Research Article

Graph of BCI-Algebras

Omid Zahiri and Rajab Ali Borzooei

Department of Mathematics, Shahid Beheshti University, 1983963113 Evin, Tehran, Iran

Correspondence should be addressed to Omid Zahiri, o_zahiri@sbu.ac.ir

Received 5 July 2011; Accepted 6 October 2011

Academic Editor: Alexander Rosa

We associate a graph to any subset \( Y \) of a BCI-algebra \( X \) and denote it by \( G(Y) \). Then we find the set of all connected components of \( G(X) \) and verify the relation between \( X \) and \( G(X) \), when \( X \) is commutative BCI-algebra or \( G(X) \) is complete graph or \( n \)-star graph. Finally, we attempt to investigate the relation between some operations on graph and some operations on BCI-algebras.

1. Introduction

BCK- and BCI-algebras are two classes of abstract algebras were introduced by Imai and Iséki [1, 2], in 1966. The notion of BCK-algebras is originated from two different ways. One of the motivations is from classical and nonclassical propositional logic. Another motivation is based on set theory. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Many authors studied the graph theory in connection with semigroups and rings. For example, Beck [3] associated to any commutative ring \( R \) its zero divisors graph \( G(R) \), whose vertices are the zero divisors of \( R \), with two vertices \( a, b \) jointed by an edge in case \( ab = 0 \). In [4], Jun and Lee defined the notion of zero divisors and quasi-ideals in BCI-algebra and show that all zero divisors are quasi-ideal. Then, they introduced the concept of associated graph of BCK / BCI-algebra and verified some properties of this graph and proved that if \( X \) is a BCK-algebra, then associated graph of \( X \) is a connected graph. Moreover, if \( X \) is a BCI-algebra and \( x \in X \) such that \( x \) is not contained in BCK-part of \( X \), then there is not any edge connecting \( x \) and \( y \), for any \( y \in X \). In this paper, we associate new graph to a BCI-algebras \( X \) which is denoted it by \( G(X) \). This definition based on branches of \( X \). If \( X \) is a BCK-algebras, then this definition and last definition, which was introduced by Jun and Lee, are the same. Then, for any \( a \in P \), we defined the concept of \( a \)-divisor, where \( P \) is a \( p \)-semisimple part of BCI-algebra \( X \) and show that it is quasi-ideal of \( X \). Then, we explain some properties of this graph as mentioned in the abstract.
2. Preliminaries

Definition 2.1 (see [1, 2]). A BCI-algebra is an algebra \((X, \ast, 0)\) of type \((2, 0)\) satisfying the following conditions:

\begin{enumerate}
\item[(BCI1)] \(((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0;\)
\item[(BCI2)] \(x \ast 0 = x;\)
\item[(BCI3)] \(x \ast y = 0\) and \(y \ast x = 0\) imply \(y = x.\)
\end{enumerate}

Moreover, the relation \(\leq\) was defined by \(x \leq y \Leftrightarrow x \ast y = 0,\) for any \(x, y \in X,\) is a partial-order on \(X,\) which is called BCI-ordering of \(X.\) The set \(B = \{x \in X \mid 0 \ast x = 0\}\) is called BCK-part of \(X.\) A BCI-algebra \(X\) is called a BCK-algebra if \(B = X.\) A BCK-chain is a BCK-algebra such that \((X, \leq)\) is a chain, where \(\leq\) is the BCI-ordering of \(X.\) A nonzero element \(a\) of BCK-algebra \(X\) is called an atom of \(X\) if \(x \ast a = 0\) implies \(x = a,\) for any nonzero element \(x \in X.\) Moreover, \(P = \{x \in X \mid 0 \ast (0 \ast x) = x\}\) is called \(p\)-semisimple part of a BCI-algebra \(X.\) It is the set of all minimal elements of \(X,\) with respect to the BCI-ordering of \(X.\) The BCI-algebra \(X\) is called a \(p\)-semisimple BCI-algebra if \(P = X.\) For any \(a \in P,\) we use the notation \(V_X(a)\) or simply \(V(a)\) to denote the set \(\{x \in X \mid a \ast x = 0\}\) which is called the branch of \(X\) with respect to \(a.\) It is easy to see that \(X = \bigcup_{a \in P} V(a),\) and for any distinct elements \(a, b \in P,\) we have

\begin{enumerate}
\item[(P1)] \(V(a) \cap V(b) = \emptyset;\)
\item[(P2)] if \(x \in V(a)\) and \(y \in V(b),\) then \(x \ast y \in V(a \ast b);\)
\item[(P3)] if \(x \in V(a),\) then \(\{y \in X \mid x \leq y\} \cup \{y \in X \mid y \leq x\} \subseteq V(a);\)
\item[(P4)] for all \(x \in X,\) \(x \ast a = 0\) implies \(x = a.\)
\end{enumerate}

A BCI-algebra \(X\) is called commutative if \(x \leq y\) implies \(x = y \ast (y \ast x),\) for any \(x, y \in X.\) Moreover, \(X\) is called branchwise commutative if \(x \ast (x \ast y) = y \ast (y \ast x),\) for all \(x, y \in V(a)\) and all \(a \in P.\) For more details, we refer to [5–11].

Lemma 2.2 (see [9]). A BCI-algebra \(X\) is commutative if and only if it is branchwise commutative.

Theorem 2.3 (see [10]). Let \(X = \{x_0, x_1, \ldots, x_n\}\) and \((X, \ast, 0)\) be a BCK-chain with \(x_0\) as the zero element, whose BCI-ordering is supposed as follows: \(x_0 < x_1 < \cdots < x_n.\) Then, \(X\) is commutative if and only if the relation \(*\) on \(X\) is given by \(x_i * x_j = x_{i \wedge j},\) where \(i * j = \max\{0, i - j\},\) for any \(i, j \in \{0, 1, \ldots, n\}\) (see [10, Theorem 2.3.3]).

A partial-order set \((P, \leq)\) is said to be of finite length if the lengths of all chains of \(P\) are bounded. Let \(x, y \in P.\) Then, a chain of length \(n\) between \(x\) and \(y\) is a chain \(a_0 < a_1 < \cdots < a_{n-1} < a_n\) such that \(\{a_0, a_n\} = \{x, y\}.\) For any \(x, y \in P, l(x, y)\) is the greatest number in the lengths of all chain between \(a\) and \(b.\)

A BCI-algebra \(X\) is said to be finite length if it is finite length as a partial-order set.
Note 1 (see [4]). Let $X$ be a $BCI$-algebra and $A \subseteq X$. We will use the notations $U_X(A)$ and $L_X(A)$ or simply $U(A)$ and $L(A)$ to denote the sets $\{x \in X \mid a * x = 0, \forall a \in A\}$ and $\{x \in X \mid x * a = 0, \forall a \in A\}$, respectively, that is, $U(A) = \{x \in X \mid a \leq x, \forall a \in A\}$ and $L(A) = \{x \in X \mid x \leq a, \forall a \in A\}$.

