Research Article

Some Properties of Multiple Generalized $q$-Genocchi Polynomials with Weight $\alpha$ and Weak Weight $\beta$

J. Y. Kang

Department of Mathematics, Hannam University, Daejeon 306-791, Republic of Korea

Correspondence should be addressed to J. Y. Kang, rkdwjdnnr2002@yahoo.co.kr

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The present paper deals with the various $q$-Genocchi numbers and polynomials. We define a new type of multiple generalized $q$-Genocchi numbers and polynomials with weight $\alpha$ and weak weight $\beta$ by applying the method of $p$-adic $q$-integral. We will find a link between their numbers and polynomials with weight $\alpha$ and weak weight $\beta$. Also we will obtain the interesting properties of their numbers and polynomials with weight $\alpha$ and weak weight $\beta$. Moreover, we construct a Hurwitz-type zeta function which interpolates multiple generalized $q$-Genocchi polynomials with weight $\alpha$ and weak weight $\beta$ and find some combinatorial relations.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$ (see [1–21]). When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < 1$.

Throughout this paper, we use the following notation:

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}. \quad (1.1)$$

Hence $\lim_{q \to 1} [x]_q = x$ for all $x \in \mathbb{Z}_p$ (see [1–14, 16, 18, 20, 21]).
We say that \( g : \mathbb{Z}_p \to \mathbb{C}_p \) is uniformly differentiable function at a point \( a \in \mathbb{Z}_p \) and we write \( g \in \text{UD}(\mathbb{Z}_p) \) if the difference quotients \( \Phi_g : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{C}_p \) such that

\[
\Phi_g(x, y) = \frac{g(x) - g(y)}{x - y}
\]

have a limit \( g'(a) \) as \( (x, y) \to (a, a) \).

Let \( d \) be a fixed integer, and let \( p \) be a fixed prime number. For any positive integer \( N \), we set

\[
X = X_d = \lim_{N \to \infty} \left[ \frac{\mathbb{Z}}{dp^N \mathbb{Z}} \right], \quad X_1 = \mathbb{Z}_p,
\]

\[
X^* = \bigcup_{0 < a < dp} (a + dp\mathbb{Z}_p),
\]

\[
a + dp^N\mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \pmod{dp^N} \right\},
\]

where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp^N \).

For any positive integer \( N \),

\[
\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q} \tag{1.4}
\]

is known to be a distribution on \( X \).

For \( g \in \text{UD}(\mathbb{Z}_p) \), Kim defined the \( q \)-deformed fermionic \( p \)-adic integral on \( \mathbb{Z}_p \):

\[
I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x. \tag{1.5}
\]

(see \([1-13]\)), and note that

\[
\int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \int_{\times} g(x) d\mu_{-q}(x). \tag{1.6}
\]

We consider the case \( q \in (-1, 0) \) corresponding to \( q \)-deformed fermionic certain and annihilation operators and the literature given there in \([9, 13, 14]\).

In \([9, 12, 14, 19]\), we introduced multiple generalized Genocchi number and polynomials. Let \( \chi \) be a primitive Dirichlet character of conductor \( f \in \mathbb{N} \). We assume that \( f \)}
is odd. Then the multiple generalized Genocchi numbers, $G^{(r)}_{n,\chi}$, and the multiple generalized Genocchi polynomials, $G^{(r)}_{n,\chi}(x)$, associated with $\chi$, are defined by

$$F^{(r)}_{\chi}(t) = \left( \frac{2t \sum_{a=0}^{r-1} X(a)(-1)^a e^{at}}{e^{at} + 1} \right)^r = \sum_{n=0}^{\infty} G^{(r)}_{n,\chi} \frac{t^n}{n!},$$

$$F^{(r)}_{\chi}(t,x) = \left( \frac{2t \sum_{a=0}^{r-1} X(a)(-1)^a e^{at}}{e^{at} + 1} e^{tx} \right)^r = \sum_{n=0}^{\infty} G^{(r)}_{n,\chi}(x) \frac{t^n}{n!}. \tag{1.7}$$

In the special case $x = 0$, $G^{(r)}_{n,\chi} = G^{(r)}_{n,\chi}(0)$ are called the $n$th multiple generalized Genocchi numbers attached to $\chi$.

