Research Article

Two Sufficient Conditions for Hamilton and Dominating Cycles

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We prove that if G is a 2-connect graph of size q (the number of edges) and minimum degree δ with $\delta \geq \sqrt{2q/3 + \epsilon/12} - 1/2$, where $\epsilon = 11$ when $\delta = 2$ and $\epsilon = 31$ when $\delta \geq 3$, then each longest cycle in G is a dominating cycle. The exact analog of this theorem for Hamilton cycles follows easily from two known results according to Dirac and Nash-Williams: each graph with $\delta \geq \sqrt{q + 5/4} - 1/2$ is hamiltonian. Both results are sharp in all respects.

1. Introduction

Only finite undirected graphs without loops or multiple edges are considered. We reserve n , q , δ , and κ to denote the number of vertices (order), the number of edges (size), the minimum degree, and the connectivity of a graph, respectively. A graph G is hamiltonian if G contains a hamiltonian cycle, that is, a cycle of length n . Further, a cycle C in G is called a dominating cycle if the vertices in $G \setminus C$ are mutually nonadjacent. A good reference for any undefined terms is [1].

The following two well-known theorems provide two classic sufficient conditions for Hamilton and dominating cycles by linking the minimum degree δ and order n .

Theorem A (see [2]). *Every graph with $\delta \geq (1/2)n$ is hamiltonian.*

Theorem B (see [3]). *If G is a 2-connect graph with $\delta \geq (1/3)(n + 2)$, then each longest cycle in G is a dominating cycle.*

The exact analog of Theorem A that links the minimum degree δ and size q easily follows from Theorem A and a particular result according to Nash-Williams [4] (see Theorem 1.1 below).

Theorem 1.1. *Every graph is hamiltonian if*

$$\delta \geq \sqrt{q + \frac{5}{4}} - \frac{1}{2}. \quad (1.1)$$

The hypothesis in Theorem 1.1 is equivalent to $q \leq \delta^2 + \delta - 1$ and cannot be relaxed to $q \leq \delta^2 + \delta$ due to the graph $K_1 + 2K_\delta$ consisting of two copies of $K_{\delta+1}$ and having exactly one vertex in common. Hence, Theorem 1.1 is best possible.

The main goal of this paper is to prove the exact analog of Theorem B for dominating cycles based on another similar relation between δ and q .

Theorem 1.2. *Let G be a 2-connect graph with*

$$\delta \geq \sqrt{\frac{2q}{3} + \frac{\epsilon}{12}} - \frac{1}{2}, \quad (1.2)$$

where $\epsilon = 11$ when $\delta = 2$ and $\epsilon = 31$ when $\delta \geq 3$. Then each longest cycle in G is a dominating cycle.

To show that Theorem 1.2 is sharp, suppose first that $\delta = 2$, implying that the hypothesis in Theorem 1.2 is equivalent to $q \leq 8$. The graph $K_1 + 2K_2$ shows that the connectivity condition $\kappa \geq 2$ in Theorem 1.2 cannot be relaxed by replacing it with $\kappa \geq 1$. The graph with vertex set $\{v_1, v_2, \dots, v_8\}$ and edge set

$$\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_1v_7, v_7v_8, v_8v_4\} \quad (1.3)$$

shows that the size bound $q \leq 8$ cannot be relaxed by replacing it with $q \leq 9$. Finally, the graph $K_2 + 3K_1$ shows that the conclusion “each longest cycle in G is a dominating cycle” cannot be strengthened by replacing it with “ G is hamiltonian.” Analogously, we can use $K_1 + 2K_\delta$, $K_2 + 3K_{\delta-1}$, and $K_\delta + (\delta + 1)K_1$, respectively, to show that Theorem 1.2 is sharp when $\delta \geq 3$. So, Theorem 1.2 is best possible in all respects.

To prove Theorems 1.1 and 1.2, we need two known results, the first of which is belongs Nash-Williams [4].

Theorem C (see [4]). *If $\delta = (n - 1)/2$, then either G is hamiltonian or $G = K_1 + 2K_\delta$, or $G = \overline{K}_{\delta+1} + G_\delta$, where G_δ denote an arbitrary graph on δ vertices.*

The next theorem provides a lower bound for the length of a longest cycle in 2-connected graphs according to Dirac [2].

Theorem D (see [2]). *Every 2-connected graph either has a hamiltonian cycle or has a cycle of length at least 2δ .*

2. Notations and Preliminaries

The set of vertices of a graph G is denoted by $V(G)$ and the set of edges by $E(G)$. For S , a subset of $V(G)$, we denote by $G \setminus S$ the maximum subgraph of G with vertex set $V(G) \setminus S$.

We write $G[S]$ for the subgraph of G induced by S . For a subgraph H of G , we use $G \setminus H$ short for $G \setminus V(H)$. The neighborhood of a vertex $x \in V(G)$ will be denoted by $N(x)$. Set $d(x) = |N(x)|$. Furthermore, for a subgraph H of G and $x \in V(G)$, we define $N_H(x) = N(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$.

A simple cycle (or just a cycle) C of length t is a sequence $v_1 v_2 \cdots v_t v_1$ of distinct vertices $v_1, \dots, v_t$ with $v_i v_{i+1} \in E(G)$ for each $i \in \{1, \dots, t\}$, where $v_{t+1} = v_1$. When $t = 2$, the cycle $C = v_1 v_2 v_1$ on two vertices v_1, v_2 coincides with the edge $v_1 v_2$, and when $t = 1$, the cycle $C = v_1$ coincides with the vertex v_1 . So, all vertices and edges in a graph can be considered as cycles of lengths 1 and 2, respectively.

Paths and cycles in a graph G are considered as subgraphs of G . If Q is a path or a cycle, then the length of Q , denoted by $|Q|$, is $|E(Q)|$. We write $\vec{Q}$ with a given orientation by $\vec{Q}$. For $x, y \in V(Q)$, we denote by $x\vec{Q}y$ the subpath of Q in the chosen direction from x to y . For $x \in V(Q)$, we denote the h th successor and the h th predecessor of x on $\vec{Q}$ by x^{+h} and x^{-h} , respectively. We abbreviate x^{+1} and x^{-1} by x^+ and x^- , respectively.

Special Definitions

Let G be a graph, C a longest cycle in G , and $P = x\vec{P}y$ a longest path in $G \setminus C$ of length $\bar{p} \geq 0$. Let $\xi_1, \xi_2, \dots, \xi_s$ be the elements of $N_C(x) \cup N_C(y)$ occurring on C in a consecutive order. Set

$$I_i = \xi_i \vec{C} \xi_{i+1}, \quad I_i^* = \xi_i^+ \vec{C} \xi_{i+1}^- \quad (i = 1, 2, \dots, s), \quad (2.1)$$

where $\xi_{s+1} = \xi_1$.

(*1) We call $I_1, I_2, \dots, I_s$ elementary segments on C created by $N_C(x) \cup N_C(y)$.

(*2) We call a path $L = z\vec{L}w$ an intermediate path between two distinct elementary segments I_a and I_b if

$$z \in V(I_a^*), \quad w \in V(I_b^*), \quad V(L) \cap V(C \cup P) = \{z, w\}. \quad (2.2)$$

(*3) The set of all intermediate paths between elementary segments $I_{i_1}, I_{i_2}, \dots, I_{i_t}$ will be denoted by $Y(I_{i_1}, I_{i_2}, \dots, I_{i_t})$.

Lemma 2.1. Let G be a graph, C a longest cycle in G , and $P = x\vec{P}y$ a longest path in $G \setminus C$ of length $\bar{p} \geq 1$. If $|N_C(x)| \geq 2$, $|N_C(y)| \geq 2$ and $N_C(x) \neq N_C(y)$, then

$$|C| \geq \begin{cases} 3\delta + \max\{\sigma_1, \sigma_2\} - 1 \geq 3\delta, & \text{if } \bar{p} = 1, \\ \max\{2\bar{p} + 8, 4\delta - 2\bar{p}\}, & \text{if } \bar{p} \geq 2, \end{cases} \quad (2.3)$$

where $\sigma_1 = |N_C(x) \setminus N_C(y)|$ and $\sigma_2 = |N_C(y) \setminus N_C(x)|$.

Lemma 2.2. Let G be a graph, C a longest cycle in G , and $P = x\bar{P}y$ a longest path in $G \setminus C$ of length $\bar{p} \geq 0$. If $N_C(x) = N_C(y)$, $|N_C(x)| \geq 2$ and I_a, I_b are elementary segments induced by $N_C(x) \cup N_C(y)$, then

(a1) if L is an intermediate path between I_a and I_b , then

$$|I_a| + |I_b| \geq 2\bar{p} + 2|L| + 4, \quad (2.4)$$

(a2) if $Y(I_a, I_b) \subseteq E(G)$ and $|Y(I_a, I_b)| = i$ for some $i \in \{1, 2, 3\}$, then

$$|I_a| + |I_b| \geq 2\bar{p} + i + 5. \quad (2.5)$$

Lemma 2.3. Let G be a graph, S a cut set in G , and H a connected component of $G \setminus S$ of order h . Then

$$q_H \geq \frac{h(2\delta - h + 1)}{2}, \quad (2.6)$$

where $q_H = |\{xy \in E(G) : \{x, y\} \cap V(H) \neq \emptyset\}|$.

Lemma 2.4. Let G be a 2-connect graph. If $\delta \geq (n - 2)/3$, then either

$$q \geq \begin{cases} 9 & \text{when } \delta = 2, \\ \frac{3(\delta - 1)(\delta + 2)}{2} & \text{when } \delta \geq 3, \end{cases} \quad (2.7)$$

or each longest cycle in G is a dominating cycle.