Definition 2.4 (see [4]). A nonempty subset $I$ of $BCI$-algebra $X$ is called a quasi-ideal of $X$ if $x * y = 0$ implies $x \in I$, for any $y \in I$ and $x \in X$.

Proposition 2.5 (see [12]). Let $(X, *, 0)$ and $(Y, *, 0)$ be two $BCI$-algebras such that $X \cap Y = \{0\}$ and $*$ be the binary operation on $U \cap Y$ as follows: for any $a, b \in U \cap Y$;

$$a * b = \begin{cases} 
  a *_1 b & \text{if } a, b \in X, \\
  a *_2 b & \text{if } a, b \in Y, \\
  a & \text{otherwise.}
\end{cases}$$

Then, $(U \cup Y, *, 0)$ is a $BCI$-algebra. We denote $(U \cup Y, *, 0)$ by $U \oplus Y$.

Definition 2.6 (see [4]). For any $x \in X$, the set $\{y \in X \mid L(\{x, y\}) = \{0\}\}$ is called the set of all zero divisors of $x$.

Theorem 2.7 (see [4]). For any element $x$ of $BCI$-algebra $X$, the set of all zero divisors of $x$ is a quasi-ideal of $X$ containing the zero element $\{0\}$.

Let $G$ be a graph, $E(G)$ be the set of all edges of $G$ and $V(G)$ be the set of all vertexes of $G$. For any $S \subseteq E(G)$, the graph with vertex set $V(G)$ and edge set $E(G) - S$ is denoted by $G - S$. The edge which connect two vertexes $x, y$ is denoted by $xy$. Note that, $xy$ and $yx$ are the same. For any $T \subseteq V(G)$, the graph with vertex set $V(G) - T$ and edge set $E(G) - T'$ is denoted by $G - T$, where $T' = \{xy \in E(G) \mid x \in T, y \in G\}$. A graph $H$ is called a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph $G = (V, E)$ is connected, if any vertices $x, y$ of $G$ linked by a path in $G$, otherwise the graph is disconnected. A tree is a connected graph with no cycles. The degree of a vertex $v$ in a graph $G$, denoted by $\deg(v)$, is the number of edges of $G$ incident with $v$ and $\deg(G) = \sum_{v \in V} \deg(v) = 2|E|$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of $G$, respectively. If $G$ and $H$ are two graphs such that $V(G) \cap V(H) = \emptyset$, then the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ is called the disjoint union of $G$ and $H$ and we denote it by $G \cup H$. Any graph $G = (V, E)$ may be expressed uniquely as a disjoint union of connected graphs. These graphs are called the connected components or simply components of $G$. The number of components of $G$ is denoted by $c(G)$. A graph $G$ is called empty graph if $E(G) = \emptyset$. Moreover, a graph $K$ is called complete graph if $xy \in E(G)$, for any distinct elements $x, y \in V(G)$. For more details we refer to [13, 14].

Definition 2.8. Let $G$ and $H$ be two graphs such that $V(G) \cap V(H) = \emptyset$. We denote the graph $G + H$, for the graph, whose vertex set and edge set are $V(G) \cup V(H)$ and $E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$, respectively.

Definition 2.9 (see [4]). By the associated graph of $BCI$ $BCI$-algebra $X$, denoted $\Gamma(X)$, we means the graph whose vertices are just the elements of $X$, and for distinct $x, y \in \Gamma(X)$, there is an edge connecting $x$ and $y$ if and only if $L(\{x, y\}) = \{0\}$.
Proof. Let $G$ be a finite graph with $|V(G)| = n$. Then, the adjacency matrix of $G$ is an $n \times n$ matrix $A_G = [a_{ij}]$ such that $a_{ij}$ is the number of edges joining $v_i$ and $v_j$. Moreover, we denote the characteristic polynomial of the matrix $A_G$, by $\chi(G, \lambda)$. That is $\chi(G, \lambda) = \det(\lambda I - A_G)$, where $I$ is a $n \times n$ identity matrix.

**Proposition 2.11** (see [15, Proposition 2.3]). Let $G$ be a graph and $\chi(G, \lambda) = \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_n$, be the characteristic polynomial of the adjacent matrix $G$. Then, $c_1 = 0$, $-c_2$ is the number of edges of $G$ and $-c_3$ is twice the number of triangles in $G$.

3. Graph Based on BCI-Algebras

From now on, in this paper, $(X, \ast, 0)$ or simply $X$ is a BCI-algebra, $B$ is a BCK-part and $P$ is a $p$-semisimple part of $X$, unless otherwise state. For all $x, y \in X$, we use $x \leq y$ to denote $x \leq y$ and $x \neq y$.

**Definition 3.1.** Note that the set of all 0-divisors of $x$ and the set of all zero divisors of $x$ are the same. Let $x \in X$. Then, there exists $a \in P$ such that $x \in V(a)$. We will use the notation $Z_x$ to denote the set of all $y \in X$ such that $L((x, y)) = \{a\}$, that is, $Z_x = \{y \in X \mid L((x, y)) = \{a\}\}$, which is called the set of $a$-divisor of $X$.

Note that the set of all 0-divisors of $x$ and the set of all zero divisors of $x$ are the same.

**Lemma 3.2.** Let $a, b \in P$ and $x, y \in X$. Then

(i) one has

$$L((x, a)) = \begin{cases} a & \text{if } x \in V(a), \\ \emptyset & \text{otherwise}, \end{cases}$$

(ii) if $a \neq b$, $x \in V(a)$ and $y \in V(b)$, then $L((x, y)) = \emptyset$.

**Proof.** (i) Clearly, if $x \in V(a)$, then $L((x, a)) = \{a\}$. Let $x \notin V(a)$. Then, $y \in L((x, a))$ implies $y \leq x$ and $y \leq a$. Since $a \in P$, we have $a = y$ and so $a \leq x$, which is impossible. Therefore, $L((x, a)) = \emptyset$.

(ii) Since $a \neq b$, we have $L((x, y)) \subseteq V(a) \cap V(b) = \emptyset$. \qed

**Lemma 3.3.** For any $x, y \in X$, if $x \ast y = 0$, then $L((x)) \subseteq L((y))$ and $Z_y \subseteq Z_x$.

**Proof.** Let $x \ast y = 0$, for some $x, y \in X$. Then, $x \leq y$. Clearly, $L(x) \subseteq L(y)$. If $x \in V(a)$, where $a \in P$, then $a \leq x \leq y$ and so by (P3), $y \in V(a)$, too. Now, let $u \in Z_y$. Then, $L((u, y)) = \{a\}$ and so $a \in L((u, x)) \subseteq L((u, y)) = \{a\}$. Hence, $L((u, x)) = a$ and so $u \in Z_x$. Therefore, $Z_y \subseteq Z_x$. \qed

Jun and Lee in [4] proved that if $x$ is an element of BCK-algebra $X$, then the set of all zero divisors of $x$ is a quasi-ideal of $X$. In Theorem 3.4, we will show that if $x$ is an element of BCI-algebra $X$, then $Z_x$ is a quasi-ideal of $X$.

