Now, $N \neq M$ implies $\chi_N \neq \chi_M$. Since $\mu \ll_f M$, so we must have $\mu + \chi_N \neq \chi_M$.

$$\begin{aligned}
&\Rightarrow \text{there exists } x_0 \in M \text{ such that } (\mu + \chi_N)(x_0) < 1,\\
&\Rightarrow \vee\{\mu(y) \wedge \chi_N(z) \mid y, z \in M,\ y + z = x_0\} < 1,\\
&\Rightarrow \text{either } \mu(y) < 1 \text{ or } z \notin N, \text{ for all } y, z \in M,\ y + z = x_0,\\
&\Rightarrow \text{either } y \notin \mu_* \text{ or } z \notin N, \text{ for all } y, z \in M,\ y + z = x_0,\\
&\Rightarrow x_0 = y + z \ \notin \mu_* + N,\\
&\Rightarrow \mu_* + N \neq M,\\
&\Rightarrow \mu_* \ll M.
\end{aligned}$$

Conversely, we assume $\mu_* \ll M$. Let $\nu \in F(M)$ be such that $\nu \neq \chi_M$. This implies that $\nu_* \neq M$. Thus $\mu_* + \nu_* \neq M$ (since $\mu_* \ll M$). This implies that there exists $x_0 \in M$ such that $x \notin \mu_* + \nu_*$. Thus for every $y, z \in M$ with $y + z = x_0$ implies either $y \notin \mu_*$ or $z \notin \nu_*$ otherwise $x_0 \in \mu_* + \nu_*$. Therefore,

$$\begin{aligned}
&\mu(y) < 1 \text{ or } \nu(z) < 1 \text{ for every } y,\ z \in M \text{ and } y + z = x_0,\\
&\Rightarrow \mu(y) \wedge \nu(z) < 1, \text{ for every } y,\ z \in M \text{ and } y + z = x_0,\\
&\Rightarrow \vee\{\mu(y) \wedge \nu(z) \mid y, z \in M, y + z = x_0\} < 1,\\
&\Rightarrow (\mu + \nu)(x_0) < 1,\\
&\Rightarrow \mu + \nu \neq \chi_M,\\
&\Rightarrow \mu \ll_f M.
\end{aligned}$$

□

Corollary 4.10. *Let $\mu, \sigma \in F(M)$. Then $\mu \ll_f \sigma$ if and only if $\mu_* \ll \sigma^*$.*

Proof. Let $\mu \ll_f \sigma \ \Rightarrow\ \mu \ll_f \sigma^*$. So, by above theorem we get $\mu_* \ll \sigma^*$. Conversely if $\mu_* \ll \sigma^*$. So, by above theorem $\mu \ll_f \sigma^*$. Hence $\mu \ll_f \sigma$. □

Theorem 4.11 (see [8]). *Let $\mu, \nu \in F(M)$. Then $\mu \ll_f M$, $\nu \ll_f M$ if and only if $\mu + \nu \ll_f M$.*

As a consequence, we obtain the following.

Theorem 4.12. *Any finite sum of fuzzy small submodules of M is also a fuzzy small submodule in M.*

Theorem 4.13. *Let $N \leq M$ and $\mu \in F(M)$ be such that $\mu \subseteq \chi_N$. If $\mu_{|_N} \ll_f N$, then $\mu \ll_f M$.*

Proof. Let $\nu \in F(M)$ be such that $\mu + \nu = \chi_M$. Let $x \in N$. Then

$$\begin{aligned}
&(\mu_{|N} + (\nu_{|N} \cap \chi_N))(x),\\
&= \vee\{\mu(y) \wedge (\nu_{|N} \cap \chi_N)(z) \mid y,\ z \in N,\ y + z = x\},\\
&= \vee\{\mu(y) \wedge \nu(z) \mid y,\ z \in N,\ y + z = x\},\\
&= (\mu + \nu)(x) = 1 \text{ (since } \mu + \nu = \chi_M).
\end{aligned}$$

Therefore, $(\mu_{|N} + (\nu_{|N} \cap \chi_N)) = \chi_N$. Since $\mu_{|N} \ll_f N$, so $\nu_{|N} \cap \chi_N = \chi_N$. This implies that $\chi_N \subseteq \nu$ and $\mu \subseteq \chi_N \subseteq \nu$. Thus $\mu + \nu \subseteq \nu \Rightarrow \chi_M \subseteq \nu \Rightarrow \chi_M = \nu$. Hence $\mu \ll_f M$. □