3. Proofs

Proof of Lemma 2.1. Put

$$A_1 = N_C(x) \setminus N_C(y), \quad A_2 = N_C(y) \setminus N_C(x), \quad M = N_C(x) \cap N_C(y). \quad (3.1)$$

By the hypothesis, $N_C(x) \neq N_C(y)$, implying that

$$\max\{|A_1|, |A_2|\} \geq 1. \quad (3.2)$$

Let $\xi_1, \xi_2, \dots, \xi_s$ be the elements of $N_C(x) \cup N_C(y)$ occurring on C in a consecutive order. Put $I_i = \xi_i \bar{C} \xi_{i+1}$ ($i = 1, 2, \dots, s$), where $\xi_{s+1} = \xi_1$. Clearly, $s = |A_1| + |A_2| + |M|$. Since C is extreme, $|I_i| \geq 2$ ($i = 1, 2, \dots, s$). Next, if $\{\xi_i, \xi_{i+1}\} \cap M \neq \emptyset$ for some $i \in \{1, 2, \dots, s\}$, then $|I_i| \geq \bar{p} + 2$. Further, if either $\xi_i \in A_1, \xi_{i+1} \in A_2$ or $\xi_i \in A_2, \xi_{i+1} \in A_1$, then again $|I_i| \geq \bar{p} + 2$.

Case 1. ($\bar{p} = 1$).

Case 1.1 ($|A_i| \geq 1 (i = 1, 2)$). It follows that among $I_1, I_2, \dots, I_s$ there are $|M| + 2$ segments of length at least $\bar{p} + 2$. Observing also that each of the remaining $s - (|M| + 2)$ segments has a length at least 2, we have

$$\begin{aligned} |C| &\geq (\bar{p} + 2)(|M| + 2) + 2(s - |M| - 2) \\ &= 3(|M| + 2) + 2(|A_1| + |A_2| - 2) \\ &= 2|A_1| + 2|A_2| + 3|M| + 2. \end{aligned} \quad (3.3)$$

Since $|A_1| = d(x) - |M| - 1$ and $|A_2| = d(y) - |M| - 1$,

$$|C| \geq 2d(x) + 2d(y) - |M| - 2 \geq 3\delta + d(x) - |M| - 2. \quad (3.4)$$

Recalling that $d(x) = |M| + |A_1| + 1$, we get

$$|C| \geq 3\delta + |A_1| - 1 = 3\delta + \sigma_1 - 1. \quad (3.5)$$

Analogously, $|C| \geq 3\delta + \sigma_2 - 1$. So,

$$|C| \geq 3\delta + \max\{\sigma_1, \sigma_2\} - 1 \geq 3\delta. \quad (3.6)$$

Case 1.2 (either $|A_1| \geq 1, |A_2| = 0$ or $|A_1| = 0, |A_2| \geq 1$). Assume without loss of generality that $|A_1| \geq 1$ and $|A_2| = 0$, that is, $|N_C(y)| = |M| \geq 2$ and $s = |A_1| + |M|$. Hence, among $I_1, I_2, \dots, I_s$ there are $|M| + 1$ segments of length at least $\bar{p} + 2 = 3$. Taking into account that each of the remaining $s - (|M| + 1)$ segments has a length at least 2 and $|M| + 1 = d(y)$, we get

$$\begin{aligned} |C| &\geq 3(|M| + 1) + 2(s - |M| - 1) = 3d(y) + 2(|A_1| - 1) \\ &\geq 3\delta + |A_1| - 1 = 3\delta + \max\{\sigma_1, \sigma_2\} - 1 \geq 3\delta. \end{aligned} \quad (3.7)$$

Case 2 ($\bar{p} \geq 2$). We first prove that $|C| \geq 2\bar{p} + 8$. Since $|N_C(x)| \geq 2$ and $|N_C(y)| \geq 2$, there are at least two segments among $I_1, I_2, \dots, I_s$ of length at least $\bar{p} + 2$. If $|M| = 0$, then clearly $s \geq 4$ and

$$|C| \geq 2(\bar{p} + 2) + 2(s - 2) \geq 2\bar{p} + 8. \quad (3.8)$$

Otherwise, since $\max\{|A_1|, |A_2|\} \geq 1$, there are at least three elementary segments of length at least $\bar{p} + 2$, that is,

$$|C| \geq 3(\bar{p} + 2) \geq 2\bar{p} + 8. \quad (3.9)$$

So, in any case, $|C| \geq 2\bar{p} + 8$.

To prove that $|C| \geq 4\delta - 2\bar{p}$, we distinguish two main cases.

Case 2.1 ($|A_i| \geq 1 (i = 1, 2)$). It follows that among $I_1, I_2, \dots, I_s$ there are $|M| + 2$ segments of length at least $\bar{p} + 2$. Further, since each of the remaining $s - (|M| + 2)$ segments has a length at least 2, we get

$$\begin{aligned} |C| &\geq (\bar{p} + 2)(|M| + 2) + 2(s - |M| - 2) \\ &= (\bar{p} - 2)|M| + (2\bar{p} + 4|M| + 4) + 2(|A_1| + |A_2| - 2) \\ &\geq 2|A_1| + 2|A_2| + 4|M| + 2\bar{p}. \end{aligned} \quad (3.10)$$

Observing also that

$$|A_1| + |M| + \bar{p} \geq d(x), \quad |A_2| + |M| + \bar{p} \geq d(y), \quad (3.11)$$

we have

$$2|A_1| + 2|A_2| + 4|M| + 2\bar{p} \geq 2d(x) + 2d(y) - 2\bar{p} \geq 4\delta - 2\bar{p}, \quad (3.12)$$

implying that $|C| \geq 4\delta - 2\bar{p}$.

Case 2.2 (either $|A_1| \geq 1, |A_2| = 0$ or $|A_1| = 0, |A_2| \geq 1$). Assume without loss of generality that $|A_1| \geq 1$ and $|A_2| = 0$, that is, $|N_C(y)| = |M| \geq 2$ and $s = |A_1| + |M|$. It follows that among $I_1, I_2, \dots, I_s$ there are $|M| + 1$ segments of length at least $\bar{p} + 2$. Observing also that $|M| + \bar{p} \geq d(y) \geq \delta$, that is, $2\bar{p} + 4|M| \geq 4\delta - 2\bar{p}$, we get

$$\begin{aligned} |C| &\geq (\bar{p} + 2)(|M| + 1) \geq (\bar{p} - 2)(|M| - 1) + 2\bar{p} + 4|M| \\ &\geq 2\bar{p} + 4|M| \geq 4\delta - 2\bar{p}. \end{aligned} \quad (3.13)$$

□

Proof of Lemma 2.2. Let $\xi_1, \xi_2, \dots, \xi_s$ be the elements of $N_C(x)$ occurring on C in a consecutive order. Put $I_i = \xi_i \vec{C} \xi_{i+1} (i = 1, 2, \dots, s)$, where $\xi_{s+1} = \xi_1$. To prove (a1), let $L = z \vec{L} w$ be an intermediate path between elementary segments I_a and I_b with $z \in V(I_a^*)$ and $w \in V(I_b^*)$. Put

$$\left| \xi_a \vec{C} z \right| = d_1, \quad \left| z \vec{C} \xi_{a+1} \right| = d_2, \quad \left| \xi_b \vec{C} w \right| = d_3, \quad \left| w \vec{C} \xi_{b+1} \right| = d_4, \quad (3.14)$$

$$C' = \xi_a x \vec{P} y \xi_b \overleftarrow{C} z \vec{L} w \vec{C} \xi_a.$$

Clearly,

$$|C'| = |C| - d_1 - d_3 + |L| + |P| + 2. \quad (3.15)$$

Since C is extreme, we have $|C| \geq |C'|$, implying that $d_1 + d_3 \geq \bar{p} + |L| + 2$. By a symmetric argument, $d_2 + d_4 \geq \bar{p} + |L| + 2$. Hence

$$|I_a| + |I_b| = \sum_{i=1}^4 d_i \geq 2\bar{p} + 2|L| + 4. \quad (3.16)$$

The proof of (a1) is complete. To prove (a2), let $Y(I_a, I_b) \subseteq E(G)$ and $|Y(I_a, I_b)| = i$ for some $i \in \{1, 2, 3\}$.

Case 1 ($i = 1$). It follows that $Y(I_a, I_b)$ consists of a unique intermediate edge $L = zw$. By (a1),

$$|I_a| + |I_b| \geq 2\bar{p} + 2|L| + 4 = 2\bar{p} + 6. \quad (3.17)$$

Case 2 ($i = 2$). It follows that $Y(I_a, I_b)$ consists of two edges e_1, e_2 . Put $e_1 = z_1w_1$ and $e_2 = z_2w_2$, where $\{z_1, z_2\} \subseteq V(I_a^*)$ and $\{w_1, w_2\} \subseteq V(I_b^*)$.

Case 2.1 ($z_1 \neq z_2$ and $w_1 \neq w_2$). Assume without loss of generality that z_1 and z_2 occur in this order on I_a .

Case 2.1.1. w_2 and w_1 occur in this order on I_b .

Put

$$\begin{aligned} |\xi_a \vec{C} z_1| &= d_1, & |z_1 \vec{C} z_2| &= d_2, & |z_2 \vec{C} \xi_{a+1}| &= d_3, \\ |\xi_b \vec{C} w_2| &= d_4, & |w_2 \vec{C} w_1| &= d_5, & |w_1 \vec{C} \xi_{b+1}| &= d_6, \end{aligned} \quad (3.18)$$

$$C' = \xi_a \vec{C} z_1 w_1 \overleftarrow{C} w_2 z_2 \vec{C} \xi_b x \vec{P} y \xi_{b+1} \vec{C} \xi_a.$$

Clearly,

$$\begin{aligned} |C'| &= |C| - d_2 - d_4 - d_6 + |\{e_1\}| + |\{e_2\}| + |P| + 2 \\ &= |C| - d_2 - d_4 - d_6 + \bar{p} + 4. \end{aligned} \quad (3.19)$$

Since C is extreme, $|C| \geq |C'|$, implying that $d_2 + d_4 + d_6 \geq \bar{p} + 4$. By a symmetric argument, $d_1 + d_3 + d_5 \geq \bar{p} + 4$. Hence

$$|I_a| + |I_b| = \sum_{i=1}^6 d_i \geq 2\bar{p} + 8. \quad (3.20)$$

Case 2.1.2. w_1 and w_2 occur in this order on I_b .

Putting

$$C' = \xi_a \vec{C} z_1 w_1 \vec{C} w_2 z_2 \vec{C} \xi_b x \vec{P} y \xi_{b+1} \vec{C} \xi_a, \quad (3.21)$$

we can argue as in Case 2.1.1.