**Theorem 3.4.** $Z_x$ is a quasi-ideal of $X$, for any $x \in X$. 

Proof. Let $x \in X$ such that $x \in V(a)$, for some $a \in P$ and $u, v \in X$ such that $u \ast v = 0$ and $v \in Z_x$. Then, $u \leq v$ and $L(\{v, x\}) = \{a\}$. Hence, $v \in V(a)$ and so by (P3), $u \in V(a)$. Since $a \in L(\{u, x\}) \subseteq L(\{v, x\}) = \{a\}$, then $L(\{u, x\}) = \{a\}$. Therefore, $u \in Z_x$ and so $Z_x$ is a quasi-ideal of $X$. \[\square\]

Definition 3.5. Let $Y \subseteq X$, and $G(Y)$ be a simple graph, whose vertices are just the elements of $Y$ and for distinct $x, y \in Y$, there is an edge connecting $x$ and $y$, denoted by $xy$ if and only if $L(\{x, y\}) = \{a\}$, for some $a \in P$. If $Y = X$, then $G(X)$ is called a BCI-graph of $X$. 

Clearly, if $X$ is a BCK-algebra, then $G(X) = \Gamma(X)$. But it is not true, in general.

Example 3.6. (i) Let $X = \{0, 1, a, b, c\}$. Define the binary operation “$\ast$” on $X$ by the following table:

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>0</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>a</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then, $(X, \ast, 0)$ is a BCI-algebra, $P = \{0, a\}$, $V(0) = \{0, 1\}$, and $V(a) = \{a, b, c\}$. Moreover, $L(\{1, 0\}) = \{0\}$ and $L(\{a, b\}) = L(\{a, c\}) = L(\{b, c\}) = \{a\}$ and so $E(G(X)) = \{10, ac, bc, ab\}$. Therefore, the graphs $G(X)$ and $\Gamma(X)$ which are given by Figure 1 are different.

(ii) Let $(X, \ast, 0)$ be the BCI-algebra in Example 3.6(i) and $Y = \{0, 1, b, c\}$. Then, $G(Y)$ is given by Figure 2.

Example 3.7 (see [4]). Let $X = \{0, a, b, c\}$. Define the binary operation “$\ast$” on $X$ by the following table:
which is impossible. Therefore, for some

Let $L$ be a directed graph. Conversely, let $L$ be a directed graph. Then, by Lemma 3.2(ii) $xy \in E(G(X))$. If there is a path $x, x_1, x_2, \ldots, x_n, y$, which link $x$ to $y$, then $a = L(\{x, x_1\}) \neq \emptyset$ and so by Lemma 3.2(ii), $x_1 \in V(a)$. By a similar way, we have $x_2, \ldots, x_n, y \in V(a)$, which is impossible. Hence, there is not any path between $x$ and $y$. Therefore, by (i), $G(X)$ is a graph with $|P|$ components.

**Theorem 3.9.** $X$ is a p-semisimple BCI-algebra if and only if $G(X)$ is an empty graph.

**Proof.** Clearly, if $X$ is a p-semisimple BCI-algebra, then by Lemma 3.2(ii), $G(X)$ is an empty graph. Conversely, let $G(X)$ be an empty graph and $x \in X$. Since, $0 * (0 * x) \in P$ and $x \in V(0 * (0 * x))$, then $L(\{0 * (0 * x), x\}) = \{0 * (0 * x)\}$. If $0 * (0 * x) \subseteq x$, then there exists an edge between $x$ and $0 * (0 * x)$ in $G(X)$, which is impossible. Hence, $x = 0 * (0 * x)$ and so $X$ is a p-semisimple BCI-algebra.

**Definition 3.10.** Let $a \in P$. The element $x \in V(a)$ is called an $a$-atom if $x \neq a$ and $y * x = 0$ implies $y = a$ or $y = x$, for all $y \in X$.

Note that if $X$ is a BCK-algebra then the concept of 0-atom and atom are the same.

**Lemma 3.11.** Let $X$ be a BCI-algebra of finite order. Then, for any $x \in X - P$, there is an $a$-atom $b \in X$, such that $b \leq x$, for some $a \in P$.

**Proof.** Let $x \in X - P$. Then, there is $a \in P$ such that $x \in V(a)$. Let $S = \{L(x, u) \mid u * x = 0, u \in V(a) - \{a\}\}$. Clearly, $0 \in S$. Since $X$ is of finite length, then $S$ has the greatest element. Let $L(x, u)$ be the greatest element of $S$. Then, we show that $v$ is an $a$-atom of $X$. Let $w * v = 0$, for some $w \in X$. Then, $w \leq v$ and so by (P3), $w \in V(a)$. If $w \not\leq v$, then $1 + L(x, u) \leq L(x, w)$, which is impossible. Therefore, $w = v$ or $w = a$ and so $v$ is an atom of $X$.

**Theorem 3.12.** Let $X$ be a finite length BCI-algebra and $a \in P$. Then

(i) $G(V(a))$ is a tree if and only if $V(a) = \{a\}$ or $V(a)$ has only one $a$-atom.

(ii) $G(X)$ is a tree if and only if $X = \{0\}$ or $X$ is a BCK-algebra with only one atom.
Proof. (i) Let $G(V(a))$ be a tree. If $V(a) = \{a\}$, we do not have any thing to prove. Let $V(a) \neq \{a\}$ and $A$ be the set of all $a$-atoms of $V(a)$. Then, by Lemma 3.11, we have $1 \leq |A|$. Let $x, y \in A, u \leq x$ and $u \leq y$, for some $u \in X$. Then, by (P3), $u \in V(a)$. Since $x, y$ are $a$-atoms of $V(a)$, then $u = a$ or $x = u = y$. Hence, $x = y$ or $xy \in E(G(X))$. If $xy \in E(G(X))$, then by Lemma 3.2(i), we have $xy, ax, ay \in E(G(X))$. Hence, $G(X)$ has a cycle, which is impossible. Therefore, $x = y$ and so $|A| = 1$. Conversely, let $V(a)$ has only one $a$-atom. By Proposition 3.8, $G(V(a))$ is a connected graph. If $V(a) = \{a\}$, then clearly, $G(V(a))$ is a tree. Let $V(a) \neq \{a\}$, $x, y \in V(a) - \{a\}$ and $u$ be an $a$-atom of $X$. Then, by Lemma 3.11, $u \in L(\{x, y\})$ and so $L(\{x, y\}) \neq \{a\}$. Hence, $E(G(X)) = \{xa \mid x \in V(a) - \{a\}\}$ and so $G(V(a))$ does not have any cycle. Therefore, $G(V(a))$ is a tree.