Corollary 4.14. *Let $\mu, \nu \in F(M)$ and $\mu \subseteq \nu$. If $\mu \ll_f \nu$, then $\mu \ll_f M$.*

Proof. By definition, $\mu \ll_f \nu$ means $\mu \ll_f \nu^*$. Therefore, from above theorem we get $\mu \ll_f M$. □

Theorem 4.15. *Let $\mu, \nu \in F(M)$. Then $\mu \ll_f \nu$ if and only if $\mu_* \ll \nu_*$.*

Proof. Suppose, $\mu \ll_f \nu$. Let $N \leq \nu_*$ and $N \neq \nu_*$. This implies $\chi_N \neq \chi_{\nu_*}$. Since $\mu \ll_f \nu$, so $\mu + \chi_N \neq \chi_{\nu_*}$. This ensures that there exists x_0 in ν_* such that $x_0 \notin \mu_* + N$. Thus $\mu_* + N \neq \nu_*$ and hence $\mu_* \ll \nu_*$.

Conversely, we assume $\mu_* \ll \nu_*$. This implies $\mu_* \ll \nu_* \leq \nu^*$. Therefore, $\mu_* \ll \nu^*$ (Lemma 3.1(b)). So, by Corollary 4.10 we have $\mu \ll_f \nu$. □

Definition 4.16. A fuzzy submodule σ in M is called a fuzzy direct sum of two fuzzy submodules μ and ν if $\sigma = \mu + \nu$ and $\mu \cap \nu = \chi_\theta$.

Definition 4.17. Any $\mu \in F(M)$ is called a fuzzy direct summand of M if there exists $\nu \in F(M)$ such that χ_M is a fuzzy direct sum of μ, ν.

Theorem 4.18. *Let μ, ν be fuzzy submodules of M with $\mu \subseteq \nu$ and ν be a direct summand of M. Then $\mu \ll_f M$ if and only if $\mu \ll_f \nu$.*

Proof. Suppose, $\mu \ll_f M$. Since ν is a direct summand of M, so there exists $\gamma \in F(M)$ such that

$$\chi_M = \nu + \gamma, \qquad \nu \cap \gamma = \chi_\theta. \tag{4.4}$$

First we prove $M = \nu^* \oplus \gamma^*$. Now, $\nu \cap \gamma = \chi_\theta$ implies $\nu^* \cap \gamma^* = \theta$. Also, from $\chi_M = \nu + \gamma$ we have $M = (\nu + \gamma)^*$. We claim $(\nu + \gamma)^* = \nu^* + \gamma^*$. For this let $x \in (\nu + \gamma)^*$. Then

$$\begin{aligned}
&(\nu + \gamma)(x) = \vee\{\nu(y) \wedge \gamma(z) \mid y,\ z \in M, y + z = x\} > 0,\\
&\Rightarrow \nu(y) > 0 \text{ and } \gamma(z) > 0 \text{ for some } y, z \in M,\ y + z = x,\\
&\Rightarrow x = y + z, \text{ for some } y \in \nu^*, z \in \gamma^*,\\
&\Rightarrow x \in \nu^* + \gamma^*,\\
&\Rightarrow (\nu + \gamma)^* \subseteq \nu^* + \gamma^*,
\end{aligned}$$

On the other hand if $x \in \nu^* + \gamma^*$, then $x = y + z$ for some $y,\ z \in M$ with $\nu(y) > 0$, $\gamma(z) > 0$. This implies $0 < \vee\{\nu(y) \wedge \gamma(z) \mid y,\ z \in M,\ y + z = x\} = (\nu + \gamma)(x)$. Thus $x \in (\nu + \gamma)^*$ and so, we have $(\nu + \gamma)^* = \nu^* + \gamma^*$ and hence $M = (\nu + \gamma)^* = \nu^* + \gamma^*$. Therefore, ν^* is a direct summand

of M. But, we know $\mu \ll_f M$ if and only if $\mu_* \ll M$. Thus $\mu_* \ll \nu^*$ and ν^* is a direct summand of M and so, by Lemma 3.1(c) we get $\mu_* \ll \nu^*$. Hence, by Corollary 4.10 we have $\mu \ll_f \nu$.