Case 2.2 (either $z_1 = z_2, w_1 \neq w_2$ or $z_1 \neq z_2, w_1 = w_2$). Assume without loss of generality that $z_1 \neq z_2, w_1 = w_2$ and z_1, z_2 occur in this order on I_a . Put

$$\begin{aligned} \left| \xi_a \vec{C} z_1 \right| &= d_1, & \left| z_1 \vec{C} z_2 \right| &= d_2, & \left| z_2 \vec{C} \xi_{a+1} \right| &= d_3, \\ \left| \xi_b \vec{C} w_1 \right| &= d_4, & \left| w_1 \vec{C} \xi_{b+1} \right| &= d_5, \\ C' &= \xi_a x \vec{P} y \xi_b \overleftarrow{C} z_1 w_1 \vec{C} \xi_a, \\ C'' &= \xi_a \vec{C} z_2 w_1 \overleftarrow{C} \xi_{a+1} x \vec{P} y \xi_{b+1} \vec{C} \xi_a. \end{aligned} \quad (3.22)$$

Clearly,

$$\begin{aligned} |C'| &= |C| - d_1 - d_4 + |\{e_1\}| + |P| + 2 = |C| - d_1 - d_4 + \bar{p} + 3, \\ |C''| &= |C| - d_3 - d_5 + |\{e_2\}| + |P| + 2 = |C| - d_3 - d_5 + \bar{p} + 3. \end{aligned} \quad (3.23)$$

Since C is extreme, $|C| \geq |C'|$ and $|C| \geq |C''|$, implying that

$$d_1 + d_4 \geq \bar{p} + 3, \quad d_3 + d_5 \geq \bar{p} + 3. \quad (3.24)$$

Hence,

$$|I_a| + |I_b| = \sum_{i=1}^5 d_i \geq d_1 + d_3 + d_4 + d_5 + 1 \geq 2\bar{p} + 7. \quad (3.25)$$

Case 3 ($i = 3$). It follows that $Y(I_a, I_b)$ consists of three edges e_1, e_2, e_3 . Let $e_i = z_i w_i$ ($i = 1, 2, 3$), where $\{z_1, z_2, z_3\} \subseteq V(I_a^*)$ and $\{w_1, w_2, w_3\} \subseteq V(I_b^*)$. If there are two independent edges among e_1, e_2, e_3 , then we can argue as in Case 2.1. Otherwise, we can assume without loss of generality that $w_1 = w_2 = w_3$ and z_1, z_2, z_3 occur in this order on I_a . Put

$$\begin{aligned} \left| \xi_a \vec{C} z_1 \right| &= d_1, & \left| z_1 \vec{C} z_2 \right| &= d_2, & \left| z_2 \vec{C} z_3 \right| &= d_3, \\ \left| z_3 \vec{C} \xi_{a+1} \right| &= d_4, & \left| \xi_b \vec{C} w_1 \right| &= d_5, & \left| w_1 \vec{C} \xi_{b+1} \right| &= d_6, \\ C' &= \xi_a x \vec{P} y \xi_b \overleftarrow{C} z_1 w_1 \vec{C} \xi_a, \\ C'' &= \xi_a \vec{C} z_3 w_1 \overleftarrow{C} \xi_{a+1} x \vec{P} y \xi_{b+1} \vec{C} \xi_a. \end{aligned} \quad (3.26)$$

Clearly,

$$\begin{aligned} |C'| &= |C| - d_1 - d_5 + |\{e_1\}| + \bar{p} + 2, \\ |C''| &= |C| - d_4 - d_6 + |\{e_3\}| + \bar{p} + 2. \end{aligned} \quad (3.27)$$

Since C is extreme, we have $|C| \geq |C'|$ and $|C| \geq |C''|$, implying that

$$d_1 + d_5 \geq \bar{p} + 3, \quad d_4 + d_6 \geq \bar{p} + 3. \quad (3.28)$$

Hence,

$$|I_a| + |I_b| = \sum_{i=1}^6 d_i \geq d_1 + d_4 + d_5 + d_6 + 2 \geq 2\bar{p} + 8. \quad (3.29)$$

□

Proof of Lemma 2.3. Put

$$V(H) = \{v_1, \dots, v_h\}, \quad |N(v_i) \cap S| = \beta_i \quad (i = 1, \dots, h). \quad (3.30)$$

Observing that $h \geq d(v_i) - \beta_i + 1 \geq \delta - \beta_i + 1$ for each $i \in \{1, 2, \dots, h\}$, we have $\beta_i \geq \delta - h + 1$ ($i = 1, 2, \dots, h$). Therefore,

$$\begin{aligned} q_H &= q(H) + \sum_{i=1}^h \beta_i = \frac{1}{2} \sum_{i=1}^h d_H(v_i) + \sum_{i=1}^h \beta_i, \\ &= \frac{1}{2} \sum_{i=1}^h (d_H(v_i) + \beta_i) + \frac{1}{2} \sum_{i=1}^h \beta_i = \frac{1}{2} \sum_{i=1}^h d(v_i) + \frac{1}{2} \sum_{i=1}^h (\delta - h + 1), \\ &\geq \frac{1}{2} h\delta + \frac{1}{2} h(\delta - h + 1) = \frac{h(2\delta - h + 1)}{2}. \end{aligned} \quad (3.31)$$

□

Proof of Lemma 2.4. Let C be a longest cycle in G and $P = x_1 \vec{P} x_2$ a longest path in $G \setminus C$ of length $\bar{p}$. If $|V(P)| \leq 1$, then C is a dominating cycle and we are done. Let $|V(P)| \geq 2$, that is, $\bar{p} \geq 1$. By the hypothesis, $|C| + \bar{p} + 1 \leq n \leq 3\delta + 2$. Further, by Theorem D, $|C| \geq 2\delta$. From these inequalities, we get

$$n \leq 3\delta + 2, \quad |C| \leq 3\delta - \bar{p} + 1, \quad 1 \leq \bar{p} \leq \delta + 1. \quad (3.32)$$

Let $\xi_1, \xi_2, \dots, \xi_s$ be the elements of $N_C(x_1) \cup N_C(x_2)$ occurring on C in a consecutive order. Put

$$I_i = \xi_i \vec{C} \xi_{i+1}, \quad I_i^* = \xi_i^+ \vec{C} \xi_{i+1}^- \quad (i = 1, 2, \dots, s), \quad (3.33)$$

where $\xi_{s+1} = \xi_1$.

Case 1 ($\delta = 2$). Let Q be a longest path in G with $Q = \xi \vec{Q} \eta$ and $V(Q) \cap V(C) = \{\xi, \eta\}$. Since C is extreme, we have $|\xi \vec{C} \eta| \geq |Q|$ and $|\eta \vec{C} \xi| \geq |Q|$, implying that

$$|C| = |\xi \vec{C} \eta| + |\eta \vec{C} \xi| \geq 2|Q|. \quad (3.34)$$

Since $\kappa \geq 2$ and $\bar{p} \geq 1$, we have $|Q| \geq 3$. By (3.34), $|C| \geq 2|Q| \geq 6$, implying that $q \geq |C| + |Q| \geq 9$.

Case 2. ($\delta \geq 3$).

Case 2.1 ($\bar{p} = 1$). By (3.32),

$$|C| \leq 3\delta. \quad (3.35)$$

Case 2.1.1 ($N_C(x_1) \neq N_C(x_2)$). It follows that $\max\{\sigma_1, \sigma_2\} \geq 1$, where

$$\sigma_1 = |N_C(x_1) \setminus N_C(x_2)|, \quad \sigma_2 = |N_C(x_2) \setminus N_C(x_1)|. \quad (3.36)$$

By Lemma 2.1, $|C| \geq 3\delta$. Recalling (3.35), we get $|C| = 3\delta$. If $\max\{\sigma_1, \sigma_2\} \geq 2$, then by Lemma 2.1, $|C| \geq 3\delta + 1$, contradicting (3.35). Let $\max\{\sigma_1, \sigma_2\} = 1$. Clearly, $s \geq \delta$ and $|I_i| \geq 3$ ($i = 1, 2, \dots, s$). Further, if $s \geq \delta + 1$, then $|C| \geq 3s \geq 3\delta + 3$, again contradicting (3.35). Let $s = \delta$, implying that $|I_i| = 3$ ($i = 1, 2, \dots, s$). By Lemma 2.2, $Y(I_1, I_2, \dots, I_s) = \emptyset$. Let $H_1, H_2, \dots, H_{s+1}$ be the connected components of $G \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$ with $V(H_i) = V(I_i^*)$ ($i = 1, 2, \dots, s$) and $V(H_{s+1}) = \{x_1, x_2\}$. For each $i \in \{1, 2, \dots, s+1\}$, put

$$h_i = |V(H_i)|, \quad q_i = |\{xy \in E(G) : \{x, y\} \cap V(H_i) \neq \emptyset\}|. \quad (3.37)$$

Clearly, $h_i = 2$ ($i = 1, 2, \dots, s+1$). By Lemma 2.3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} = 2\delta - 1 \quad (i = 1, 2, \dots, s+1), \quad (3.38)$$

implying that

$$q \geq \sum_{i=1}^{s+1} q_i \geq (s+1)(2\delta - 1) = (\delta+1)(2\delta - 1) > \frac{3(\delta-1)(\delta+2)}{2}. \quad (3.39)$$

Case 2.1.2 ($N_C(x_1) = N_C(x_2)$). Clearly, $s \geq \delta - 1$. If $s \geq \delta$, then we can argue as in Case 2.1.1. Let $s = \delta - 1$. Further, if $|I_i| + |I_j| \geq 10$ for some distinct $i, j \in \{1, 2, \dots, s\}$, then $|C| \geq 10 + 3(s-2) = 3\delta + 1$, contradicting (3.35). Hence

$$|I_i| + |I_j| \leq 9 \quad \text{for each distinct } i, j \in \{1, 2, \dots, s\}. \quad (3.40)$$

Claim 1. $Y(I_1, I_2, \dots, I_s) \subseteq E(G)$ and

- (1) if $\max_i |I_i| \leq 4$ then $|Y(I_1, I_2, \dots, I_s)| \leq 3$,
- (2) if $\max_i |I_i| = 5$ then $|Y(I_1, I_2, \dots, I_s)| \leq \delta - 1$,
- (3) if $\max_i |I_i| = 6$ then $|Y(I_1, I_2, \dots, I_s)| \leq 2(\delta - 2)$.