(ii) Let $G(X)$ be a tree. Then, $G(X)$ is a connected graph and so by Proposition 3.8, $|P| = 1$. Hence, $P = \{0\}$ and so, $X$ is a $BCK$-algebra. Since $X$ is a $BCK$-algebra, we have $X = V(0)$ and so by (i), $X = \{0\}$ or $X$ is a $BCK$-algebra with only one atom. The converse is straight consequent of (i). \hfill \Box

Example 3.13. Let $X = \mathbb{R}^+ = \{x \in \mathbb{R} \mid 0 \leq x\}$ and $x*y = \max\{x - y, 0\}$, for any $x, y \in X$. Then, $(X, \ast, 0)$ is a $BCK$-algebra. Clearly, $X$ does not have any atoms. Since $X$ is a $BCK$-algebra, then by Proposition 3.8, $G(X)$ is a connected graph. Moreover, $L(\{x, y\}) \neq \{0\}$, for any $x, y \in X - \{0\}$. Hence, $E(G(X)) = \{x0 \mid x \in X - \{0\}\}$. Therefore, $G(X)$ is a tree.

Example 3.14. Let “∗” be the binary operation in Example 3.13, $A = \{a_i \mid i \in \mathbb{R}^+\}$ and $B = \{b_i \mid i \in \mathbb{R}^+\}$, where $a_0 = b_0$ and $A \cap B = \{0\}$. Let a binary operation $\ast'$ on $X$ is defined as follows: for all $i, j \in \mathbb{R}^+$,

\begin{align*}
a_i \ast' a_j &= a_{i+j}; \\
b_i \ast' b_j &= b_{i+j}; \\
a_i \ast' b_j &= a_i; \\
b_i \ast' a_j &= b_i.
\end{align*}

(3.2)

By Example 3.13, $(A, \ast_1, 0)$ and $(B, \ast_2, 0)$ are two $BCK$-algebras, where $a_i \ast_1 a_j = a_{i+j}$ and $b_i \ast_2 b_j = b_{i+j}$, for any $i, j \in \mathbb{R}^+$. Hence, by Proposition 2.5, $(X, \ast', 0)$ is a $BCK$-algebra and so by Proposition 3.8, $G(X)$ is a connected graph. Let $a_i \in A - \{a_0\}$ and $b_j \in B - \{a_0\}$. Since $X$ is a $BCK$-algebra, then $0 \in L(\{a_i, b_j\})$. Let $u \in L(\{a_i, b_j\})$, for some $u \in X$. If $u = b_j$, for some $j \in \mathbb{R}^+$, then $u = u \ast' a_i = 0$. If $u = a_j$, for some $j \in \mathbb{R}^+$, then $u = u \ast' b_i = 0$. Hence,
**Proposition 3.15.** Let $X$ be a BCI-algebra such that the set $\{|V(a)| : a \in P\}$ is bounded. Then, $\Delta(G(X)) = \max\{|V(a)| - 1 : a \in P\}$.

**Proof.** Since the set $\{|V(a)| : a \in P\}$ is bounded, then there is a $u \in P$, such that $|V(u)| - 1 = \max\{|V(a)| - 1 : a \in P\}$. We show that $\deg(x) \leq \deg(u)$, for any $x \in X$. Let $x \in X$. Then, there is $a \in P$ such that $x \in V(a)$. Since $V(a) \cap V(b) = \emptyset$, for any $b \in P - \{a\}$, then by Proposition 3.8(ii) and Lemma 3.2(i), $\deg(x) \leq |V(a)| - 1 \leq |V(u)| - 1 = \deg(u)$. Therefore, $\Delta(G(X)) = \deg(u) = |V(u)| - 1 = \max\{|V(a)| - 1 : a \in P\}$. \hfill \Box

**Corollary 3.16.** (i) If $X$ is a finite BCI-algebra, then $\Delta(G(X)) = \max\{|V(a)| - 1 : a \in P\}$.

(ii) If $X$ is a finite BCK-algebra, then $\Delta(G(X)) = |X| - 1$.

**Proof.** (i) If $X$ is a finite BCI-algebra, then $\{|V(a)| : a \in P\}$ is bounded. Therefore, by Proposition 3.15, $\Delta(G(X)) = \max\{|V(a)| - 1 : a \in P\}$.

(ii) If $X$ is a BCK-algebra, then $P = \{0\}$ and so $\{|V(a)| - 1 : a \in P\} = \{|V(0)| - 1\} = \{|X| - 1\}$. Hence, by Proposition 3.15, $\Delta(G(X)) = |X| - 1$. \hfill \Box

**Definition 3.17.** Let $G$ be a graph with $n + 1$ vertices. Then $G$ is called an $n$-star graph if it has Figure 4.

**Theorem 3.18.** Let $X$ be a BCK-algebra with $m + 1$ elements and only one atom. Then $G(X)$ is an $m$-star graph. Moreover, if $G$ is an $m$-star graph, then there is a BCK-algebra $X$ such that $G(X) = G$.

**Proof.** Since, $X$ is a BCK-algebra, then $G(X)$ is a connected graph. Let $a$ be an atom of $X$. Then by Lemma 3.11, $a \leq x$, for all $x \in X - \{0\}$ and so $E(G(X)) = \{0x \mid x \in X - \{0\}\}$. Therefore, $G(X)$ is a $m$-star graph. Now, let $G$ be a $m$-star graph, for some $m \in \mathbb{N}$ and $X = V(G) = \{x_0, x_1, \ldots, x_m\}$ be a chain such that $x_i < x_{i+1}$, for any $i \in \{0,1,\ldots,m-1\}$. Define the binary operation $*$ on $X$ by $x_i * x_j = x_{n_{ij}}$, where $i * j = \max\{0,i - j\}$, for any $i,j \in \{0,1,\ldots,m\}$. By Theorem 2.3, $X$ is a commutative BCK-chain and so $L(\{x,y\}) \neq \{0\}$, for any $x,y \in X - \{0\}$. Hence, by Lemma 3.2(i), $E(G(X)) = \{x_0 x_1, x_0 x_2, \ldots, x_0 x_m\}$ and so $G(X)$ is an $m$-star graph. \hfill \Box