The proof of the converse part follows from Corollary 4.14. □

Theorem 4.19. *Let $\sigma_1, \sigma_2 \in F(M)$ and $\chi_M = \sigma_1 \oplus \sigma_2$. Also, let $\mu_1, \mu_2 \in F(M)$ be such that $\mu_1 \subseteq \sigma_1$ and $\mu_2 \subseteq \sigma_2$. Then $\mu_1 \ll_f \sigma_1$ and $\mu_2 \ll_f \sigma_2$ if and only if $\mu_1 \oplus \mu_2 \ll_f \sigma_1 \oplus \sigma_2$, that is, if and only if $\mu_1 \oplus \mu_2 \ll_f M$.*

Proof. Suppose, $\mu_1 \ll_f \sigma_1$ and $\mu_2 \ll_f \sigma_2$. Then $\mu_{1_*} \ll_f \sigma_1^*$ and $\mu_{2_*} \ll_f \sigma_2^*$ (Corollary 4.10). Therefore, we get $\mu_{1_*} \oplus \mu_{2_*} \ll \sigma_1^* \oplus \sigma_2^*$ (Lemma 3.1(d)). Now, $\chi_M = \sigma_1 \oplus \sigma_2$ implies $M = (\sigma_1 \oplus \sigma_2)^* = \sigma_1^* \oplus \sigma_2^*$ and since $\mu_{1_*} \oplus \mu_{2_*} = (\mu_1 \oplus \mu_2)_*$, therefore we get $(\mu_1 \oplus \mu_2)_* \ll (\sigma_1 \oplus \sigma_2)^*$. Hence $\mu_1 \oplus \mu_2 \ll_f \sigma_1 \oplus \sigma_2$ (Corollary 4.10).

Conversely, we assume $\mu_1 \oplus \mu_2 \ll \sigma_1 \oplus \sigma_2$ if and only if $\mu_1 \oplus \mu_2 \ll_f M$. Now, $\mu_1 \leq \mu_1 \oplus \mu_2$ and $\mu_1 \oplus \mu_2 \ll_f M$ imply $\mu_1 \ll_f M$. Again, since σ_1 is a fuzzy direct summand of M and $\mu_1 \subseteq \sigma_1$, so by Theorem 4.18 we get $\mu_1 \ll_f \sigma_1$. Similarly, it can be proved that $\mu_2 \ll_f \sigma_2$. □

Corollary 4.20. *Let $M_1 \leq M$, $M_2 \leq M$ and $M = M_1 \oplus M_2$. Let $\mu_1 \in F(M_1)$, $\mu_2 \in F(M_2)$. Define*

$$\overline{\mu_i}(x) = \begin{cases} \mu_i(x) & \text{if } x \in M_i, \\ 0 & \text{otherwise}. \end{cases} \tag{4.5}$$

for all $x \in M$, $i = 1, 2$. Then $\overline{\mu_1} \oplus \overline{\mu_2} \ll_f M$ if and only if $\mu_1 \ll_f M_1$ and $\mu_2 \ll_f M_2$.

Theorem 4.21 (see [8]). *Let M, $\overline{M}$ be any two modules over the same ring R and let $f : M \to \overline{M}$ be a module homomorphism. If $\mu \ll_f M$, then $f(\mu) \ll_f \overline{M}$.*

Theorem 4.22. *Let $\mu, \nu \in F(M)$ be such that $\mu \subseteq \nu$. Then $\nu \ll_f M$ if and only if $\mu \ll_f M$ and $\nu/\mu \ll_f \chi_M/\mu$, that is, if and only if $\mu \ll_f M$ and $\nu/\mu \ll_f M/\mu^*$.*

Proof. It is obvious that, $\chi_M/\mu = \chi_{M/\mu^*}$. So, it is sufficient to show $\nu \ll_f M$ if and only if $\mu \ll_f M$ and $\nu/\mu \ll_f M/\mu^*$.