Proof. If $Y(I_1, I_2, \dots, I_s) = \emptyset$ then we are done. Otherwise, $Y(I_a, I_b) \neq \emptyset$, for some distinct $a, b \in \{1, 2, \dots, s\}$. By definition, there is an intermediate path L between I_a and I_b . If $|L| \geq 2$, then by Lemma 2.2,

$$|I_a| + |I_b| \geq 2\bar{p} + 2|L| + 4 \geq 10, \quad (3.41)$$

contradicting (3.40). Otherwise, $|L| = 1$ and therefore, $Y(I_1, I_2, \dots, I_s) \subseteq E(G)$. By Lemma 2.2, $|I_a| + |I_b| \geq 2\bar{p} + 6 = 8$. Combining this with (3.40), we have

$$8 \leq |I_a| + |I_b| \leq 9. \quad (3.42)$$

Furthermore, if $|Y(I_a, I_b)| \geq 3$, then by Lemma 2.2, $|I_a| + |I_b| \geq 2\bar{p} + 8 = 10$, contradicting (3.42). So,

$$1 \leq |Y(I_i, I_j)| \leq 2 \quad \text{for each distinct } i, j \in \{1, 2, \dots, s\}. \quad (3.43)$$

Put $r = |\{i : |I_i| \geq 4\}|$. If $r \geq 4$, then $|C| \geq 16 + 3(s - 4) = 3\delta + 1$, contradicting (3.35). Further, if $r = 0$, then by Lemma 2.2, $Y(I_1, I_2, \dots, I_s) = \emptyset$. Let $1 \leq r \leq 3$.

Case a1 ($r = 3$). It follows that $|I_i| \geq 4$ ($i = a, b, c$) for some distinct $a, b, c \in \{1, 2, \dots, s\}$ and $|I_i| = 3$ for each $i \in \{1, 2, \dots, s\} \setminus \{a, b, c\}$. Recalling that $s = \delta - 1$ and $|C| = 3\delta$, we have $|I_a| = |I_b| = |I_c| = 4$, that is, $\max_i |I_i| = 4$. By Lemma 2.2, $|Y(I_i, I_j)| \leq 1$ for each distinct $i, j \in \{a, b, c\}$. Moreover, we have $|Y(I_i, I_j)| = 0$ if either $i \notin \{a, b, c\}$ or $j \notin \{a, b, c\}$. So,

$$|Y(I_1, I_2, \dots, I_s)| = |Y(I_a, I_b, I_c)| \leq 3. \quad (3.44)$$

Case a2 ($r = 2$). It follows that $|I_a| \geq 4$ and $|I_b| \geq 4$ for some distinct $a, b \in \{1, 2, \dots, s\}$ and $|I_i| = 3$ for each $i \in \{1, 2, \dots, s\} \setminus \{a, b\}$. By (3.42), we can assume without loss of generality that either $|I_a| = |I_b| = 4$ or $|I_a| = 5, |I_b| = 4$.

Case a2.1 ($|I_a| = |I_b| = 4$). It follows that $\max_i |I_i| = 4$. By Lemma 2.2, $|Y(I_a, I_b)| \leq 1$ and $Y(I_i, I_j) = \emptyset$ if $\{i, j\} \neq \{a, b\}$, implying that $|Y(I_1, I_2, \dots, I_s)| = |Y(I_a, I_b)| \leq 1$.

Case a2.2 ($|I_a| = 5, |I_b| = 4$). It follows that $\max_i |I_i| = 5$. By Lemma 2.2, we have $|Y(I_a, I_b)| \leq 2$ and $|Y(I_a, I_i)| \leq 1$ for each $i \in \{1, 2, \dots, s\} \setminus \{a, b\}$. Furthermore, $Y(I_i, I_j) = \emptyset$ if $a \notin \{i, j\}$. Thus, $|Y(I_1, I_2, \dots, I_s)| \leq \delta - 1$.

Case a3 ($r = 1$). It follows that $|I_a| \geq 4$ for some $a \in \{1, 2, \dots, s\}$ and $|I_i| = 3$ for each $i \in \{1, 2, \dots, s\} \setminus \{a\}$. By (3.42), $4 \leq |I_a| \leq 6$.

Case a3.1 ($|I_a| = 4$). It follows that $\max_i |I_i| = 4$. By Lemma 2.2, $Y(I_a, I_i) = \emptyset$ for each $i \in \{1, 2, \dots, s\} \setminus \{a\}$, implying that $|Y(I_1, I_2, \dots, I_s)| = 0$.

Case a3.2 ($|I_a| = 5$). It follows that $\max_i |I_i| = 5$. By Lemma 2.2, $|Y(I_a, I_i)| \leq 1$ for each $i \in \{1, 2, \dots, s\} \setminus \{a\}$ and $Y(I_i, I_j) = \emptyset$ if $a \notin \{i, j\}$, that is, $|Y(I_1, I_2, \dots, I_s)| \leq \delta - 2$.

Case a3.3 ($|I_a| = 6$). It follows that $\max_i |I_i| = 6$. By Lemma 2.2, $|Y(I_a, I_i)| \leq 2$ for each $i \in \{1, 2, \dots, s\} \setminus \{a\}$ and $Y(I_i, I_j) = \emptyset$ if $a \notin \{i, j\}$, that is, $|Y(I_1, I_2, \dots, I_s)| \leq 2(\delta - 2)$. Claim 1 is proved.

Let $e \in Y(I_1, I_2, \dots, I_s)$ and let $e = zw$, where $z \in V(I_a^*)$ and $w \in V(I_b^*)$ for some distinct $a, b \in \{1, 2, \dots, s\}$. Put $G' = G \setminus e$. Form a graph G'' in the following way. If $d(z) \geq \delta$ and $d(w) \geq \delta$ in G' then we take $G'' = G'$. Next, suppose that $d(z) = \delta - 1$ and $d(w) \geq \delta$ in G' . Put

$$U_1 = (\{\xi_1, \xi_2, \dots, \xi_s\} \cup V(I_a^*)) \setminus \{z\}, \quad U_2 = (\{\xi_1, \xi_2, \dots, \xi_s\} \cup V(I_b^*)) \setminus \{w\}. \quad (3.45)$$

If $U_1 \subseteq N(z)$, then clearly $d(z) \geq |U_1| = \delta$ in G' , contradicting the hypothesis. Otherwise, $zv \notin E(G')$ for some $v \in U_1$ and we take $G'' = G' + \{zv\}$. Finally, if $d(z) = d(w) = \delta - 1$, then as above, $zv \notin E(G')$ and $wu \notin E(G')$ for some $v \in U_1, u \in U_2$ and we take $G'' = G' + \{zv, wu\}$. Clearly, $\delta(G'') = \delta(G)$ and $q = q(G) \geq q(G'') - 1$. This procedure may be repeated for all edges of $Y(I_1, I_2, \dots, I_s)$. The resulting graph G^* satisfies the following conditions:

$$\delta(G^*) = \delta(G), \quad q(G) \geq q(G^*) - |Y(I_1, I_2, \dots, I_s)|. \quad (3.46)$$

In fact,

$$G^* = (G \setminus Y(I_1, I_2, \dots, I_s)) + E^*, \quad (3.47)$$

where E^* consists of at most $2|Y(I_1, I_2, \dots, I_s)|$ appropriate new edges such that $G^* \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$ is disconnected. Let $H_1, H_2, \dots, H_t$ be the connected components of $G^* \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$ with $V(I_i^*) \subseteq V(H_i)$ ($i = 1, 2, \dots, s$) and $V(H_{s+1}) = \{x_1, x_2\}$. For each $i \in \{1, 2, \dots, s+1\}$, put

$$h_i = |V(H_i)|, \quad q_i = |\{xy \in E(G^*) : \{x, y\} \cap V(H_i) \neq \emptyset\}|. \quad (3.48)$$

Clearly, $h_i \geq 2$ ($i = 1, 2, \dots, s+1$). If $h_i \geq 6$ for some $i \in \{1, 2, \dots, s\}$, then

$$n \geq \sum_{i=1}^{s+1} h_i + |\{\xi_1, \xi_2, \dots, \xi_s\}| \geq 6 + 3s = 3\delta + 3, \quad (3.49)$$

contradicting (3.32). Otherwise, $2 \leq h_i \leq 5 \leq 2\delta - 1$ ($i = 1, 2, \dots, s+1$). It follows that $(h_i - 2)(2\delta - h_i - 1) \geq 0$ which is equivalent to

$$\frac{h_i(2\delta - h_i + 1)}{2} \geq 2\delta - 1 \quad (i = 1, 2, \dots, s+1). \quad (3.50)$$

Case 2.1.2.1 ($\max_i |I_i| \leq 4$). By (3.50) and Lemma 2.3, $q_i(G^*) \geq 2\delta - 1$ ($i = 1, 2, \dots, s+1$). Hence

$$q(G^*) \geq \sum_{i=1}^{s+1} q_i(G^*) \geq (s+1)(2\delta - 1) = \delta(2\delta - 1). \quad (3.51)$$

Using (3.46) and Claim 1, we have

$$q \geq q(G^*) - 3 \geq \delta(2\delta - 1) - 3 \geq \frac{3(\delta - 1)(\delta + 2)}{2}. \quad (3.52)$$

Case 2.1.2.2 ($\max_i |I_i| = 5$). Assume without loss of generality that $\max_i |I_i| = |I_1| = 5$, that is, $4 \leq h_1 \leq 5$. By (3.50) and Lemma 2.3, $q_i(G^*) \geq 2\delta - 1$ ($i = 2, \dots, s+1$) and

$$q_1(G^*) \geq \frac{h_1(2\delta - h_1 + 1)}{2} \geq 2(2\delta - 3). \quad (3.53)$$

Hence

$$q(G^*) \geq s(2\delta - 1) + 2(2\delta - 3) = 2\delta^2 + \delta - 5. \quad (3.54)$$

By (3.46) and Claim 1,

$$q \geq q(G^*) - (\delta - 1) \geq 2\delta^2 - 4 > \frac{3(\delta - 1)(\delta + 2)}{2}. \quad (3.55)$$

Case 2.1.2.3 ($\max_i |I_i| = 6$). Assume without loss of generality that $\max_i |I_i| = |I_1| = 6$, that is, $h_1 = 5$. By (3.50) and Lemma 2.3, $q_i(G^*) \geq 2\delta - 1$ ($i = 2, \dots, s+1$) and

$$q_1(G^*) \geq \frac{h_1(2\delta - h_1 + 1)}{2} = 5(\delta - 2). \quad (3.56)$$

Hence

$$q(G^*) \geq s(2\delta - 1) + 5(\delta - 2) = 2\delta^2 + 2\delta - 9. \quad (3.57)$$

By (3.46) and Claim 1,

$$q \geq q(G^*) - 2(\delta - 2) \geq 2\delta^2 - 5 > \frac{3(\delta - 1)(\delta + 2)}{2}. \quad (3.58)$$

Case 2.2 ($\bar{p} \geq 2$). According to (3.32), we can distinguish five main cases, namely, $2 \leq \bar{p} \leq \delta - 3$, $\bar{p} = \delta - 2$, $\bar{p} = \delta - 1$, $\bar{p} = \delta$, and $\bar{p} = \delta + 1$.