**Theorem 3.19.** $X$ is a commutative BCI-algebra and $x * y = x * a$, for any $a \in P$ and $x,y \in V(a)$, where $x \neq y$ if and only if $G(X)$ is a graph with complete components.
Proof. Let $G(X)$ be a graph with complete components, $a \in P$ and $x, y \in V(a)$ such that $x \neq y$. Since $x \ast y \in V(0)$ by $(BCI6)$, we get $x \ast (x \ast y) \in L(\{x, y\})$. Now, since $x, y \in V(a)$ and $G(X)$ is a graph with complete components, then by Proposition 3.8, $xy \in E(G(X))$ and so $L(\{x, y\}) = \{a\}$. Hence, $x \ast (x \ast y) = a$ and so by $(BCI6)$ and $(BCI4)$, we get $x \ast a \leq x \ast y$. On the other hand, since $a \leq y$, then by $(BCI7)$, $x \ast y \leq x \ast a$. Therefore, $x \ast y = x \ast a$. Now, we show that $X$ is a commutative $BCI$-algebra. Let $a \in P$ and $x, y \in V(a)$. Clearly, if $x = y$, then $x \ast (x \ast y) = x \ast (x \ast a) \leq a$. Since $a \in P$, we get $x \ast (x \ast a) = a$. By the similar way, we get $y \ast (x \ast y) = a$ and so $x \ast (x \ast y) = x \ast (x \ast a)$. Hence, $X$ is branchwise commutative. Therefore, by Lemma 2.2, $X$ is commutative $BCI$-algebra. Conversely, let $X$ be a commutative $BCI$-algebra such that $x \ast y = x \ast a$, for any $a \in P$ and distinct elements $x, y \in V(a)$. Let $a \in P$ and $x, y \in V(a)$. If $u \in L(\{x, y\})$, then by $(P3)$, $u \ast a = u \ast x = 0$ and so $u \leq a$. Since $u \in P$, we have $a = u$. Hence, $L(\{x, y\}) = \{a\}$ and so $xy \in E(G(X))$. Therefore, by Proposition 3.8, all components of $G(X)$ are complete graph.

**Corollary 3.20.** The graph $G(X)$ is a complete graph if and only if $X$ is a $BCI$-algebra and

$$x \ast y = \begin{cases} 0 & \text{if } y = x, \\ x & \text{otherwise.} \end{cases}$$

(3.3)

**Definition 3.21** (see [10]). Let $(X_1, \ast_1, 0)$ be a $BCI$-algebra and $(X_2, \ast_2, 0)$ be a $BCI$-algebra such that $X_1 \cap X_2 = \{0\}$. Define the binary operation “$\ast$” on $X_1 \cup X_2$ by

$$x \ast y = \begin{cases} x \ast_1 y & \text{if } x, y \in X_1; \\ x \ast_2 y & \text{if } x, y \in X_2; \\ 0 \ast_2 y & \text{if } x \in X_1, y \in X_2 - \{0\}; \\ x & \text{if } x \in X_2, y \in X_1. \end{cases}$$

(3.4)

The $(X_1 \cup X_2, \ast, 0)$ is a $BCI$-algebra, whose $BCI$-part $B$ contains $X_1$ and $p$-semisimple part $P$ is contained in $X_2$. This algebra is called the Li’s union of $X_1$ and $X_2$ and is denoted by $X_1 \cup X_2$.

**Lemma 3.22.** Let $(X_1, \ast_1, 0)$ be a $BCI$-algebra, $(X_2, \ast_2, 0)$ be a $BCI$-algebra such that $X_1 \cap X_2 = \{0\}$ and $Y$ be the Li’s union of $X_1$ and $X_2$. Then

$$L_Y(\{x, y\}) = \begin{cases} L_{X_1}(\{x, y\}) & \text{if } x, y \in X_1; \\ L_{X_2}(\{x, y\}) & \text{if } x, y \in X_2 - V_{X_2}(0); \\ L_{X_1}(\{x, y\}) \cup X_1 & \text{if } x, y \in V_{X_2}(0) - \{0\}; \\ \emptyset & \text{otherwise.} \end{cases}$$

(3.5)

Proof. If $x, y \in X_1$, then

$$u \in L_{X_1}(\{x, y\}) \iff u \ast_1 x = u \ast x = 0 = u \ast y = u \ast_1 y \iff u \in L_Y(\{x, y\}).$$

(3.6)

Hence, $L_{X_1}(\{x, y\}) = L_Y(\{x, y\})$. Let $x, y \in V_{X_2}(a)$, for some $a \in P - \{0\}$. Then by Definition 3.21, $a \in Q - \{0\}$, where $Q$ is a $p$-semisimple part of $Y$. Let $u \in L_Y(\{x, y\})$. Then
Proof. Clearly, \( V(G(X_1 \cup X_2)) = X_1 \cup X_2 = V((G(X_1) \cup G(X_2)) \setminus S) \). Let \( Y = X_1 \cup X_2 \), \( Q \) be a \( p \)-semisimple part of \( Y \) and \( x \) \( y \) \( E \(G(Y)) \). Then \( x \neq y \) and \( L_Y \{ x, y \} = \{ a \} \), for some \( a \) \( Q \). If \( a \neq 0 \), then by Definition 3.21, \( x, y \) \( X_2 \) and so by Lemma 3.22, \( L_X \{ x, y \} = \{ a \} \). Hence, \( x, y \in E(G(X_2) - S) \subseteq E((G(X_1) \cup G(X_2)) - S) \). Let \( a = 0 \). Then \( L_Y \{ x, y \} = \{ 0 \} \) and so \( x, y \in V_Y(0) \) \( V_X(0) \). If \( x, y \in V_X(0) \), then by Lemma 3.22, \( xy \in E(G(X_1)) \subseteq E((G(X_1) \cup G(X_2)) - S) \). If \( x = 0 \) or \( y = 0 \), then clearly, \( xy \in E(G(X_1) \cup G(X_2) - S) \). If \( x, y \in V_X(0) \), then by Lemma 3.22, \( L_X \{ x, y \} \subseteq L_Y \{ x, y \} \) \( \{ 0 \} \) and so \( X = 0 \), which is impossible. If \( x \in V_X(0) \) \( \{ 0 \} \) and \( y \in V_X(0) \) \( \{ 0 \} \), then \( x, y \in E(G(Y)) \subseteq E((G(X_1) \cup G(X_2)) - S) \). Hence, \( xy \in E((G(X_1) \cup G(X_2)) - S) \). If \( a = 0 \), then \( x \neq y \) and \( xy \in E(G(Y)) \). Hence, \( E(G(Y)) \subseteq E((G(X_1) \cup G(X_2)) - S) \). Then \( xy \in E(G(X_1)) \) \( E(G(X_2)) \). By Lemma 3.22, \( E(G(X_1)) \subseteq E(G(Y)) \). Hence, \( xy \in E(G(Y)) \). Therefore, \( E((G(X_1) \cup G(X_2)) - S) \subseteq E(G(Y)) \) and so \( E((G(X_1) \cup G(X_2)) - S) = E(G(Y)) \).

Corollary 3.24. Let \( (X_1, *_1, 0) \) be a BCK-algebra, \( (X_2, *_2, 0) \) be a BCI-algebra such that \( X_1 \cap X_2 = \emptyset \) and the graph \( G(V_X(0)) \) be a tree. Then \( G(X_1 \cup X_2) = G(X_1) \cup G(X_2) \).