Now, having discussed the multiple generalized Genocchi numbers and polynomials, we were ready to multiple-generalize them to their $q$-analogues. In generalizing the generating functions of the Genocchi numbers and polynomials to their respective $q$-analogues; it is more useful than defining the generating function for the Genocchi numbers and polynomials (see [12]).

Our aim in this paper is to define multiple generalized $q$-Genocchi numbers $G^{(a,\beta,r)}_{n,\chi,q}$ and polynomials $G^{(a,\beta,r)}_{n,\chi,q}(x)$ with weight $\alpha$ and weak weight $\beta$. We investigate some properties which are related to multiple generalized $q$-Genocchi numbers $G^{(a,\beta,r)}_{n,\chi,q}$ and polynomials $G^{(a,\beta,r)}_{n,\chi,q}(x)$ with weight $\alpha$ and weak weight $\beta$. We also derive the existence of a specific interpolation function which interpolate multiple generalized $q$-Genocchi numbers $G^{(a,\beta,r)}_{n,\chi,q}$ and polynomials $G^{(a,\beta,r)}_{n,\chi,q}(x)$ with weight $\alpha$ and weak weight $\beta$ at negative integers.

### 2. The Generating Functions of Multiple Generalized $q$-Genocchi Numbers and Polynomials with Weight $\alpha$ and Weak Weight $\beta$

Many mathematicians constructed various kinds of generating functions of the $q$-Genocchi numbers and polynomials by using $p$-adic $q$-Vokenborn integral. First we introduce multiple generalized $q$-Genocchi numbers and polynomials with weight $\alpha$ and weak weight $\beta$.

Let us define the generalized $q$-Genocchi numbers $G^{(a,\beta)}_{n,\chi,q}$ and polynomials $G^{(a,\beta)}_{n,\chi,q}(x)$ with weight $\alpha$ and weak weight $\beta$, respectively,

$$F_{\chi,q}^{(a,\beta)}(t) = \sum_{n=0}^{\infty} G^{(a,\beta)}_{n,\chi,q} \frac{t^n}{n!} = \int_{X} t^\chi(x) e^{[x]_{\alpha,t}} d \mu_{-\varphi}(x),$$

$$F_{\chi,q}^{(a,\beta)}(t,x) = \sum_{n=0}^{\infty} G^{(a,\beta)}_{n,\chi,q}(x) \frac{t^n}{n!} = \int_{X} t^\chi(y) e^{[x+y]_{\alpha,t}} d \mu_{-\varphi}(y). \tag{2.1}$$

By using the Taylor expansion of $e^{[x]_{\alpha,t}}$, we have

$$\sum_{n=0}^{\infty} \int_{X} \chi(x) [x]_{\alpha,t}^n d \mu_{-\varphi}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} G^{(a,\beta)}_{n,\chi,q} \frac{t^{n-1}}{n!} = G^{(a,\beta)}_{0,\chi,q} + \sum_{n=0}^{\infty} G^{(a,\beta)}_{n+1,\chi,q} \frac{t^n}{n+1} \frac{1}{n!}. \tag{2.2}$$
By comparing the coefficient of both sides of $t^n/n!$ in (2.2), we get

$$G^{(\alpha, \beta)}_{n+1, X,q} \frac{n}{n+1} = \frac{[2]_q}{(1 - q^n)^2} \sum_{a=0}^{n-1} (-1)^a q^a \chi(a) \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{l+a} \left( 1 + q^{(l+a)} \right).$$

From (2.2) and (2.3), we can easily obtain that

$$\sum_{n=0}^{\infty} G^{(\alpha, \beta)}_{n, X,q} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( t \int_X \chi(x) [x]_q^n \mu_{q-\phi} (x) \right) \frac{t^n}{n!} = [2]_q \sum_{l=0}^{\infty} (-1)^l q^l \chi(l) e^{l \cdot \phi}.$$  