Case 2.2.1 ($2 \leq \bar{p} \leq \delta - 3$). It follows that $|N_C(x_i)| \geq \delta - \bar{p} \geq 3$ ($i = 1, 2$) and

$$\delta \geq 5, \quad \delta - \bar{p} \geq 3. \quad (3.59)$$

If $N_C(x_1) \neq N_C(x_2)$, then by (3.59) and Lemma 2.1, $|C| \geq 4\delta - 2\bar{p} \geq 3\delta - \bar{p} + 3$, contradicting (3.32). Let $N_C(x_1) = N_C(x_2)$. Clearly, $s \geq |N_C(x_1)| - (|V(P)| - 1) \geq \delta - \bar{p}$ and $|I_i| \geq \bar{p} + 2$ ($i = 1, 2, \dots, s$). If $s \geq \delta - \bar{p} + 1$, then

$$\begin{aligned} |C| &\geq s(\bar{p} + 2) \geq (\delta - \bar{p} + 1)(\bar{p} + 2), \\ &= (\delta - \bar{p} - 1)(\bar{p} - 1) + 3\delta - \bar{p} + 1 \geq 3\delta - \bar{p} + 3, \end{aligned} \quad (3.60)$$

again contradicting (3.32). Let $s = \delta - \bar{p}$. It means that $x_1x_2 \in E(G)$, that is, $G[V(P)]$ is hamiltonian. By symmetric arguments, $N_C(y) = N_C(x_1)$ for each $y \in V(P)$. Assume that $Y(I_1, I_2, \dots, I_s) \neq \emptyset$, that is, $Y(I_a, I_b) \neq \emptyset$ for some elementary segments I_a and I_b . By the definition, there is an intermediate path L between I_a and I_b . If $|L| \geq 2$, then by Lemma 2.2

$$|I_a| + |I_b| \geq 2\bar{p} + 2|L| + 4 \geq 2\bar{p} + 8. \quad (3.61)$$

Hence

$$\begin{aligned} |C| &= |I_a| + |I_b| + \sum_{i \in \{1, \dots, s\} \setminus \{a, b\}} |I_i| \geq 2\bar{p} + 8 + (s - 2)(\bar{p} + 2), \\ &= (\delta - \bar{p} - 2)(\bar{p} - 1) + 3\delta - \bar{p} + 2 \geq 3\delta - \bar{p} + 3, \end{aligned} \quad (3.62)$$

contradicting (3.32). Thus, $|L| = 1$, that is, $Y(I_1, I_2, \dots, I_s) \subseteq E(G)$. By Lemma 2.2,

$$|I_a| + |I_b| \geq 2\bar{p} + 2|L| + 4 = 2\bar{p} + 6, \quad (3.63)$$

which yields

$$\begin{aligned} |C| &= |I_a| + |I_b| + \sum_{i \in \{1, \dots, s\} \setminus \{a, b\}} |I_i| \geq 2\bar{p} + 6 + (s - 2)(\bar{p} + 2) \\ &= (s - 2)(\bar{p} - 2) + 4\delta - 2\bar{p} - 2 \geq 3\delta - \bar{p} - 2 + (\delta - \bar{p}). \end{aligned} \quad (3.64)$$

If $\delta - \bar{p} \geq 4$, then $|C| \geq 3\delta - \bar{p} + 2$, contradicting (3.32). Let $\delta - \bar{p} \leq 3$. Recalling (3.59), we have $\delta - \bar{p} = 3$, that is, $\bar{p} = \delta - 3$ and $s = \delta - \bar{p} = 3$. Hence, $|C| \geq s(\bar{p} + 2) = 3(\delta - 1)$. On the other hand, by (3.32) and the fact that $\bar{p} \geq 2$, we have $|C| \leq 3\delta - \bar{p} + 1 \leq 3\delta - 1$. Thus

$$3\delta - 3 \leq |C| \leq 3\delta - 1. \quad (3.65)$$

Put $G' = G \setminus Y(I_1, I_2, I_3)$. As in Case 2.1.2, form a graph G^* by adding at most $2|Y(I_1, I_2, I_3)|$ new edges in G' such that $\delta(G^*) = \delta(G)$ and $G^* \setminus \{\xi_1, \xi_2, \xi_3\}$ are disconnected. We denote $G^* = G$ immediately if $Y(I_1, I_2, I_3) = \emptyset$. Hence

$$q(G) \geq q(G^*) - |Y(I_1, I_2, I_3)|. \quad (3.66)$$

Let $H_1, H_2, \dots, H_t$ be the connected components of $G^* \setminus \{\xi_1, \xi_2, \xi_3\}$ with $V(I_i^*) \subseteq V(H_i)$ ($i = 1, 2, 3$) and $V(P) \subseteq V(H_4)$. Since $x_1 x_2 \in E(G)$ (i.e., $G[V(P)]$ is hamiltonian) and P is extreme, we have $V(H_4) = V(P)$. Using notation (3.48) for G^* , we have $h_i \geq |I_i| - 1 \geq \bar{p} + 1 = \delta - 2$ ($i = 1, 2, 3$) and $h_4 = \delta - 2$. If $h_i \geq \delta + 1$ for some $i \in \{1, 2, 3\}$, then

$$n \geq h_1 + h_2 + h_3 + h_4 + s \geq 4\delta - 2. \quad (3.67)$$

By (3.59), $\delta \geq 5$, implying that $4\delta - 2 \geq 3\delta + 3$ and $n \geq 3\delta + 3$, contradicting (3.32). Let $\delta - 2 \leq h_i \leq \delta$ ($i = 1, 2, 3, 4$). It follows that

$$\frac{h_i(2\delta - h_i + 1)}{2} \geq \frac{(\delta - 2)(\delta + 3)}{2} \quad (i = 1, 2, 3, 4). \quad (3.68)$$

By Lemma 2.3, $q_i(G^*) \geq (\delta - 2)(\delta + 3)/2$ ($i = 1, 2, 3, 4$), implying that

$$q(G^*) \geq \sum_{i=1}^4 q_i(G^*) \geq 2(\delta - 1)(\delta + 3). \quad (3.69)$$

If $|Y(I_1, I_2, I_3)| \geq 4$, then $|Y(I_a, I_b)| \geq 2$ for some distinct $a, b \in \{1, 2, 3\}$. By Lemma 2.2

$$|I_a| + |I_b| \geq 2\bar{p} + 7 = 2\delta + 1 \quad (3.70)$$

and hence $|C| \geq 3\delta$, contradicting (3.65). So, $|Y(I_1, I_2, I_3)| \leq 3$. By (3.66) and (3.69),

$$q \geq q(G^*) - 3 \geq 2(\delta - 1)(\delta + 3) - 3 \geq \frac{3(\delta - 1)(\delta + 2)}{2}. \quad (3.71)$$

Case 2.2.2 ($\bar{p} = \delta - 2$). It follows that $|N_C(x_i)| \geq \delta - \bar{p} = 2$ ($i = 1, 2$). By (3.32),

$$|C| \leq 3\delta + 1 - \bar{p} = 2\delta + 3. \quad (3.72)$$

If $N_C(x_1) \neq N_C(x_2)$, then by Lemma 2.1, $|C| \geq 4\delta - 2\bar{p} = 2\delta + 4$, contradicting (3.72). Let $N_C(x_1) = N_C(x_2)$. Further, if $s \geq 3$, then

$$|C| \geq s(\bar{p} + 2) \geq 3\delta = 2\delta + (\bar{p} + 2) \geq 2\delta + 4, \quad (3.73)$$

again contradicting (3.72). Let $s = 2$. It follows that $x_1 x_2 \in E(G)$, that is, $G[V(P)]$ is hamiltonian. By symmetric arguments, $N_C(y) = N_C(x_1) = \{\xi_1, \xi_2\}$ for each $y \in V(P)$. Clearly, $|I_i| \geq \bar{p} + 2 = \delta$ ($i = 1, 2$).