Proof. If \( X_1 = \{ 0 \} \), then \( X_2 = X_1 \cup X_2 \) and so \( G(X_1 \cup X_2) = G(X_2) = G(X_1) \cup G(X_2) \). Now, let \( G(V_X(0)) \) be a tree. Then for any distinct elements \( x, y \in V_X(0) \), we have \( xy \notin E(G(V_X(0))) \). Hence, by Proposition 3.8(ii), \( xy \notin E(G(X_2)) \) and so \( S = \emptyset \), where \( S = \{ xy \in E(G(X_2)) \mid x, y \in V_X(0) \} \). Therefore, by Proposition 3.23, \( G(X_1 \cup X_2) = G(X_1) \cup G(X_2) \).

Example 3.25. (i) Let \( X_1 = \{ 0, a, b, c \} \) and \( X_2 = \{ 0, 1, 2, 3 \} \). Define the binary operations \( *_1 \) and \( *_2 \) on \( X_1 \) and \( X_2 \), respectively, by the following tables:

<table>
<thead>
<tr>
<th>( *_1 )</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>a</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( *_2 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>


Then \((X_1, \ast_1, 0)\) and \((X_2, \ast_2, 0)\) are BCK-algebras (see [10]). Also, \(G(X_1), G(X_2)\) and \(G(X_1) \cup G(X_2)\) are given by Figure 5.

Let \(Y = X_1 \cup X_2\). Then \(L_Y(\{x, 0\}) = \{0\}\), for any \(x \in Y\). Hence, by Lemma 3.22, \(x \in L_Y(\{x, y\})\) and \(L_Y(\{0, z\}) = \{0\} = L_Y(\{a, b\})\), for any \(x \in \{1, 2, 3\}\), \(z \in X_1\) and \(y \in \{a, b, c\}\). Moreover, \(X_1 \subseteq L_Y(\{a, b\}) \cap L_Y(\{a, c\}) \cap L_Y(\{b, c\})\). Hence, \(E(G(Y)) = \{0x \mid x \in Y - \{0\}\} \cup \{ab\}\) and so \(G(Y)\) is given by Figure 6.

If \(S\) is the set was defined in the Proposition 3.23, then \(S = \{13, 23\}\). Therefore, \(G(Y) = G(X_1) \cup (G(X_2) - S)\).

(ii) Let \(X\) be the BCI-algebra was defined in Example 3.6(i) and \(X_1\) be the BCK-algebra in (i). Then the BCK-part of \(X_2\) is a tree. Hence, by Corollary 3.24, \(G(X_1 \cup X) = G(X_1) \cup G(X)\).

**Lemma 3.26.** Let \((X, \ast_1, 0)\) and \((Y, \ast_2, 0)\) be two BCK-algebras and \(x, y \in X \oplus Y\). Then

\[
L_{X \oplus Y}(x, y) = \begin{cases} 
L_X(x, y) & \text{if } x, y \in X; \\
L_Y(x, y) & \text{if } x, y \in Y; \\
\{0\} & \text{otherwise.}
\end{cases}
\]

**Proof.** Let \(x, y \in X \cup Y\).

1. If \(x, y \in X\), then for any \(u \in L_X(x, y)\), we have \(u \ast x = 0 = u \ast y\). If \(u \in Y\), then by Proposition 2.5, \(u = u \ast x = 0\) and so \(u \in X\). Moreover, if \(u \in X\), then \(u \ast x = u \ast x = 0\) and \(u \ast y = u \ast y = 0\) and so \(u \in L_X(x, y)\). Hence, \(L_{X \oplus Y}(x, y) \subseteq L_X(x, y)\). Now, let \(u \in L_X(x, y)\). Then \(u, x, y \in X\) and \(u \ast x = 0 = u \ast y\) and so \(u \ast x = u \ast x = 0\) and \(u \ast y = u \ast y\). Hence, \(L_X(x, y) \subseteq L_{X \oplus Y}(x, y)\). Therefore, \(L_{X \oplus Y}(x, y) = L_X(x, y)\), for all \(x, y \in X\). By the similar way, we can prove that \(L_{X \oplus Y}(x, y) = L_Y(x, y)\), for all \(x, y \in Y\).

2. If \(x \in X\) and \(y \in Y\). Since \(X \oplus Y\) is a BCK-algebra, we have \(0 \in L_{X \oplus Y}(x, y)\). Let \(u \in L_{X \oplus Y}(x, y)\). Then \(u \ast x = 0 = u \ast y\). Since \(x \in X\) and \(y \in Y\), then by definition of
Let \( (X, \ast_1, 0) \) and \( (Y, \ast_2, 0) \) be two \( BCK \)-algebras. Then \( G(X \oplus Y) - \{0\} = (G(X) - \{0\}) \cup (G(Y) - \{0\}) \).

**Proof.** Clearly, \( V(G(X \oplus Y) - \{0\}) = V((G(X) - \{0\}) \cup (G(Y) - \{0\})) \). Let \( xy \in E(G(X \oplus Y) - \{0\}) \). Then \( x \neq 0, y \neq 0 \) and \( L_{X\Theta Y}(x, y) = \{0\} \). If \( x, y \in X \) or \( x, y \in Y \), then by Lemma 3.26, \( L_X(x, y) = \{0\} \) or \( L_Y(x, y) = \{0\} \) and so \( xy \in E(G(X) - \{0\}) \) or \( xy \in E(G(Y) - \{0\}) \). Hence, by Definition 2.8, \( xy \in E((G(X) - \{0\}) \cup (G(Y) - \{0\})) \). If \( x \in X \) and \( y \in Y \) or \( x \in Y \) and \( y \in X \), then \( x \in V((G(X) - \{0\})) \) and \( y \in V(G(Y) - \{0\}) \) and so \( xy \in E((G(X) - \{0\}) \cup (G(Y) - \{0\})) \).

Now, let \( xy \in E((G(X) - \{0\}) \cup (G(Y) - \{0\})) \). Then \( xy \in E(G(X) - \{0\}) \) or \( xy \in E(G(Y) - \{0\}) \) or \( x \in X - \{0\} \) and \( y \in Y - \{0\} \). If \( xy \in E(G(X) - \{0\}) \cup E(G(Y) - \{0\}) \), then \( x \neq 0, y \neq 0 \) and \( L_X(x, y) = \{0\} \) or \( L_Y(x, y) = \{0\} \). Hence, by Lemma 3.26, \( xy \in E(G(X \oplus Y) - \{0\}) \). If \( x \in X - \{0\} \) and \( y \in Y - \{0\} \), then by Lemma 3.26, \( L_{X\Theta Y} = \{0\} \) and so \( xy \in E(G(X \oplus Y) - \{0\}) \).

Therefore, \( E(G(X \oplus Y) - \{0\}) = E((G(X) - \{0\}) \cup (G(Y) - \{0\})) \).

**Corollary 3.28.** Let \( (X, \ast_1, 0) \) and \( (Y, \ast_2, 0) \) be two finite \( BCK \)-algebras. Then \( |E(X \oplus Y)| = |X| + |Y| - 1 + (|X| - 1) \cdot (|Y| - 1) \).