Therefore, we obtain

$$F^{(\alpha, \beta)}_{X,q} (t) = [2]_q \sum_{l=0}^{\infty} (-1)^l q^l \chi(l) e^{l \cdot \phi} = \sum_{n=0}^{\infty} G^{(\alpha, \beta)}_{n, X,q} \frac{t^n}{n!}.$$  

Similarly, we find the generating function of generalized $q$-Genocchi polynomials with weight $\alpha$ and weak weight $\beta$:

$$G^{(\alpha, \beta)}_{0, X,q} (x) = 0,
G^{(\alpha, \beta)}_{n+1, X,q} (x) = \int_X \chi(y) [x + y]_q^n \mu_{q-\phi} (y) = [2]_q \sum_{l=0}^{\infty} (-1)^l q^l \chi(l) [x + l]_q^n.$$  

From (2.6), we have

$$F^{(\alpha, \beta)}_{X,q} (t, x) = [2]_q \sum_{l=0}^{\infty} (-1)^l q^l \chi(l) e^{l \cdot \phi} = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} G^{(\alpha, \beta)}_{n, X,q} (x) \frac{t^n}{n!}.$$  

Observe that $F^{(\alpha, \beta)}_{X,q} (t) = F^{(\alpha, \beta)}_{X,q} (t, 0)$. Hence we have $G^{(\alpha, \beta)}_{n, X,q} = G^{(\alpha, \beta)}_{n, X,q} (0)$. If $q \to 1$ into (2.7), then we easily obtain $F^{(\alpha, \beta)}_{X,q} (t, x)$.

First, we define the multiple generalized $q$-Genocchi numbers $G^{(\alpha, \beta, r)}_{n, X,q}$ with weight $\alpha$ and weak weight $\beta$:

$$F^{(\alpha, \beta, r)}_{X,q} (t) = [2]_q \sum_{k_1, \ldots, k_r=0}^{\infty} (-1)^{\sum_{i=1}^{r} k_i} q^{\sum_{i=1}^{r} k_i} \left( \prod_{i=1}^{r} \chi(k_i) \right) e^{\sum_{i=1}^{r} k_i \cdot \phi} \int_X \chi(x_1) \cdots \chi(x_r) e^{(x_1 + \cdots + x_r) \cdot \phi} d \mu_{q-\phi} (x_1) \cdots d \mu_{q-\phi} (x_r).$$  

$$= \sum_{n=0}^{\infty} G^{(\alpha, \beta, r)}_{n, X,q} \frac{t^n}{n!}.$$
Let $$\text{Theorem 2.1.}$$

Then we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{r=1}^{\infty} G_{n,r}^{(x_1, \ldots, x_r)} \cdot \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{r=1}^{\infty} G_{n,r}^{(x_1, \ldots, x_r)} \cdot \frac{t^n}{n!} \right)$$

(2.9)

where $$\binom{n+r}{r} = \frac{(n+r)!}{n!r!}.$$

By comparing the coefficients on the both sides of (2.9), we obtain the following theorem.

**Theorem 2.1.** Let $$q \in \mathbb{C}_p$$ with $$|1-q|_p < 1$$ and $$n \in \mathbb{Z}_+.$$ Then one has

$$G_{n+r,\chi,\lambda}^{(a,\beta,r)}(x) = \sum_{r=1}^{n+r} G_{n+r,\chi,\lambda}^{(a,\beta,r)} = \cdots = G_{n+r-1,\chi,\lambda}^{(a,\beta,r)} = 0,$$

(2.10)

From now on, we define the multiple generalized $$q$$-Genocchi polynomials $$G_{n,\chi,q}^{(a,\beta,r)}(x)$$ with weight $$a$$ and weak weight $$\beta.$$

$$F_{\chi,q}^{(a,\beta,r)}(t, x) = [2]_q^r \frac{t^r}{r!} \sum_{k_1, \ldots, k_r=0}^{\infty} (-1)^{\sum_{i=1}^{r} k_i} q^{\sum_{i=1}^{r} k_i} \prod_{i=1}^{r} \chi(k_i) e^{\sum_{i=1}^{r} k_i x_i t}$$