Case 2.2.2.1 ($\Upsilon(I_1, I_2) = \emptyset$). It follows that $G \setminus \{\xi_1, \xi_2\}$ is disconnected. Let $H_1, H_2, \dots, H_t$ be the connected components of $G \setminus \{\xi_1, \xi_2\}$ with $V(I_i^*) \subseteq V(H_i)$ ($i = 1, 2$) and $V(P) \subseteq V(H_3)$. Since P is extreme and $G[V(P)]$ is hamiltonian, we have $V(H_3) = V(P)$. By notation (3.48), $h_i \geq |I_i| - 1 \geq \delta - 1$ ($i = 1, 2$) and $h_3 = \delta - 1$. If $h_i \geq \delta + 3$ for some $i \in \{1, 2\}$, then

$$n \geq h_1 + h_2 + h_3 + |\{\xi_1, \xi_2\}| \geq 3\delta + 3, \quad (3.74)$$

contradicting (3.32). So, $\delta - 1 \leq h_i \leq \delta + 2$ ($i = 1, 2, 3$). By Lemma 2.3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} \geq \frac{(\delta - 1)(\delta + 2)}{2} \quad (i = 1, 2, 3). \quad (3.75)$$

Hence,

$$q \geq \sum_{i=1}^3 q_i \geq \frac{3(\delta - 1)(\delta + 2)}{2}. \quad (3.76)$$

Case 2.2.2.2 ($\Upsilon(I_1, I_2) \neq \emptyset$). By definition, there is an intermediate path L between I_1 and I_2 . If $|L| \geq 2$, then by Lemma 2.2

$$|C| = |I_1| + |I_2| \geq 2\bar{p} + 2|L| + 4 \geq 2\delta + 4, \quad (3.77)$$

contradicting (3.72). Otherwise, $\Upsilon(I_1, I_2) \subseteq E(G)$. Further, if $|\Upsilon(I_1, I_2)| \geq 3$, then by Lemma 2.2

$$|C| = |I_1| + |I_2| \geq 2\bar{p} + 8 = 2\delta + 4, \quad (3.78)$$

again contradicting (3.72). Thus $|\Upsilon(I_1, I_2)| \leq 2$.

Case 2.2.2.2.1 ($|\Upsilon(I_1, I_2)| = 1$). Put $G' = G \setminus \Upsilon(I_1, I_2)$. As in Case 2.1.2, form a graph G^* by adding at most two new edges in G' such that $\delta(G^*) = \delta(G)$, $G^* \setminus \{\xi_1, \xi_2\}$ is disconnected and $q(G) \geq q(G^*) - 1$. Let $H_1, H_2, \dots, H_t$ be the connected components of $G^* \setminus \{\xi_1, \xi_2\}$ with $V(I_i^*) \subseteq V(H_i)$ ($i = 1, 2$) and $V(P) = V(H_3)$. Using notation (3.48) for G^* , as in Case 2.2.2.1, we have $\delta - 1 \leq h_i \leq \delta + 2$ ($i = 1, 2, 3$). By Lemma 2.2, $|I_1| + |I_2| \geq 2\bar{p} + 6 = 2\delta + 2$. It means that $\max_i |I_i| \geq \delta + 1$, that is, $\max_i h_i \geq \delta$. Assume without loss of generality that $h_1 \geq \delta$. By Lemma 2.3,

$$\begin{aligned} q_1(G^*) &\geq \frac{h_1(2\delta - h_1 + 1)}{2} \geq \frac{\delta(\delta + 1)}{2}, \\ q_i(G^*) &\geq \frac{h_i(2\delta - h_i + 1)}{2} \geq \frac{(\delta - 1)(\delta + 2)}{2} \quad (i = 2, 3), \end{aligned} \quad (3.79)$$

implying that

$$q(G^*) \geq \frac{\delta(\delta+1)}{2} + (\delta-1)(\delta+2). \quad (3.80)$$

Hence

$$q \geq q(G^*) - 1 \geq \frac{\delta(\delta+1)}{2} + (\delta-1)(\delta+2) - 1 \geq \frac{3(\delta-1)(\delta+2)}{2}. \quad (3.81)$$

Case 2.2.2.2.2 ($|Y(I_1, I_2)| = 2$). By Lemma 2.2,

$$|C| = |I_1| + |I_2| \geq 2\bar{p} + 7 = 2\delta + 3. \quad (3.82)$$

Recalling (3.72), we get $|C| = 2\delta + 3$ and $V(G) = V(P \cup C)$. Put $G' = G \setminus Y(I_1, I_2)$. As in Case 2.1.2, form a graph G^* by adding at most four new edges in G' such that $\delta(G^*) = \delta(G)$, $G^* \setminus \{\xi_1, \xi_2\}$ is disconnected and $q(G) \geq q(G^*) - 2$. Let H_1, H_2 , and H_3 be the connected components of $G^* \setminus \{\xi_1, \xi_2\}$ with $V(H_i) = V(I_i^*)$ ($i = 1, 2$) and $V(H_3) = V(P)$. Using notation (3.48) for G^* , we have as in Case 2.2.2.1, $\delta - 1 \leq h_i \leq \delta + 2$ ($i = 1, 2, 3$). Since $|I_i| \geq \delta$ ($i = 1, 2$) and $|C| = |I_1| + |I_2| = 2\delta + 3$, we can assume without loss of generality that either $|I_1| = \delta + 2$, $|I_2| = \delta + 1$ or $|I_1| = \delta + 3$, $|I_2| = \delta$.

Case 2.2.2.2.2.1 ($|I_1| = \delta + 2$, $|I_2| = \delta + 1$). It follows that $h_1 = \delta + 1$, $h_2 = \delta$ and $h_3 = \delta - 1$. By Lemma 2.3,

$$\begin{aligned} q_i(G^*) &\geq \frac{h_i(2\delta - h_i + 1)}{2} = \frac{\delta(\delta+1)}{2} \quad (i = 1, 2), \\ q_3(G^*) &\geq \frac{h_3(2\delta - h_3 + 1)}{2} = \frac{(\delta-1)(\delta+2)}{2}. \end{aligned} \quad (3.83)$$

Hence

$$q \geq \sum_{i=1}^3 q_i(G^*) - 2 \geq \delta(\delta+1) - 2 + \frac{(\delta-1)(\delta+2)}{2} = \frac{3(\delta-1)(\delta+2)}{2}. \quad (3.84)$$

Case 2.2.2.2.2.2 ($|I_1| = \delta + 3$, $|I_2| = \delta$). Let $Y(I_1, I_2) = \{e_1, e_2\}$, where

$$e_1 = y_1 z_1, \quad e_2 = y_2 z_2, \quad \{y_1, y_2\} \subseteq V(I_1^*), \quad \{z_1, z_2\} \subseteq V(I_2^*). \quad (3.85)$$

If $y_1 \neq y_2$ and $z_1 \neq z_2$, then as in proof of Lemma 2.2 (Case 2.1),

$$|C| = |I_1| + |I_2| \geq 2\bar{p} + 8 = 2(\delta-2) + 8 = 2\delta + 4, \quad (3.86)$$

contradicting (3.72). Let either $y_1 \neq y_2$ and $z_1 = z_2$ or $y_1 = y_2$ and $z_1 \neq z_2$.

Case 2.2.2.2.2.1 ($y_1 \neq y_2$ and $z_1 = z_2$). Assume without loss of generality that y_1, y_2 occur on I_1 in this order. If $y_2 = y_1^+$, then

$$|C| \geq \left| \xi_1 \vec{C} y_1 z_1 y_2 \vec{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1 \right| = 2\delta + 4, \quad (3.87)$$

contradicting (3.72). Let $y_2 \neq y_1^+$, that is, $|y_1 \vec{C} y_2| \geq 2$. Put

$$\begin{aligned} C' &= \xi_1 \vec{C} y_2 z_1 \vec{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1, \\ C'' &= \xi_1 \overleftarrow{C} z_1 y_1 \vec{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1. \end{aligned} \quad (3.88)$$

Clearly,

$$\begin{aligned} |C| \geq |C'| &= \left| \xi_1 \vec{C} y_1 \right| + \left| y_1 \vec{C} y_2 \right| + |\{e_2\}| + \left| \xi_2 \vec{C} z_1 \right| + (\bar{p} + 2), \\ |C| \geq |C''| &= \left| \xi_1 \overleftarrow{C} z_1 \right| + |\{e_1\}| + \left| y_1 \vec{C} y_2 \right| + \left| y_2 \vec{C} \xi_2 \right| + (\bar{p} + 2). \end{aligned} \quad (3.89)$$

By summing and observing that

$$\left| \xi_1 \vec{C} y_1 \right| + \left| y_1 \vec{C} y_2 \right| + \left| y_2 \vec{C} \xi_2 \right| + \left| \xi_2 \vec{C} z_1 \right| + \left| z_1 \vec{C} \xi_1 \right| = |C|, \quad (3.90)$$

we get

$$2|C| \geq |C| + \left| y_1 \vec{C} y_2 \right| + 2(\bar{p} + 2) + 2 \geq |C| + 2\delta + 4. \quad (3.91)$$

Hence $|C| \geq 2\delta + 4$, again contradicting (3.72).

Case 2.2.2.2.2.2 ($y_1 = y_2$ and $z_1 \neq z_2$). Assume without loss of generality that z_2, z_1 occur on I_2 in this order.

Case 2.2.2.2.2.2.1 ($\delta \geq 6$). If $|\xi_1 \vec{C} y_1| \geq \delta - 1$ and $|y_1 \vec{C} \xi_2| \geq \delta - 1$, then $|I_1| \geq 2\delta - 2 \geq \delta + 4$, contradicting the hypothesis. Thus, we can assume without loss of generality that $|\xi_1 \vec{C} y_1| \leq \delta - 2$. If $y_1^- = \xi_1$, then

$$|C| \geq \left| \xi_1 \overleftarrow{C} z_2 y_1 \vec{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1 \right| \geq 2\delta + 5, \quad (3.92)$$

contradicting (3.72). Let $y_1^- \neq \xi_1$, that is, $y_1^- \in V(I_1^*)$. Since $\Upsilon(I_1, I_2) = \{y_1 z_1, y_1 z_2\}$, we have $N(y_1^-) \subset V(I_1)$. If $N(y_1^-) \cap V(y_1^+ \vec{C} \xi_2^-) = \emptyset$, then $|N(y_1^-)| \leq \delta - 1$, a contradiction. Otherwise, $y_1^- w \in E(G)$ for some $w \in V(y_1^+ \vec{C} \xi_2^-)$. Put

$$\begin{aligned} R &= \xi_1 \vec{C} y_1^- w \overleftarrow{C} y_1, \\ C' &= \xi_1 \vec{R} y_1 z_1 \overleftarrow{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1, \\ C'' &= \xi_1 \overleftarrow{C} z_2 y_1 \vec{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1. \end{aligned} \quad (3.93)$$

Clearly,

$$\begin{aligned} |C| &\geq |C'| = |R| + |\{y_1 z_1\}| + |z_1 \overleftarrow{C} \xi_2| + (\bar{p} + 2), \\ |C| &\geq |C''| = |\xi_1 \overleftarrow{C} z_1| + |z_1 \overleftarrow{C} z_2| + |\{y_1 z_2\}| + |y_1 \vec{C} \xi_2| + (\bar{p} + 2). \end{aligned} \quad (3.94)$$

By summing and observing that $|R| \geq |\xi_1 \vec{C} y_1| + 1$, we get

$$2|C| \geq (|\xi_1 \vec{C} y_1| + |y_1 \vec{C} \xi_2| + |\xi_2 \vec{C} z_1| + |z_1 \vec{C} \xi_1|) + 2(\bar{p} + 2) + 4 = |C| + 2\delta + 4. \quad (3.95)$$

Hence $|C| \geq 2\delta + 4$, contradicting (3.72).