**Proof.** Straightforward.

**Example 3.29.** Let \( X = \{a_0, a_1, a_2, a_3\} \), \( Y = \{b_0, b_1, b_2\} \), \( a_0 = b_0 \) and “\( \ast \)” be the operation was defined in Example 3.13. Then \( (X, \ast_1, a_0) \) and \( (Y, \ast_2, b_0) \) are \( BCK \)-chain, where \( a_1 \ast_1 a_i = a_{i+1} \) and \( b_1 \ast_2 b_i = b_{i+1} \). Since they are \( BCK \)-chain, then by Lemma 3.2, \( G(X) \) is 3-star graph and \( G(Y) \) is 2-star graph and so \( G(X) - \{0\} \) and \( G(Y) - \{0\} \) have Figure 7.

Hence, by Definition 2.8, \( (G(X) - \{0\}) \cup (G(Y) - \{0\}) \) is given by Figure 8.

On the other hand, by Lemma 3.26, \( G(X \oplus Y) \) is given by Figure 9 and so \( G(X \oplus Y) - \{0\} \) is the graph on Figure 8.

Let \( I \) be an ideal of \( X \). Define a binary relation \( \theta \) on \( X \) as follows: \( (x, y) \in \theta \) if and only if \( x \ast y, y \ast x \in I \), for all \( x, y \in X \). Then, \( \theta \) is a congruence relation and it is called the
equivalence relation induced by $I$. If $X/I = \{[x] \mid x \in X\}$, then $(X/I, \ast, [0])$ is a BCI-algebra, where $[x] \ast [y] = [x \ast y]$, for all $x, y \in X$ (see [10]). Moreover, let $G = (V, E)$ be a graph and $\Pi$ be a partition of $X$. The graph whose vertexes are the elements of $\Pi$ and for distinct elements $u, v \in \Pi$, there is an edge connecting $u$ and $v$ if and only if $xy \in E$, for some $x \in u$ and $y \in v$, is denoted by $G/\Pi$. Now, we want to verify the relation between the $G(X/I)$ and $G(X)/\Pi$, where $I$ is an ideal of $X$, $\theta$ is a congruence relation induced by $I$ and $\Pi$ is a partition induced by $\theta$.

**Theorem 3.30.** Let $I$ be an ideal of $X$ and $\Pi$ be the partition of $X$ induced by $I$.

(i) If $X$ is a BCK-algebra, then $G(X)/\Pi$ is a subgraph of $G(X/I)$.

(ii) If $X$ is a commutative BCI-algebra, then $G(X)/\Pi$ is a subgraph of $G(X/I)$.

**Proof.** (i) Clearly, $V(G(X/I)) = \{[x] \mid x \in X\} = V(G(X)/\Pi)$. Let $[x][y] \in E(G(X)/\Pi)$. Then there are $u \in [x]$ and $v \in [y]$, such that $uv \in E(G(X))$. Hence, $L_X\{u, v\} = \emptyset$. Let $[z] \in L_{X/I}\{[x], [y]\}$. Since $[u] = [x]$ and $[v] = [y]$, then $[z \ast u] = [z] \ast [u] = [0] = [z] \ast [v] = [z \ast v]$ and so $z \ast u \in I$ and $z \ast v \in I$. Hence, $z \ast u = a$ and $z \ast v = b$, for some $a, b \in I$. Since $X$ is a BCK-algebra, then by (BCI6), $(z \ast a) \ast b \in L_X\{u, v\} = \emptyset$. Since $I$ is an ideal of $X$ and $a, b \in I$, then $z \in I$ and so $z \ast 0 \in I$ and $0 \ast z = 0 \in I$. Hence, $[z] = [0]$ and so $L_{X/I}\{[x], [y]\} = \{[0]\}$. Therefore, $[x][y] \in E(G(X/I))$ and so $G(X)/\Pi$ is a subgraph of $G(X/I)$.

(ii) Clearly, $V(G(X/I)) = \{[x] \mid x \in X\} = V(G(X)/\Pi)$. Let $[x][y] \in E(G(X)/\Pi)$. Then there are $u \in [x]$ and $v \in [y]$, such that $uv \in E(G(X))$ and so $L_X\{u, v\} = \emptyset$, for some $a \in P$. Since $u, v \in V(a)$, then by (P2), $v \ast u, u \ast v \in V(0)$ and so by (BCI6). $v \ast (v \ast u) = u \ast (u \ast v) \in L_X\{u, v\} = \{a\}$. Hence, $v \ast (v \ast u) = a = u \ast (u \ast v)$ and so $v \ast (v \ast [u]) = [a] = [u] \ast ([u] \ast [v])$. Let $[z] \in L_{X/I}\{[x], [y]\}$. Then by (P3), $[z] \in V_{X/I}\{[a]\}$. Since $X/I$ is a commutative BCI-algebra, then by Lemma 2.2, $[v] \ast ([v] \ast [z]) = [z] \ast ([z] \ast [v]) = [z]$ and so by (BCI7), we have $[z] = [v] \ast ([v] \ast [z]) \leq [v] \ast ([v] \ast [u]) = [a]$. Clearly, $a \in Q$, where $Q$ is a $p$-semisimple part of $X/I$. Hence, $[a] = [z]$ and so $L_{X/I}\{[x], [y]\} = \{[a]\}$. Therefore, $[x][y] \in E(G(X/I))$ and so $G(X)/\Pi$ is a subgraph of $G(X/I)$.

**Example 3.31.** Let $Y = \{0, 1, 2, a, b\}$ and $X = \{0, 1, 2, 3, 4\}$. Define the binary operations “$*_1$” and “$*_2$” on $Y$ and $X$, respectively by the following tables:

<table>
<thead>
<tr>
<th>$*_1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>a</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$*_2$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>a</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>b</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>
(i) It is easy to see that \((Y, \ast_1, 0)\) is a BCI-algebra and \(J = \{0, 1\}\) is an ideal of \(Y\) (see [10, Example 2.5.5]). Moreover, \([0] = [0, 1, a, b]\) and \([2] = [2]\). Hence, \(Y/J = \{[0], [2]\}\) is a commutative BCI-algebra and so \(E(G(Y/I)) = \{[0][2]\}\). Moreover, Since \(L_Y[2, 0] = 0\), then \([0][2] \in E(G(X)/\Phi)\), where \(\Phi\) is the partition of \(Y\) induced by \(J\). Therefore, \(G(X/J) = G(X)/\Phi\).

(ii) We can prove that \((X, \ast_2, 0)\) is a BCK-algebra and \(J = \{0, 1, 2\}\) is an ideal of \(X\) (see [10]). Moreover, \(X/I = \{[0], [3], [4]\}\), where \([0] = \{0, 1, 2\}, [3] = \{3\}\) and \([4] = \{4\}\). It is easy to check that \(E(G(X/I)) = \{[0][3], [0][4], [3][4]\}\) and \(E(G(X)/\Pi) = \{[0][3], [0][4]\}\).