Suppose, $\nu \ll_f M$. Then since $\mu \subseteq \nu$, so $\mu \ll_f M$. Next, we will prove $\nu/\mu \ll_f M/\mu^*$. Consider, the natural homomorphism, $f : M \to M/\mu^*$, defined by $f(x) = [x]$, where $[x]$ denotes the coset $x + \mu^*$. Since $\nu \ll_f M$, so by Theorem 4.21 we get, $f(\nu) \ll_f M/\mu^*$. Now, $f(\nu)([x])$

$$\begin{aligned}
&= \vee\{\nu(y) \mid y \in M,\ f(y) = [x]\}, \\
&= \vee\{\nu(y) \mid y \in M,\ [y] = [x]\}, \\
&= \vee\{\nu(y) \mid y \in M,\ y - x \in \mu^*\}, \\
&= \vee\{\nu(y) \mid y \in M,\ y - x = m,\ m \in \mu^*\}, \\
&= \vee\{\nu(x + m) \mid m \in \mu^*\}, \\
&= \vee\{\nu(u) \mid u = x + m,\ m \in \mu^*\}, \\
&= \vee\{\nu(y) \mid u \in [x]\}, \\
&= \nu/\mu([x]) \text{ for all } x \in \nu^*.
\end{aligned}$$

Thus $\nu/\mu = f(\nu)$ on ν^* and $f(\nu) \ll_f M/\mu^*$. Therefore, $\nu/\mu \ll_f M/\mu^*$.

Conversely, we assume $\mu \ll_f M$ and $\nu/\mu \ll_f M/\mu^*$. To show $\nu \ll_f M$, let $\sigma \in F(M)$ be such that $\sigma \neq \chi_M$. Since $\mu \ll_f M$ and $\sigma \neq \chi_M$, so we have $\mu + \sigma \neq \chi_M$. This implies $(\mu + \sigma)/\mu \neq \chi_M/\mu (=\chi_{M/\mu^*})$. Again, since $\nu/\mu \ll_f M/\mu^*$ and $(\mu + \sigma)/\mu \neq \chi_{M/\mu^*}$, so we must have

$$
\begin{aligned}
&(\nu/\mu) + (\mu + \sigma)/\mu \neq \chi_{M/\mu^*},\\
&\Rightarrow (\nu + \mu + \sigma)/\mu \neq \chi_{M/\mu^*},\\
&\Rightarrow (\mu + \nu + \sigma)/\mu \neq \chi_{M/\mu^*},\\
&\Rightarrow (\nu + \sigma)/(\mu \cap (\nu + \sigma)) \neq \chi_{M/\mu^*}, \text{ (see [9, Theorem 4.2.5])}\\
&\Rightarrow (\nu + \sigma)/\mu \neq \chi_{M/\mu^*}, \text{ (since } \mu \subseteq \nu)\\
&\Rightarrow \nu + \sigma \neq \chi_M, \text{ (since } \chi_M/\mu = \chi_{M/\mu^*}).
\end{aligned}
$$

Therefore, $\nu \ll_f M$. □

Corollary 4.23. *Let $\mu, \nu \in F(M)$ be such that $\mu \subseteq \nu$. If $\nu \ll_f M$, then μ is a coessential extension of ν in M.*

Theorem 4.24. *Let $\mu, \nu \in F(M)$ be such that $\mu \subseteq \nu$. Then μ is a coessential extension of ν in M if and only if $\mu + \sigma = \chi_M$ holds for all $\sigma \in F(M)$ with $\nu + \sigma = \chi_M$.*

Proof. Suppose, μ is a coessential extension of ν, that is, $\nu/\mu \ll_f \chi_M/\mu$. If for all $\sigma \in F(M)$ with $\nu + \sigma = \chi_M$, then

$$\frac{\chi_M}{\mu} = \frac{(\nu + \sigma)}{\mu} = \frac{(\nu + \sigma + \mu)}{\mu} = \frac{\nu}{\mu} + \frac{(\sigma + \mu)}{\mu}. \tag{4.6}$$

So, $\chi_M/\mu = (\sigma + \mu)/\mu$ (since $\nu/\mu \ll_f \chi_{M/\mu}$). Thus $\mu + \sigma = \chi_M$.

Conversely, we assume $\mu + \sigma = \chi_M$ holds for all $\sigma \in F(M)$ with $\nu + \sigma = \chi_M$. If there exists $\sigma \in F(M)$ containing μ such that $\nu/\mu + \sigma/\mu = \chi_M/\mu$, then $\chi_M = \nu + \sigma$. This yields $\chi_M = \mu + \sigma = \sigma$, (since $\mu \subseteq \sigma$) and so $\nu/\mu \ll_f \chi_M/\mu$. Hence μ is a coessential extension of ν. □

As a consequence, we obtain the following.

