Research Article

Generalized Derivations in Semiprime Gamma Rings

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Let M be a 2-torsion-free semiprime Γ -ring satisfying the condition $aab\beta c = a\beta bac$ for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$, and let $D : M \rightarrow M$ be an additive mapping such that $D(x\alpha x) = D(x)\alpha x + x\alpha d(x)$ for all $x \in M$, $\alpha \in \Gamma$ and for some derivation d of M . We prove that D is a generalized derivation.

1. Introduction

Hvala [1] first introduced the generalized derivations in rings and obtained some remarkable results in classical rings. Daif and Tammam El-Sayiad [2] studied the generalized derivations in semiprime rings. The authors consider an additive mapping $G : R \rightarrow R$ of a ring R with the property $G(x^2) = G(x)x + xD(x)$ for some derivation D of R . They prove that G is a Jordan generalized derivation.

Aydin [3] studied generalized derivations of prime rings. The author proved that if I is an ideal of a noncommutative prime ring R , a is a fixed element of R and F is an generalized derivation on R associated with a derivation d then the condition $F([x, a]) = 0$ or $[F(x), a] = 0$ for all $x \in I$ implies $d(x) = \lambda[x, a]$.

Çeven and Öztürk [4] have dealt with Jordan generalized derivations in Γ -rings and they proved that every Jordan generalized derivation on some Γ -rings is a generalized derivation.

Generalized derivations of semiprime rings have been treated by Ali and Chaudhry [5]. The authors proved that if F is a commuting generalized derivation of a semiprime ring

R associated with a derivation d then $d(x)[y, z] = 0$ for all $x, y, z \in R$ and d is central. They characterized a decomposition of R relative to the generalized derivations.

Atteya [6] proved that if U is nonzero ideal of a semiprime ring R and R admits a generalized derivation D such that $D(xy) - xy \in Z(R)$ then R contains a nonzero central ideal.

Rehman [7, 8] studied the commutativity of a ring R by means of generalized derivations acting as homomorphisms and antihomomorphisms.

In this paper, we prove the following results

Let M be a 2-torsion-free semiprime Γ -ring satisfying the following assumption:

$$aab\beta c = a\beta bac \quad \forall a, b, c \in M, \alpha, \beta \in \Gamma, \quad (*)$$

and $D : M \rightarrow M$ be an additive mapping. If there exists a derivation d of M such that $D(xax) = D(x)ax + xad(x)$ for all $x \in M, \alpha \in \Gamma$, then D is a Jordan generalized derivation.

2. Preliminaries

Let M and Γ be additive abelian groups. M is called a Γ -ring if there exists a mapping $M \times M \times M \rightarrow M$ such that for all $a, b, c \in M, \alpha, \beta \in \Gamma$ the following conditions are satisfied:

$$(i) \quad a\beta b \in M,$$

$$(ii) \quad (a+b)\alpha c = a\alpha c + b\alpha c, a(\alpha+\beta)b = a\alpha b + a\beta b, a\alpha(b+c) = a\alpha b + a\alpha c, (aab)\beta c = a\alpha(b\beta c).$$

For any $a, b \in M$ and for $\alpha, \beta \in \Gamma$ the expressions $aab - b\alpha a$ is denoted by $[a, b]_\alpha$ and $\alpha a\beta - \beta a\alpha$ are denoted by $[\alpha, \beta]_a$. Then one has the following identities:

$$\begin{aligned} [a\beta b, c]_\alpha &= a\beta[b, c]_\alpha + [a, c]_\alpha\beta b + a[\beta, \alpha]_c b, \\ [a, b\beta c]_\alpha &= b\beta[a, c]_\alpha + [a, b]_\alpha\beta c + b[\beta, \alpha]_a c, \end{aligned} \quad (2.1)$$

for all $a, b, c \in M$ and for all $\alpha, \beta \in \Gamma$. Using the assumption $(*)$ the above identities reduce to

$$\begin{aligned} [a\beta b, c]_\alpha &= a\beta[b, c]_\alpha + [a, c]_\alpha\beta b, \\ [a, b\beta c]_\alpha &= b\beta[a, c]_\alpha + [a, b]_\alpha\beta c, \end{aligned} \quad (2.2)$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

Further M stands for a prime Γ -ring with center $Z(M)$. The ring M is n -torsion-free if $nx = 0, x \in M$ implies $x = 0$, where n is a positive integer, M is prime if $a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$, and it is semiprime if $a\Gamma M\Gamma a = 0$ implies $a = 0$. An additive mapping $T : M \rightarrow M$ is called a left (right) centralizer if $T(x\alpha y) = T(x)\alpha y$ ($T(x\alpha y) = x\alpha T(y)$) for $x, y \in M, \alpha \in \Gamma$ and it is called a Jordan left (right) centralizer if

$$T(x\alpha x) = T(x)\alpha x \quad (T(x\alpha x) = x\alpha T(x)) \quad \forall x \in M, \alpha \in \Gamma. \quad (2.3)$$