(2.11)
Then we have

\[
\sum_{n=0}^{\infty} \int_X \cdots \int_X x_1^n \cdots x_r^n [x + y_1 + \cdots + y_r]_q^n d\mu_{-\varphi} \left( (y_1) \cdots d\mu_{-\varphi} \left( (y_r) \right) \right) \\
= \sum_{n=0}^{\infty} G_n(x) \frac{t^{n-r}}{n!} = \sum_{n=0}^{r-1} G_{n-1} \left( x \right) \frac{t^n}{n!} + \sum_{n=0}^{\infty} G_{n+r} \left( x \right) \frac{t^n}{(n+r)! n!}
\]

(2.12)

where \( \binom{n+r}{r} = \frac{(n+r)!}{n!r!} \).

By comparing the coefficients on the both sides of (2.12), we have the following theorem.

**Theorem 2.2.** Let \( q \in \mathbb{C} \) with \( |1 - q|_p < 1 \) and \( n \in \mathbb{Z}_+ \). Then one has

\[
G_n^{(a, \beta, r)} (x) = G_{n-1}^{(a, \beta, r)} (x) = \cdots = G_{r-1}^{(a, \beta, r)} (x) = 0,
\]

\[
\frac{G_n^{(a, \beta, r)} (x)}{(n+r)!} = \int_X \cdots \int_X x_1^n \cdots x_r^n [x + y_1 + \cdots + y_r]_q^n d\mu_{-\varphi} \left( (y_1) \cdots d\mu_{-\varphi} \left( (y_r) \right) \right)
\]

(2.13)

In (2.11), we simply identify that

\[
\lim_{q \to 1} F_n^{(a, \beta, r)} (t, x) = 2^r t^r \sum_{k_1, \ldots, k_r=0}^{\infty} (-1)^{\sum_{i=1}^r k_i} \left( \prod_{i=1}^r \chi(k_i) \right) e^t \left( \sum_{i=1}^r k_i + x \right) t^r
\]

(2.14)

So far, we have studied the generating functions of the multiple generalized \( q \)-
Genocchi numbers \( G_n^{(a, \beta, r)} \) and polynomials \( G_n^{(a, \beta, r)} (x) \) with weight \( \alpha \) and weak weight \( \beta \).
3. Modified Multiple Generalized $q$-Genocchi Polynomials with Weight $\alpha$ and Weak Weight $\beta$

In this section, we will investigate about modified multiple generalized $q$-Genocchi numbers and polynomials with weight $\alpha$ and weak weight $\beta$. Also, we will find their relations in multiple generalized $q$-Genocchi numbers and polynomials with weight $\alpha$ and weak weight $\beta$.

Firstly, we modify generating functions of $G_{n,x,q}^{(a,\beta,r)}$ and $G_{n,x,q}^{(a,\beta,r)}(x)$. We access some relations connected to these numbers and polynomials with weight $\alpha$ and weak weight $\beta$. For this reason, we assign generating function of modified multiple generalized $q$-Genocchi numbers and polynomials with weight $\alpha$ and weak weight $\beta$ which are implied by $G_{n,x,q}^{(a,\beta,r)}$ and $G_{n,x,q}^{(a,\beta,r)}(x)$. We give relations between these numbers and polynomials with weight $\alpha$ and weak weight $\beta$.

We modify (2.11) as follows:

$$\tilde{G}^{(a,\beta,r)}_{n,x,q}(t, x) = F^{(a,\beta,r)}_{n,x,q}(q^{-ax}t, x), \tag{3.1}$$

where $F^{(a,\beta,r)}_{n,x,q}(t, x)$ is defined in (2.11).

From the above we know that

$$\tilde{G}^{(a,\beta,r)}_{n,x,q}(t, x) = \sum_{n=0}^{\infty} q^{-ax} G^{(a,\beta,r)}_{n,x,q}(x) \frac{t^n}{n!}. \tag{3.2}$$

After some elementary calculations, we attain

$$\tilde{G}^{(a,\beta,r)}_{n,x,q}(t, x) = q^{-ax} e^{q^{-ax}[x]_{q^a} t} F^{(a,\beta,r)}_{n,x,q}(t), \tag{3.3}$$

where $F^{(a,\beta,r)}_{n,x,q}(t)$ is defined in (2.8).