Theorem 4.25. *Let $\gamma, \mu, \nu \in F(M)$ be such that $\gamma \subseteq \mu \subseteq \nu$. Then γ is a coessential extension of ν in M if and only if μ is a coessential extension of ν in M and γ is a coessential extension of μ in M.*

5. Jacobson L-Radical

Definition 5.1. Let $\mu \in L(M)$. Then μ is called a maximal L-submodule of M if $\mu \neq \chi_M$ (i.e., μ is proper L-submodule of M) and if σ any other proper L-submodules of M containing μ, then $\mu = \sigma$. Equivalently μ is a maximal element in the set of all nonconstant L-submodules of M under point wise partial ordering.

The intersection of all maximal L-submodules of M is known as Jacobson L-radical of M and is denoted by JLR(M).

Theorem 5.2. *Let $\mu \in L^M$. Then μ is a maximal L-submodule of M if and only if μ can be expressed as $\mu = \chi_{\mu_*} \cup \alpha_M$, where μ_* is a maximal submodule of M and α is a maximal element of L-$\{1\}$.*

Proof. Proof is similar to the proof of Theorem 3.4.3 of [9] and so it is omitted. □

Lemma 5.3. *Let M be a module over R and let $x \in M$. Then $\chi_R \odot \chi_{\{x\}} \ll_L M$ if and only if $\chi_{\{x\}}$ is in the sum of all small L-submodules of M.*

Proof. Suppose, $\chi_R \odot \chi_{\{x\}} \ll_L M$. Then by Theorem 4.9 we have, $(\chi_R \odot \chi_{\{x\}})_* \ll M$. But, from Remark 2.10(b) we have

$$\chi_R \odot \chi_{\{x\}} = \langle \chi_{\{x\}} \rangle = \chi_{\langle \{x\} \rangle} = \chi_{Rx}. \tag{5.1}$$

Therefore, $\chi_{\{x\}} \subseteq \chi_R \odot \chi_{\{x\}}$ as $\chi_{\{x\}} \subseteq \chi_{Rx}$. This implies $\chi_{\{x\}}$ is in the sum of all small L-submodules of M.

Conversely, we assume $\chi_{\{x\}}$ is in the sum of all small L-submodules of M. Then $\chi_{\{x\}} \subseteq \sum \mu_i$ (finite), where $\mu_i \ll_L M$ and so, $x \in \sum (\mu_i)_*$(finite), $(\mu_i)_* \ll M \Rightarrow x = \sum x_i$(finite), where $x_i \in (\mu_i)_*$. Now,

$$(\chi_R \odot \chi_{\{x_i\}})_* = (\chi_{Rx_i})_* = Rx_i \leq (\mu_i)_* \tag{5.2}$$

for all i. Therefore, $(\chi_R \odot \chi_{\{x_i\}})_* \ll M$ (since $(\mu_i)_* \ll M$). Therefore, by Theorem 4.9 we have $\chi_R \odot \chi_{\{x_i\}} \ll_L M$. This implies $\sum \chi_R \odot \chi_{\{x_i\}}$(finite) $\ll_L M$. But, $\chi_{\{x\}} \subseteq \sum \chi_R \odot \chi_{\{x_i\}}$(finite), and so, we must have, $\chi_R \odot \chi_{\{x\}} \subseteq \sum \chi_R \odot \chi_{\{x_i\}}$(finite). Since $\sum \chi_R \odot \chi_{\{x_i\}}$(finite) $\ll_L M$. Therefore, we have $\chi_R \odot \chi_{\{x\}} \ll_L M$. □

Definition 5.4 (see [9]). Let L be a complete Heyting algebra. Then $a \in L$-$\{1\}$ is called a maximal element, if there does not exist $c \in L$-$\{1\}$ such that $a < c < 1$.

Theorem 5.5. *For any module M, JLR(M) (the Jacobson L-radical of M), is the sum of all small L-submodules of M.*

Proof. In view of Lemma 5.3, it is sufficient to show that $\chi_R \odot \chi_{\{x\}} \ll_L M$ if and only if $\chi_{\{x\}} \subseteq$ JLR(M); or equivalently: $\chi_{\{x\}}$ is not a subset of JLR(M) if and only if $\chi_R \odot \chi_{\{x\}}$ is not a small L-submodule of M. We will proof the later one.