A mapping $\theta : M \times M \rightarrow M$ is called biadditive if it is additive in both arguments. An additive mapping $D : M \rightarrow M$ is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ for all

$x, y \in M, \alpha \in \Gamma$ and it is called a Jordan derivation if $D(x\alpha x) = D(x)\alpha x + x\alpha D(x)$ for all $x \in M, \alpha \in \Gamma$. A derivation D is inner if there exists $a \in M$, such that $D(x) = a\alpha x - x\alpha a$ holds for all $x \in M, \alpha \in \Gamma$. Every derivation is a Jordan derivation. The converse is in general not true. An additive mapping $D : M \rightarrow M$ is said to be a generalized derivation if there exists a derivation $d : M \rightarrow M$ such that $D(x\alpha y) = D(x)\alpha y + x\alpha d(y)$ for all $x, y \in M, \alpha \in \Gamma$. The maps of the form $x \rightarrow a\alpha x + x\alpha b$ where a, b are fixed elements in M and for all $\alpha \in \Gamma$ called the generalized inner derivation. An additive mapping $D : M \rightarrow M$ is said to be a Jordan generalized derivation if there exists a derivation $d : M \rightarrow M$ such that $D(x\alpha x) = D(x)\alpha x + x\alpha d(x)$ for all $x \in M, \alpha \in \Gamma$. Hence the concept of a generalized derivation covers both the concepts of a derivation and a left centralizers and the concept of a Jordan generalized derivation covers both the concepts of a Jordan derivation and a left Jordan centralizers. An example of a generalized derivation and a Jordan generalized derivation is given in [4].

3. Main Results

We start from the following subsidiary results.

Lemma 3.1. *Let M be a semiprime Γ -ring. If $a, b \in M$ are such that $a\alpha x\beta b = 0$ for all $x \in M, \alpha, \beta \in \Gamma$, then $a\alpha b = b\alpha a = 0$.*

Proof. Let $x \in M$. Then

$$\begin{aligned} a\alpha b\beta x\gamma a\alpha b &= a\alpha(b\beta x\gamma a)\alpha b = 0, \\ b\alpha a\beta x\gamma b\alpha a &= b\alpha(a\beta x\gamma b)\alpha a = 0. \end{aligned} \quad (3.1)$$

By semiprimeness of M with respect to $\beta, \gamma \in \Gamma$, it follows that $a\alpha b = b\alpha a = 0$. $\square$

Lemma 3.2. *Let M be a semiprime Γ -ring and $\theta, \varphi : M \times M \rightarrow M$ biadditive mappings. If $\theta(x, y)\alpha\omega\beta\varphi(x, y) = 0$ for all $x, y, \omega \in M$, then $\theta(x, y)\alpha\omega\beta\varphi(u, v) = 0$ for all $x, y, u, v, \omega \in M, \alpha, \beta \in \Gamma$.*

Proof. First we replace x with $x + u$ in the relation $\theta(x, y)\alpha\omega\beta\varphi(x, y) = 0$, and use the biadditivity of the θ and φ . Then we have

$$\theta(x + u, y)\alpha\omega\beta\varphi(x + u, y) = 0 \implies \theta(x, y)\alpha\omega\beta\varphi(u, y) = -\theta(u, y)\alpha\omega\beta\varphi(x, y). \quad (3.2)$$

Then

$$\begin{aligned} &(\theta(x, y)\alpha\omega\beta\varphi(u, y))\gamma z\delta(\theta(x, y)\alpha\omega\beta\varphi(u, y)) \\ &= -(\theta(u, y)\alpha\omega\beta\varphi(x, y))\gamma z\delta(\theta(u, y)\alpha\omega\beta\varphi(u, y)) = 0. \end{aligned} \quad (3.3)$$

Hence $\theta(x, y)\alpha\omega\beta\varphi(u, y) = 0$ by semiprimeness of M with respect to $\gamma, \delta \in \Gamma$.

Now we replace y by $y + v$ and obtain the assertion of the lemma with the similar observation as above. $\square$

Lemma 3.3. Let M be a semiprime Γ -ring satisfying the assumption $(*)$ and a be a fixed element of M . If $a\beta[x, y]_\alpha = 0$ for all $x, y \in M, \alpha, \beta \in \Gamma$, then $a \in Z(M)$.