Case 2.2.2.2.2.2.2.2 ($\delta = 5$). It follows that

$$|I_1| = \delta + 3 = 8, \quad |I_2| = \delta = 5, \quad |C| = 2\delta + 3 = 13. \quad (3.96)$$

If either $|\xi_1 \vec{C} y_1| \leq \delta - 2 = 3$ or $|y_1 \vec{C} \xi_2| \leq \delta - 2 = 3$, then we can argue as in Case 2.2.2.2.2.2.1. Otherwise, $|\xi_1 \vec{C} y_1| = |y_1 \vec{C} \xi_2| = 4$. If $|z_1 \overleftarrow{C} \xi_2| \geq 4$, then

$$|\xi_1 \vec{C} y_1 z_1 \overleftarrow{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1| \geq 14 > |C|, \quad (3.97)$$

a contradiction. Let $|z_1 \overleftarrow{C} \xi_2| \leq 3$. Similarly, $|\xi_1 \overleftarrow{C} z_2| \leq 3$, implying that $I_2 = \xi_2 \xi_2^+ z_2 z_1^+ \xi_1$. If $z_1^+ z_2 \in E(G)$, then

$$|\xi_1 \vec{C} y_1 z_1 z_1^+ z_2 \overleftarrow{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1| = 14 > |C|, \quad (3.98)$$

a contradiction. So, $N(z_1^+) \subseteq \{\xi_1, \xi_2, z_1, \xi_2^+\}$, again a contradiction, since $|N(z_1^+)| \geq \delta = 5$.

Case 2.2.2.2.2.2.2.3 ($\delta = 4$). It follows that

$$|I_1| = \delta + 3 = 7, \quad |I_2| = \delta = 4, \quad |C| = 2\delta + 3 = 11. \quad (3.99)$$

Since $|I_1| = 7$, we have either $|\xi_1 \vec{C} y_1| \geq 4$ or $|y_1 \vec{C} \xi_2| \geq 4$, say $|\xi_1 \vec{C} y_1| \geq 4$. Put

$$C' = \xi_1 \vec{C} y_1 z_1 \overleftarrow{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1. \quad (3.100)$$

If $|\xi_1 \vec{C} y_1| \geq 5$, then $|C'| \geq 12 > |C|$, a contradiction. This means that $|\xi_1 \vec{C} y_1| = 4$. If $|z_1 \vec{C} \xi_1| = 1$ then $|C'| \geq 12 > |C|$, a contradiction. Let $|z_1 \vec{C} \xi_1| \geq 2$. Since $|I_2| = 4$, we have $|z_1 \vec{C} \xi_1| = 2$, that is, $I_2 = \xi_2 z_2 z_1 z_1^+ \xi_1$. Further, if $z_1^+ z_2 \in E(G)$, then

$$|\xi_1 \vec{C} y_1 z_1 z_1^+ z_2 \xi_2 x_2 \overleftarrow{P} x_1 \xi_1| = 12 > |C|, \quad (3.101)$$

a contradiction. Let $z_1^+ z_2 \notin E(G)$. Since $|Y(I_1, I_2)| = 2$, we have $N(z_1^+) \subseteq \{\xi_1, \xi_2, z_1\}$, contradicting the fact that $|N(z_1^+)| \geq \delta = 4$.

Case 2.2.3 ($\bar{p} = \delta - 1$). By (3.32),

$$|C| \leq 3\delta + 1 - \bar{p} = 2\delta + 2. \quad (3.102)$$

It follows that $|N_C(x_i)| \geq \delta - \bar{p} = 1$ ($i = 1, 2$).

Case 2.2.3.1 ($|N_C(x_i)| \geq 2$ ($i = 1, 2$)). If $N_C(x_1) \neq N_C(x_2)$, then by Lemma 2.1, $|C| \geq 2\bar{p} + 8 = 2\delta + 6$, contradicting (3.102). Let $N_C(x_1) = N_C(x_2)$. If $s \geq 3$, then

$$|C| \geq s(\bar{p} + 2) \geq 3(\delta + 1) > 2\delta + 2, \quad (3.103)$$

contradicting (3.102). Let $s = 2$. It follows that $|C| \geq s(\bar{p} + 2) \geq 2(\delta + 1)$. Recalling (3.102), we get

$$|C| = 2\delta + 2, \quad |I_1| = |I_2| = \delta + 1, \quad V(G) = V(C \cup P). \quad (3.104)$$

Assume that $yz \in E(G)$ for some $y \in V(P)$ and $z \in V(C) \setminus \{\xi_1, \xi_2\}$. Besides, we can assume without loss of generality that $z \in V(I_1^*)$. Since C is extreme, we have

$$|\xi_1 \vec{C} z| \geq |x_1 \vec{P} y| + 2, \quad |z \vec{C} \xi_2| \geq |y \vec{P} x_2| + 2, \quad (3.105)$$

implying that

$$|I_1| = |\xi_1 \vec{C} z| + |z \vec{C} \xi_2| \geq |x_1 \vec{P} y| + |y \vec{P} x_2| + 4 = \bar{p} + 4 = \delta + 3, \quad (3.106)$$

a contradiction. So, $N_C(y) \subseteq \{\xi_1, \xi_2\}$ for each $y \in V(P)$. On the other hand, by Lemma 2.2, $Y(I_1, I_2) = \emptyset$ and hence $G \setminus \{\xi_1, \xi_2\}$ is disconnected. Let H_1, H_2 , and H_3 be the connected

components of $G \setminus \{\xi_1, \xi_2\}$ with $V(H_i) = V(I_i^*)$ ($i = 1, 2$) and $V(H_3) = V(P)$. Using notation (3.48), we have $h_i = \delta$ ($i = 1, 2, 3$). By Lemma 2.3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} = \frac{\delta(\delta + 1)}{2} \quad (i = 1, 2, 3), \quad (3.107)$$

implying that

$$q \geq \sum_{i=1}^3 q_i \geq \frac{3(\delta^2 + \delta)}{2} > \frac{3(\delta - 1)(\delta + 2)}{2}. \quad (3.108)$$

Case 2.2.3.2 (either $|N_C(x_1)| = 1$ or $|N_C(x_2)| = 1$). Assume without loss of generality that $|N_C(x_1)| = 1$. Put $N_C(x_1) = \{y_1\}$.

Case 2.2.3.2.1 ($N_C(x_2) \neq N_C(x_1)$). It follows that $x_2 y_2 \in E(G)$ for some $y_2 \in V(C) \setminus \{y_1\}$ and we can argue as in Case 2.2.3.1.

Case 2.2.3.2.2 ($N_C(x_2) = N_C(x_1) = \{y_1\}$). It follows that

$$N(x_i) = (V(P) \setminus \{x_i\}) \cup \{y_1\} \quad (i = 1, 2). \quad (3.109)$$

Since $\kappa \geq 2$, there is an edge zw such that $z \in V(P)$ and $w \in V(C) \setminus \{y_1\}$. Since $N_C(x_1) = N_C(x_2) = \{y_1\}$, we have $z \notin \{x_1, x_2\}$. By (3.109), $x_2 z^- \in E(G)$. Then replacing P with $x_1 \vec{P} z^- x_2 \overleftarrow{P} z$, we can argue as in Case 2.2.3.1.

Case 2.2.4 ($\bar{p} = \delta$). By (3.32), $|C| \leq 3\delta + 1 - \bar{p} = 2\delta + 1$. Let $Q = \xi \vec{Q} \eta$ be a longest path in G with $V(Q) \cap V(C) = \{\xi, \eta\}$. If $|Q| \geq \delta + 1$, then by (3.34), $|C| \geq 2|Q| \geq 2\delta + 2$, a contradiction. Let

$$|Q| \leq \delta. \quad (3.110)$$

Case 2.2.4.1 ($x_1 x_2 \notin E(G)$). It follows that $|N_C(x_i)| \geq 1$ ($i = 1, 2$). If $|N_C(x_i)| \geq 2$ for some $i \in \{1, 2\}$, then clearly $|Q| \geq \bar{p} + 2 = \delta + 2$, contradicting (3.110). Let $|N_C(x_1)| = |N_C(x_2)| = 1$. Further, if $N_C(x_1) \neq N_C(x_2)$, then again $|Q| \geq \delta + 2$, contradicting (3.110). Let $N_C(x_1) = N_C(x_2) = \{z_1\}$ for some $z_1 \in V(C)$. Since $\kappa \geq 2$, there is a path $L = y z_2$ connecting P and C such that $y \in V(P)$ and $z_2 \in V(C) \setminus \{z_1\}$. Clearly, $y \notin \{x_1, x_2\}$. If $x_2 y^- \in E(G)$, then

$$|Q| \geq |z_1 x_1 \vec{P} y^- x_2 \overleftarrow{P} y z_2| = \delta + 2, \quad (3.111)$$

contradicting (3.110). Let $x_2 y^- \notin E(G)$. Further, if $y^- \neq x_1$, then recalling that $x_2 x_1 \notin E(G)$ we have $|N_C(x_2)| \geq 2$, a contradiction. Otherwise, $y^- = x_1$ and $|Q| \geq |z_1 x_2 \overleftarrow{P} y z_2| = \delta + 1$, contradicting (3.110).