From the above, we can assign the modified multiple generalized $q$-Genocchi polynomials $e_{n,x,q}^{(a,\beta,r)}(x)$ with weight $\alpha$ and weak weight $\beta$ as follows:

$$\tilde{G}^{(a,\beta,r)}_{n,x,q}(t, x) = \sum_{n=0}^{\infty} e_{n,x,q}^{(a,\beta,r)}(x) \frac{t^n}{n!}. \tag{3.4}$$

Then we have

$$e_{n,x,q}^{(a,\beta,r)}(x) = q^{-ax} G^{(a,\beta,r)}_{n,x,q}(x). \tag{3.5}$$

**Theorem 3.1.** For $r \in \mathbb{N}$ and $n \in \mathbb{Z}$, one has

$$e_{n,x,q}^{(a,\beta,r)}(x) = q^{-(n+r)x} \sum_{i=0}^{n} \binom{n}{i} q^{ax} [x]_{q^a}^{n-i} G^{(a,\beta,r)}_{i,x,q}. \tag{3.6}$$
Corollary 3.2. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, by using (3.7), one easily obtains

$$
\mathcal{G}^{(\alpha,\beta,r)}_{n,\chi,q}(x) = q^{-(n+r)x} \sum_{m=0}^{\infty} \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{j,l,n-j-l} \binom{n-j+m-1}{m} (-1)^{\ell} q^{a_{(j+l)x+m}} G^{(\alpha,\beta,r)}_{j,l,q}. \tag{3.7}
$$

Secandly, by using generating function of the multiple generalized $q$-Genocchi polynomials with weight $\alpha$ and weak weight $\beta$, which is defined by (2.11), we obtain the following identities.

By using (2.13), we find that

$$
\begin{align*}
\mathcal{G}^{(\alpha,\beta,r)}_{n+r,\chi,q}(x) &= [2]^i q^r \sum_{m=0}^{\infty} \sum_{a_1,\ldots,a_n=0}^{f-1} \left( m + r - 1 \right) \frac{(-1)^{\sum a_i+m}}{m} \\
& \times q^{\beta \sum a_i + fm} \left( \prod_{i=1}^{r} \chi(a_i) \right) \left[ \sum_{i=1}^{r} a_i + fm + x \right]^{n} q^{x} \\
& = [2]^i q^r \sum_{a_1,\ldots,a_n=0}^{f-1} \sum_{a_0=0}^{n} \left( a_i - a_i, n-l \right) (-1)^{\sum a_i} q^{\alpha(a+n-l)+\beta} \sum_{i=1}^{r} a_i \\
& \times \left( \prod_{i=1}^{r} \chi(a_i) \right) \frac{[x]^{n-l}}{(1-q^{x})(1+q^{f(a+n-l)+\beta})^r}. \tag{3.8}
\end{align*}
$$

Thus we have the following theorem.

Theorem 3.3. Let $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $r \in \mathbb{N}$. Then one has

$$
\begin{align*}
\mathcal{G}^{(\alpha,\beta,r)}_{n+r,\chi,q}(x) &= [2]^i q^r \sum_{a_1,\ldots,a_n=0}^{f-1} \sum_{a_0=0}^{n} \sum_{a_i=0}^{l} \left( a_i - a_i, n-l \right) (-1)^{\sum a_i} q^{\alpha(a+n-l)+\beta} \sum_{i=1}^{r} a_i \\
& \times \left( \prod_{i=1}^{r} \chi(a_i) \right) \frac{[x]^{n-l}}{(1-q^{x})(1+q^{f(a+n-l)+\beta})^r}. \tag{3.9}
\end{align*}
$$