4. Characteristic Polynomials of Graphs of BCI-Algebras

In this section, we verify the characteristic polynomials of graph of BCI-algebras. Then we refined the relation between Characteristic polynomial of \(G(X)\) and Characteristic polynomial of graphs of branches of \(X\), for any BCI-algebra \(X\).

**Theorem 4.1.** Let \(X\) be a finite BCI-algebra. Then \(\chi(G(X), \lambda) = \prod_{a \in P} \chi(G(V(a)), \lambda)\).

**Proof.** Let \(m \in \mathbb{N}\), \(P = \{a_1, \ldots, a_m\}\), \(V(a_i) = \{x_{i1}, \ldots, x_{in}\}\) and \(A_t = [b_{ij}]\) be the adjacency matrix of \(G(V(a_i))\) for all \(t \in \{1, 2, \ldots, m\}\). Then \(X = \{x_{i1}, x_{i2}, \ldots, x_{in}, x_{11}, x_{12}, x_{22}, \ldots, x_{2n}, x_{mn}, \ldots, x_{nn}\}\). Since \(V(a_i) \cap V(a_j) = \emptyset\), for all distinct \(i, j \in \{1, 2, \ldots, m\}\), then by Proposition 3.8(ii), the adjacent matrix of \(G(X)\) is of the form

\[
\begin{bmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_m
\end{bmatrix},
\]

(4.1)

where \(A_j\) is a \(m_j \times m_j\) matrix, for all \(j \in \{1, 2, \ldots, m\}\). By the properties of determinant, we have \(\chi(G(X), \lambda) = \det(\Lambda - A) = \det(\Lambda I_1 - A_1) \times \det(\Lambda I_2 - A_2) \times \cdots \times \det(\Lambda I_m - A_m) = \prod_{a \in P} \chi(G(V(a)), \lambda)\), where \(I_j\) is a \(m_j \times m_j\) identity matrix, for all \(j \in \{1, 2, \ldots, m\}\).

**Corollary 4.2.** Let \(X\) be a finite BCI-algebra and \(t\) is the number of elements \(a \in P\) such that \(|V(a)| = 1\).

(i) \(\chi(G(X), \lambda) = \lambda^t \times f(\lambda)\), for some polynomial \(f(\lambda)\).

(ii) \(X\) is a \(p\)-semisimple BCI-algebra if and only if \(\chi(G(X), \lambda) = \lambda^n\), for some \(n \in \mathbb{N}\).

**Proof.** (i) Let \(|X| = n\) and \(\{a_1, \ldots, a_t\}\) be the set of all elements of \(P\) such that \(|V(a_i)| = 1\), for all \(i \in \{1, 2, \ldots, t\}\). Then by using the proof of Theorem 4.1, the adjacent matrix of \(G(X)\) is of the form

\[
\begin{bmatrix}
B & 0 \\
0 & 0
\end{bmatrix}_{n \times n},
\]

(4.2)
where $B$ is an $(n-t \times n-t)$ matrix. Hence, by properties of determinant, we have $\chi(G(X), \lambda) = \det(\lambda I_t) \times \det(\lambda I_t - B) = \lambda^t \times \det(\lambda I_t - B)$. Let $f(\lambda) = \det(\lambda I_t - B)$, then $\chi(G(X), \lambda) = \lambda^t \times f(\lambda)$.

(ii) Since $X$ is a $p$-semisimple BCI-algebra, then $|V(a)| = 1$, for all $a \in X$. Therefore, by (i), $\chi(G(X), \lambda) = \lambda^n$, where $|X| = n$. Conversely, let $\chi(G(X), \lambda) = \lambda^n$, for some $n \in \mathbb{N}$. Then by Proposition 2.11, $G(X)$ is an empty graph. Therefore, by Theorem 3.9, $X$ is a $p$-semisimple algebra.

\[ \text{Proof.} \] By Proposition 2.11, $c_1 = 0$. If $x \in X - P$, then by Lemma 3.2,

\[ \deg(x) = \left| \left\{ y \in X - \{x\} \mid L(x, y) = a, \text{ for some } a \in P \right\} \right| = |Z_x|. \]  

Now, if $x \in P$, then by Lemma 3.2, $\deg(x) = \left| \left\{ y \in X - \{x\} \mid L(x, y) = a, \text{ for some } a \in P \right\} \right| = |Z_x| - 1$. Hence, by Proposition 2.11, we have

\[ 2c_2 = -2|E(G(X))| = -\deg(G) = \left( \sum_{x \in X-P} |Z_x| + \sum_{x \in P} (|Z_x| - 1) \right) = -\sum_{x \in X} |Z_x| + |P|. \]  

(ii) By Proposition 2.11, $c_3 = -2t$, where $t$ is the number of triangles of $G(X)$. Since by Proposition 3.8, $G(X) = \bigcup_{a \in P} G(V(a))$ and $V(a) \cap V(b) = \emptyset$, for any distinct elements $a, b \in P$, then $t = \sum_{a \in P} t_a$, where $t_a$ is the number of triangles of $G(V(a))$, for all $a \in P$. Now, let $a \in P$ and $x, y, z$ be three vertices of a triangle of $G(V(a))$. Then $x, y, z$ are distinct elements of $X$ and $L(x, y) = L(x, z) = L(y, z) = \{a\}$. Since $x \ast (x \ast y) \in L(x, y)$, then $x \ast (x \ast y) = a$ and so by (BC14) and (BC16), $x \ast a \leq x \ast y$. Moreover, $a \leq y$ and (BC17), imply $x \ast y \leq x \ast a$ and so $x \ast y = x \ast a$. By the similar way, we can prove that $x \ast z = x \ast a$, $y \ast z = y \ast a = y \ast x$ and $z \ast x = z \ast a = z \ast y$. Hence, $\{x, y, z\} \in A. \text{ Conversely, } \{x, y, z\} \in A. \text{ Then there exists } a \in P \text{ such that } u \ast v = u \ast a, \text{ for any } u, v \in \{x, y, z\}. \text{ Let } u, v \in \{x, y, z\} \text{ and } w \in L[u, v]. \text{ Then by (P2), } w \in V(a). \text{ Since } X \text{ is commutative, then by Lemma 2.2, (BC16) and (BC17), } w = w \ast 0 = w \ast (w \ast u) = u \ast (u \ast w) \leq u \ast (u \ast v) = u \ast (u \ast a) \leq a. \text{ Hence, } w = a \text{ (since } a \in P), \text{ and so } L[u, v] = \{a\}. \text{ Therefore, } xy, xz, yz \in E(G(V(a))) \text{ and so } x, y, z \text{ are three vertices of a triangle of } G(V(a)) \text{, for some } a \in P. \text{ Hence, } t = \sum_{a \in P} t_a = |A|. \text{ This complete the proof.} \]  

References


Submit your manuscripts at
http://www.hindawi.com