Case 2.2.4.2 ($x_1x_2 \in E(G)$). Put $C' = x_1\vec{P}x_2x_1$. Since $\kappa \geq 2$, there are two disjoint paths L_1, L_2 connecting C' and C . Further, since P is extreme, we have $|L_1| = |L_2| = 1$. Let $L_1 = y_1z_1$ and $L_2 = y_2z_2$, where $y_1, y_2 \in V(C')$ and $z_1, z_2 \in V(C)$. Since C' is a hamiltonian cycle in $G[V(P)]$, we can assume that P is chosen such that $x_1 = y_1$. If $|x_1\vec{P}y_2| \leq 2$, then $|Q| \geq |z_1x_1x_2\overleftarrow{P}y_2z_2| \geq \delta + 1$, contradicting (3.110). Let $|x_1\vec{P}y_2| \geq 3$. If $x_2v \in E(G)$ for some $v \in \{y_2^{-1}, y_2^{-2}\}$, then

$$|Q| \geq |z_1x_1\vec{P}v\overleftarrow{P}y_2z_2| \geq \delta + 1, \quad (3.112)$$

contradicting (3.110). Otherwise, $|N_C(x_2)| \geq 2$, implying that $x_2z_3 \in E(G)$ for some $z_3 \in V(C) \setminus \{z_1\}$. Then

$$|Q| \geq |z_1x_1\vec{P}x_2z_3| \geq \delta + 2, \quad (3.113)$$

again contradicting (3.110).

Case 2.2.5 ($\bar{p} = \delta + 1$). By (3.32), $|C| \leq 3\delta + 1 - \bar{p} = 2\delta$. On the other hand, by Theorem D, $|C| \geq 2\delta$, implying that $|C| = 2\delta$ and $V(G) = V(C \cup P)$. Let $Q = \xi\vec{Q}\eta$ be a longest path in G with $V(Q) \cap V(C) = \{\xi, \eta\}$. If $|Q| \geq \delta + 1$, then by (3.35), $|C| \geq 2|Q| \geq 2\delta + 2$, a contradiction. Let

$$|Q| \leq \delta. \quad (3.114)$$

Case 2.2.5.1 ($x_1x_2 \in E(G)$). Put $C' = x_1\vec{P}x_2x_1$. Since $\kappa \geq 2$, there are two disjoint edges z_1w_1 and z_2w_2 connecting C' and C such that $z_1, z_2 \in V(C')$ and $w_1, w_2 \in V(C)$. Since C' is a hamiltonian cycle in $G[V(P)]$, we can assume without loss of generality that P is chosen such that $z_1 = x_1$. If $|x_1\vec{P}z_2| \leq 3$, then $|Q| \geq |w_1x_1x_2\overleftarrow{P}z_2w_2| \geq \delta + 1$, contradicting (3.114). Let $|x_1\vec{P}z_2| \geq 4$. Further, if $x_2v \in E(G)$ for some $v \in \{z_2^{-1}, z_2^{-2}, z_2^{-3}\}$, then

$$|Q| \geq |w_1x_1\vec{P}v\overleftarrow{P}z_2w_2| \geq \delta + 1, \quad (3.115)$$

contradicting (3.114). Now let $x_2v \notin E(G)$ for each $v \in \{z_2^{-1}, z_2^{-2}, z_2^{-3}\}$. It follows that $|N_C(x_2)| \geq 2$, that is, $x_2w_3 \in E(G)$ for some $w_3 \in V(C) \setminus \{w_1\}$. But then $|Q| \geq |w_1x_1\vec{P}x_2w_3| = \delta + 3$, contradicting (3.114).

Case 2.2.5.2 ($x_1x_2 \notin E(G)$). If $d_P(x_1) + d_P(x_2) \geq |V(P)| = \bar{p} + 1 = \delta + 2$ then by Theorem C, $G[V(P)]$ is hamiltonian and we can argue as in Case 2.2.5.1. Otherwise, $d_P(x_1) + d_P(x_2) \leq \delta + 1$, implying that

$$d_C(x_1) + d_C(x_2) \geq \delta - 1 \geq 2. \quad (3.116)$$

Assume without loss of generality that $d_C(x_1) \geq d_C(x_2)$.

Case 2.2.5.2.1 ($d_C(x_2) = 0$). It follows that $N(x_2) = V(P) \setminus \{x_1, x_2\}$. By (3.116), $d_C(x_1) \geq 2$. Put $C' = x_1^+ \vec{P} x_2 x_1^+$. Since $\kappa \geq 2$, there is a path $L = z \vec{L} w$ connecting C' and C such that $z \in V(C') \setminus \{x_1^+\}$ and $w \in V(C)$. If $x_1 \in V(L)$, that is, $x_1 z \in E(G)$, then $x_1 \vec{P} z^- x_2 \overleftarrow{P} z x_1$ is a hamiltonian cycle in $G[V(P)]$ and we can argue as in Case 2.2.5.1. Let $x_1 \notin V(L)$. Since $V(G) = V(C \cup P)$, we have $L = zw$. Further, since $d_C(x_1) \geq 2$, we have $x_1 w_1 \in E(G)$ for some $w_1 \in V(C) \setminus \{w\}$. Hence,

$$|Q| \geq \left| w_1 x_1 \vec{P} z^- x_2 \overleftarrow{P} z w \right| = \delta + 3, \quad (3.117)$$

contradicting (3.114).

Case 2.2.5.2.2 ($d_C(x_2) = 1$). Let $N_C(x_2) = \{w_1\}$. By (3.116), $d_C(x_1) \geq 1$. If either $d_C(x_1) \geq 2$ or $N_C(x_1) \neq N_C(x_2)$, then $x_1 w \in E(G)$ for some $w \in V(C) \setminus \{w_1\}$ and therefore

$$|Q| \geq \left| w x_1 \vec{P} x_2 w_1 \right| = \delta + 3, \quad (3.118)$$

contradicting (3.114). Otherwise, $N_C(x_1) = N_C(x_2) = \{w_1\}$. Since $\kappa \geq 2$, there is an edge zw such that $z \in V(P)$ and $w \in V(C) \setminus \{w_1\}$. Clearly, $z \notin \{x_1, x_2\}$. Further, we can argue as in Case 2.2.5.1.

Case 2.2.5.2.3 ($d_C(x_2) \geq 2$). Since $d_C(x_1) \geq d_C(x_2)$, we have $d_C(x_1) \geq 2$. Hence $|Q| \geq \bar{p} + 2 = \delta + 3$, contradicting (3.114). $\square$

Proof of Theorem 1.1. Let G be a graph satisfying the hypothesis of Theorem 1.1, which is equivalent to

$$q \leq \delta^2 + \delta - 1. \quad (3.119)$$

Since

$$q = \frac{1}{2} \sum_{u \in V(G)} d(u) \geq \frac{\delta n}{2}, \quad (3.120)$$

we have $\delta n/2 \leq \delta^2 + \delta - 1$ which is equivalent to

$$\delta \geq \frac{n-1}{2} - \frac{1}{2} + \frac{1}{\delta}. \quad (3.121)$$

If n is even, that is, $n = 2t$ for some integer t , then

$$\delta \geq \frac{2t-1}{2} - \frac{1}{2} + \frac{1}{\delta} = t - 1 + \frac{1}{\delta}, \quad (3.122)$$

implying that $\delta \geq t = n/2$. By Theorem A, G is hamiltonian. Let n is odd, that is, $n = 2t + 1$ for some integer t . Then $\delta \geq t - 1/2 + 1/\delta$ implying that $\delta \geq t \geq (n - 1)/2$. Recalling that G is hamiltonian when $\delta > (n - 1)/2$, we can assume that $\delta = (n - 1)/2$. By Theorem C, either G is hamiltonian or contains at least $\delta^2 + \delta$ edges, contradicting (3.119). Theorem 1.1 is proved. $\square$

Proof of Theorem 1.2. Let G be a 2-connected graph. The hypothesis of Theorem 1.2 is equivalent to

$$q \leq \begin{cases} 8 & \text{when } \delta = 2, \\ \frac{3(\delta - 1)(\delta + 2) - 1}{2} & \text{when } \delta \geq 3. \end{cases} \quad (3.123)$$

Case 1 ($\delta = 2$ and $q \leq 8$). Let C be a longest cycle in G and $P = x_1 \vec{P} x_2$ a longest path in $G \setminus C$ of length $\bar{p}$. If $\bar{p} = 0$, then C is a dominating cycle and we are done. Let $\bar{p} \geq 1$. Since $\kappa \geq 2$, there is a path $Q = \xi \vec{Q} \eta$ such that $V(Q) \cap V(C) = \{\xi, \eta\}$ and $|Q| \geq 3$. Further, since C is extreme, we have $|C| = |\xi \vec{C} \eta| + |\eta \vec{C} \xi| \geq 2|Q| \geq 6$ and therefore, $q \geq |C| + |Q| \geq 9$, contradicting the hypothesis.

Case 2 ($\delta \geq 3$ and $q \leq (3(\delta - 1)(\delta + 2) - 1)/2$). Since

$$q = \frac{1}{2} \sum_{u \in V(G)} d(u) \geq \frac{\delta n}{2}, \quad (3.124)$$

we have $\delta n/2 \leq (3(\delta - 1)(\delta + 2) - 1)/2$, which is equivalent to

$$\delta \geq \frac{n - 2}{3} - \frac{1}{3} + \frac{7}{3\delta}. \quad (3.125)$$

If $n = 3t$ for some integer t , then

$$\delta \geq \frac{3t - 2}{3} - \frac{1}{3} + \frac{7}{3\delta} = t - 1 + \frac{7}{3\delta}, \quad (3.126)$$

implying that $\delta \geq t = n/3 > (n - 2)/3$. Next, if $n = 3t + 1$ for some integer t , then

$$\delta \geq \frac{3t - 1}{3} - \frac{1}{3} + \frac{7}{3\delta} = t - \frac{2}{3} + \frac{7}{3\delta}, \quad (3.127)$$

implying that $\delta \geq t = (n - 1)/3 > (n - 2)/3$. Finally, if $n = 3t + 2$ for some integer t , then

$$\delta \geq \frac{3t}{3} - \frac{1}{3} + \frac{7}{3\delta} = t - \frac{1}{3} + \frac{7}{3\delta}, \quad (3.128)$$

implying that $\delta \geq t = (n - 2)/3$. So, $\delta \geq (n - 2)/3$, in any case. By Lemma 2.4, each longest cycle in G is a dominating cycle. $\square$

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