By using (2.13), we have

$$
\mathcal{F}^{(\alpha,\beta,r)}_{\chi,q}(t,x) = [2]^i q^r \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} (-1)^{\ell} q^{a_l x} \sum_{a_1,\ldots,a_n=0}^{f-1} (-1)^{\sum a_i} a_i \\
& \times q^{(\alpha+\beta)\sum a_i} \left( \prod_{i=1}^{r} \chi(a_i) \right) \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-q^{f(\alpha+\beta)})^{m} \frac{t^n}{n!}. \tag{3.10}
$$
Thus we have

\[
\sum_{n=0}^{\infty} G_{\alpha,\beta,r}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[ 2 \sum_{l=0}^{n} \frac{n!}{l!} \right] (-1)^l q^{alx} (1 - q^x)^{-\alpha} \sum_{a_1,\ldots,a_l=0}^{f-1} (-1) \prod_{i=1}^{r} (1 + q^{f(1+\beta)})^{-r} \frac{t^n}{n!}. \tag{3.11}
\]

By comparing the coefficients of both sides of \((n + r)!/t^{n+r}\) in the above, we arrive at the following theorem.

**Theorem 3.4.** Let \(q \in \mathbb{C}_p\) with \(|1 - q|_p < 1\), \(r \in \mathbb{N}\). Then one has

\[
\frac{C_{\alpha,\beta,r}(x)}{(n+r)!} = \left[ 2 \sum_{l=0}^{n} \frac{n!}{l!} \right] \sum_{a_1,\ldots,a_l=0}^{f-1} (-1) \prod_{i=1}^{r} (1 + q^{f(1+\beta)})^{-r} \frac{t^n}{n!}. \tag{3.12}
\]

From (2.12), we easily know that

\[
\sum_{n=0}^{\infty} G_{\alpha,\beta,r}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[ 2 \sum_{l=0}^{n} \frac{n!}{l!} \right] \sum_{k_1,\ldots,k_l=0}^{r} (-1) \prod_{i=1}^{l} (1 + q^{f(1+\beta)})^{-r} \frac{t^n}{n!}. \tag{3.13}
\]

From the above, we get the following theorem.

**Theorem 3.5.** Let \(r \in \mathbb{N}, k \in \mathbb{Z}_+\). Then one has

\[
C_{\alpha,\beta,r}(x) = C_{1,\alpha,\beta,r}(x) = \cdots = C_{r-1,\alpha,\beta,r}(x) = 0,
\]

\[
G_{l+r,\alpha,\beta,r}(x) = [2]_{q^f}^{(l+r)} \sum_{k_1,\ldots,k_l=0}^{r} (-1)^l q^{\sum_{i=1}^{l} k_i} \prod_{i=1}^{l} (1 + q^{f(1+\beta)})^{-r} \frac{t^n}{n!}. \tag{3.14}
\]
From (2.13), we have

\[
\sum_{n=0}^{\infty} G_{n,j,q}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} G_{n,j,q}(x) \frac{t^n}{n!} = 2 [q^f t^{r+s} \sum_{a_1,\ldots,a_r=0}^{f-1} \sum_{b_1,\ldots,b_s=0}^{f-1} \sum_{m=0}^{\infty} \left( \begin{array}{c} m + r - 1 \\ m \end{array} \right) (-1)^{\sum_{i=1}^{r+1} a_i + m} q^{\sum_{i=1}^{r+1} a_i + fm} \\
\times \left( \prod_{i=1}^{r} \chi(a_i) \right) e^{\sum_{i=1}^{r} a_i + fm + x} t^r \sum_{b_1,\ldots,b_s=0}^{f-1} \sum_{k=0}^{\infty} \left( \begin{array}{c} k + s - 1 \\ k \end{array} \right) (-1)^{\sum_{i=1}^{r+1} b_i + k} \\
\times q^{\sum_{i=1}^{r+1} b_i + fk} \left( \prod_{i=1}^{s} \chi(b_i) \right) e^{\sum_{i=1}^{s} b_i + fk + x} t^s \right].
\]

(3.15)