Suppose, $\chi_{\{x\}}$ is not a subset of JLR(M). Then there exists a maximal L-submodule ν of M such that $\chi_{\{x\}}$ is not a subset of ν. This implies $\chi_R \odot \chi_{\{x\}} = \langle \chi_{\{x\}} \rangle$ which is not a submodule of ν. Since ν is maximal, so $\nu \neq \chi_M$. Therefore, ν is a maximal L-submodule of M which is properly contained in $\chi_R \odot \chi_{\{x\}} + \nu$. This implies $\chi_R \odot \chi_{\{x\}} + \nu = \chi_M$. Thus $\nu \neq \chi_M$ and $\chi_R \odot \chi_{\{x\}} + \nu = \chi_M$. So, by definition we have, $\chi_R \odot \chi_{\{x\}}$ is not a small L-submodule of M.

Conversely, we assume $\chi_R \odot \chi_{\{x\}}$ is not a small L-submodule of M. Then there exists a $\nu \in L(M)$ with $\nu \neq \chi_M$ such that $\chi_R \odot \chi_{\{x\}} + \nu = \chi_M$. Let S be the collection of all such ν. Then $S \neq \Phi$ because, $\nu \in S$. Now, for each $\sigma \in S$, $\sigma \neq \chi_M$ and $\chi_{\{x\}}$ is not a subset of σ. Moreover, any

proper fuzzy submodule containing ν is also in S. Also, $(S, \subseteq)$ forms a poset and the union of members of a chain in S is again a member of S. Therefore, by Zorn's lemma, S has a maximal element, (say) μ. Since $\mu \in S$ so, $\chi_{\{x\}}$ is not a subset of μ. Now, let $\gamma \in L(M)$ be such that $\mu \subseteq \gamma$. If $\gamma \neq \chi_M$, then γ is also in S. So, by maximality of μ, we have $\mu = \gamma$. This shows that μ is a maximal L-submodule of M. Since $\chi_{\{x\}}$ is not a subset of μ, so we must have $\chi_{\{x\}}$ is not a subset of JLR(M). □

Corollary 5.6. *If JLR(M) is a small L-submodule of M, then it is the largest small L-submodule of M.*

Corollary 5.7. *Let M be finitely generated module, then JLR(M) exists and is a small L-submodule of M provided L-$\{1\}$ has a maximal element.*

Remark 5.8. However, if we take $L = [0, 1]$, then L-$\{1\}$ does not possess any maximal element, and so by Theorem 5.2, maximal L-submodule of M does not exist. Since the existence of the JLR(M) depends on the existence of maximal L-submodule of M, therefore, the assumption of the existence of a maximal element of L-$\{1\}$ in corollary 5.5 is necessary.

Example 5.9. Let $L = \{0, 0.25, 0.5, 0.75, 1\}$. Then L is a Complete Heyting algebra together with the operations minimum(meet), maximum(join) and $\leq$(partial ordering), then 0.75 is a maximal element of L-$\{1\}$. Consider $M = Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ under addition modulo 8. Then M is a module over the ring Z. Let $S = \{0, 2, 4, 6\}$. Define $\mu \in [0, 1]^M$ as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0.75, & \text{otherwise.} \end{cases} \tag{5.3}$$

Then $\mu_* = \{0, 2, 4, 6\} = S$, which is a maximal submodule of Z_8. Also, $\mu = \chi_{\mu_*} \cup 0.75_M$, where 0.75 is a maximal element of L-$\{1\}$. So, by Theorem 5.2 we have μ as a maximal L-submodule of Z_8. In fact, μ is the only maximal L-submodule of Z_8 and so JLR$(M) = \mu$. Since $\mu_* \ll Z_8$ and hence by Theorem 4.9, we get $\mu \ll_L Z_8$. Thus JLR$(M) \ll_L Z_8$. However, if we consider $L = [0, 1]$, then [0,1) does not have a maximal element and so by Theorem 5.2 there does not exist any maximal L-submodule(maximal fuzzy submodule). So, JLR(M) does not exist when $L = [0, 1]$.

6. Conclusion

In this paper some aspects and properties of fuzzy small submodules have been introduced which dualize the notion of fuzzy essential submodules. This concept has opened a new avenue toward the study of fuzzy Goldie dimension, for example using the notion of fuzzy small submodules one can define hollow submodules and discrete submodules. In our future study we may investigate various aspects of (i) spanning dimension of fuzzy submodules, (ii) corank of fuzzy submodules, (iii) fuzzy lifting modules with chain condition on fuzzy small submodules, and (iv) Noetherian and Artinian conditions on fuzzy Jacobson radical of a module.

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