By using Cauchy product in (3.15), we obtain

\[
\sum_{n=0}^{\infty} \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) G_{n,j,q}(x) G_{n-j,j,q}(x) \frac{t^n}{n!} = 2 [q^f t^{r+s} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{a_1,\ldots,a_r=0}^{j-1} \sum_{b_1,\ldots,b_s=0}^{j-1} \sum_{m=0}^{\infty} \left( \begin{array}{c} j + r - 1 \\ j \end{array} \right) \left( \begin{array}{c} n - j + s - 1 \\ n - j \end{array} \right) \\
\times (-1)^{\sum_{i=1}^{r+1} a_i + \sum_{i=1}^{s} b_i + r} q^{\sum_{i=1}^{r+1} a_i + \sum_{i=1}^{s} b_i + fm} \left( \prod_{i=1}^{r} \chi(a_i) \right) \left( \prod_{i=1}^{s} \chi(b_i) \right) \\
\times e^{\sum_{i=1}^{r} a_i + fm + x} t^r \sum_{b_1,\ldots,b_s=0}^{j-1} \sum_{k=0}^{\infty} \left( \begin{array}{c} k + s - 1 \\ k \end{array} \right) (-1)^{\sum_{i=1}^{r+1} b_i + k} \\
\times e^{\sum_{i=1}^{s} b_i + fk + x} t^s \right].
\]

(3.16)

From (3.16), we have

\[
\sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) G_{j,j,q}(x) G_{m-j,j,q}(x) \right) \frac{t^m}{m!} = 2 [q^f t^{r+s} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{a_1,\ldots,a_r=0}^{j-1} \sum_{b_1,\ldots,b_s=0}^{j-1} \sum_{m=0}^{\infty} \left( \begin{array}{c} j + r - 1 \\ j \end{array} \right) \left( \begin{array}{c} n - j + s - 1 \\ n - j \end{array} \right) \\
\times (-1)^{\sum_{i=1}^{r+1} a_i + \sum_{i=1}^{s} b_i + r} q^{\sum_{i=1}^{r+1} a_i + \sum_{i=1}^{s} b_i + fm} \left( \prod_{i=1}^{r} \chi(a_i) \right) \left( \prod_{i=1}^{s} \chi(b_i) \right) \\
\times \left[ \sum_{i=1}^{r} a_i + fj + x \right] \frac{t^r}{q^f} \sum_{b_1,\ldots,b_s=0}^{j-1} \sum_{k=0}^{\infty} \left( \begin{array}{c} k + s - 1 \\ k \end{array} \right) (-1)^{\sum_{i=1}^{r+1} b_i + k} \\
\times \left[ \sum_{i=1}^{s} b_i + f(n - j) + x \right] \frac{t^s}{q^s} \right].
\]

(3.17)

By comparing the coefficients of both sides of \( t^{m+r+s} / (m + r + s)! \) in (3.17), we have the following theorem.
Theorem 3.6. Let \( r \in \mathbb{N} \) and \( s \in \mathbb{Z}_+ \). Then one has

\[
\sum_{j=0}^{l+r+s} \binom{l+r+s}{j} G_{[r,\chi,q]}^{(\alpha,\beta,r)}(x) G_{[l+r+s-j,\chi,q]}^{(\alpha,\beta,s)}(x)
\]

\[
= [2]^r q^s \sum_{j=0}^{\infty} \sum_{j=0}^{f-1} \sum_{j=0}^{f-1} \sum_{j=0}^{r} (-1)^{l+j} \binom{j+r-1}{j} \binom{n-j}{n-j} \binom{n+j}{n+j}
\]

\[
\times \left( \sum_{i=1}^{\infty} \binom{a_i+fj+x}{r} q^s + \left[ \sum_{i=1}^{n} b_i + f(n-j) + x \right] q^s \right). \tag{3.18}
\]

Corollary 3.7. In (3.18) setting \( s = 1 \), one has

\[
\sum_{j=0}^{l+r+1} \binom{l+r+1}{j} G_{[r+1,\chi,q]}^{(\alpha,\beta,r)}(x) G_{[l+r+1-j,\chi,q]}^{(\alpha,\beta,s)}(x)
\]

\[
= [2]^{r+1} q^s \sum_{j=0}^{\infty} \sum_{j=0}^{f-1} \sum_{j=0}^{f-1} \sum_{j=0}^{r} (-1)^{l+j} \binom{j+r-1}{j} \binom{n-j}{n-j} \binom{n+j}{n+j}
\]

\[
\times \left( \chi(b_1) \prod_{i=1}^{r} \chi(a_i) \right) \left( \left[ \sum_{i=1}^{r} a_i + fj + x \right] q^s + \left[ b_i + f(n-j) + x \right] q^s \right). \tag{3.19}
\]

By using (2.13) we have the following theorem.

Theorem 3.8. Distribution theorem is as follows:

\[
G_{[n+r,\chi,q]}^{(\alpha,\beta,r)} = \left[ \frac{f^n}{[f]_{-q}^n} \right] \sum_{a_1, \ldots, a_r=0}^{\infty} (-1)^{\sum_{i=1}^{r} a_i} q^\sum_{i=1}^{r} a_i \left( \prod_{i=1}^{r} \chi(a_i) \right) c_{[n+r,\chi,q]}^{(\alpha,\beta,r)} \left( \frac{a_1 + \cdots + a_r}{f} \right),
\]

\[
G_{[n+r,\chi,q]}^{(\alpha,\beta,s)}(x) = \left[ \frac{f^n}{[f]_{-q}^n} \right] \sum_{a_1, \ldots, a_r=0}^{\infty} (-1)^{\sum_{i=1}^{r} a_i} q^\sum_{i=1}^{r} a_i \left( \prod_{i=1}^{r} \chi(a_i) \right) c_{[n+r,\chi,q]}^{(\alpha,\beta,s)} \left( x + \frac{a_1 + \cdots + a_r}{f} \right). \tag{3.20}
\]

4. Interpolation Function of Multiple Generalized \( q \)-Genocchi Polynomials with Weight \( \alpha \) and Weak Weight \( \beta \)

In this section, we see interpolation function of multiple generalized \( q \)-Genocchi polynomials with weak weight \( \alpha \) and find some relations.
Let us define interpolation function of the $G_{k+r,q}^{(a,b,r)}(x)$ as follows.

**Definition 4.1.** Let $q, s \in \mathbb{C}$ with $|q| < 1$ and $0 < x \leq 1$. Then one defines

$$
\xi_{X,q}^{(a,b,r)}(s,x) = [2]_q^{s} \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-1)^{\sum k_i} q^{\sum k_i} \left( \prod_{i=1}^{r} \chi(k_i) \right)}{[x + \sum_{i=1}^{r} k_i]_{q^s}}.
$$

(4.1)

We call $\xi_{X,q}^{(a,b,r)}(s,x)$ the multiple generalized Hurwitz type $q$-zeta function.

In (4.1), setting $r = 1$, we have

$$
\xi_{X,q}^{(a,b,1)}(s,x) = [2]_q^{s} \sum_{l=0}^{\infty} \frac{(-1)^l q^l \chi(l)}{[x + l]_{q^s}} = \xi_{X,q}^{(a,b)}(s,x).
$$

(4.2)

**Remark 4.2.** It holds that

$$
\lim_{q \to 1} \xi_{X,q}^{(a,b,r)}(s,x) = 2^r \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-1)^{\sum k_i} \left( \prod_{i=1}^{r} \chi(k_i) \right)}{[x + \sum_{i=1}^{r} k_i]_{q^s}}.
$$

(4.3)

Substituting $s = -n, n \in \mathbb{Z}_+$ into (4.1), then we have,

$$
\xi_{X,q}^{(a,b,r)}(-n,x) = [2]_q^{s} \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-1)^{\sum k_i} q^{\sum k_i} \left( \prod_{i=1}^{r} \chi(k_i) \right)}{[x + \sum_{i=1}^{r} k_i]_{q^s}}.
$$

(4.4)

Setting (3.14) into the above, we easily get the following theorem.

**Theorem 4.3.** Let $r \in \mathbb{N}, n \in \mathbb{Z}_+$. Then one has

$$
\xi_{X,q}^{(a,b,r)}(-n,x) = \frac{G_{n+r,q}^{(a,b,r)}(x)}{(-n!)^r r!}.
$$

(4.5)

**References**