Proof. First we calculate the following expressions using the assumption $(*)$,

$$\begin{aligned} [z, a]_\alpha \beta x \delta [z, a]_\alpha &= z \alpha a \beta x \delta [z, a]_\alpha - a \alpha z \beta x \delta [z, a]_\alpha \\ &= z \alpha a \beta [z, x \alpha a]_\delta - z \alpha a \beta [z, x]_\delta \alpha a - a \alpha [z, z \delta x \alpha a]_\beta + a \alpha [z, z \delta x]_\beta \alpha a. \end{aligned} \quad (3.4)$$

Since $a\beta[x, y]_\alpha = 0$ for all $x, y \in M, \alpha, \beta \in \Gamma$, we get $[z, a]_\alpha \beta x \delta [z, a]_\alpha = 0$. By the semiprimeness of M we get $[z, a]_\alpha = 0$ for all $\alpha \in \Gamma$. Hence $a \in Z(M)$. $\square$

Lemma 3.4. Let M be a Γ -ring satisfying the condition $(*)$ and $D : M \rightarrow M$ be a Jordan generalized derivation with the associated derivation d . Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$(i) \ D(x\alpha y + y\alpha x) = D(x)\alpha y + D(y)\alpha x + xad(y) + yad(x).$$

In particular, if M is 2-torsion-free, then

$$(ii) \ D(x\alpha y\beta x) = D(x)\alpha y\beta x + x\beta d(y\beta x),$$

$$(iii) \ D(x\alpha y\beta z + z\beta y\alpha x) = D(x)\alpha y\beta z + D(z)\beta y\alpha x + xad(y\beta z) + z\beta d(y\alpha x).$$

Proof. (i) We have $D(x\alpha x) = D(x)\alpha x + xad(x)$, for all $x \in M, \alpha \in \Gamma$. Then replacing x by $x + y$, and following the series of implications below we get the result:

$$\begin{aligned} &D((x + y)\alpha(x + y)) \\ &= D(x + y)\alpha(x + y) + (x + y)ad(x + y) \\ \implies &D(x\alpha x + x\alpha y + y\alpha x + y\alpha y) \\ &= D(x)\alpha x + D(y)\alpha x + D(x)\alpha y + D(y)\alpha x + xad(y) + xad(x) + yad(y) + yad(x) \\ \implies &D(x\alpha x + y\alpha y) + D(x\alpha y + y\alpha x) \\ &= (D(x)\alpha x + xad(x) + D(y)\alpha y + yad(y)) + D(x)\alpha y \\ &\quad + D(y)\alpha x + xad(y) + yad(x) \\ \implies &D(x\alpha y + y\alpha x) \\ &= D(x)\alpha y + D(y)\alpha x + xad(y) + yad(x), \quad \forall x, y \in M, \alpha \in \Gamma. \end{aligned} \quad (3.5)$$

(ii) Replace y by $x\beta y + y\beta x$ in the above relation (3.5), then we get,

$$\begin{aligned} &D(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) \\ &= D(x)\alpha(x\beta y + y\beta x) + D(x\beta y + y\beta x)\alpha x \\ &\quad + xad(x\beta y + y\beta x) + (x\beta y + y\beta x)ad(x), \quad \forall x, y, z \in M, \alpha, \beta \in \Gamma. \\ \implies &D(x\alpha x\beta y + y\beta x\alpha x) + D(x\alpha y\beta x) + D(x\beta y\alpha x) \\ &= D(x)\alpha x\beta y + D(x)\alpha y\beta x + D(x)\beta y\alpha x + D(y)\beta x\alpha x + x\beta d(y)\alpha x + y\beta d(x)\alpha x \\ &\quad + xad(x\beta y + y\beta x) + (x\beta y + y\beta x)ad(x), \quad \forall x, y, z \in M, \alpha, \beta \in \Gamma. \end{aligned} \quad (3.6)$$

Using the assumption (*), we conclude that

$$\begin{aligned} D(x\alpha x\beta y + y\beta x\alpha x) + 2D(x\alpha y\beta x) &= D(x)\alpha x\beta y + D(x)\alpha y\beta x + D(x)\beta y\alpha x + D(y)\beta x\alpha x \\ &\quad + x\beta d(y)\alpha x + y\beta d(x)\alpha x + xad(x\beta y + y\beta x) \\ &\quad + (x\beta y + y\beta x)ad(x), \quad \forall x, y, z \in M, \alpha, \beta \in \Gamma. \end{aligned} \quad (3.7)$$

Again, replacing x by $x\beta x$ in (3.5)

$$\begin{aligned} D(x\beta x\alpha y + y\alpha x\beta x) &= D(x\beta x)\alpha y + D(y)\alpha x\beta x + x\beta xad(y) \\ &\quad + yad(x\beta x), \quad \forall x, y \in M, \alpha, \beta \in \Gamma, \\ \Rightarrow D(x\beta x\alpha y + y\alpha x\beta x) &= D(x)\beta x\alpha y + x\beta d(x)\alpha y + D(y)\alpha x\beta x + x\beta xad(y) \\ &\quad + yad(x)\beta x + y\alpha x\beta d(x), \quad \forall x, y \in M, \alpha, \beta \in \Gamma. \end{aligned} \quad (3.8)$$

Adding both sides $2D(x\alpha y\beta x)$, we get,

$$\begin{aligned} &\Rightarrow D(x\beta x\alpha y + y\alpha x\beta x) + 2D(x\alpha y\beta x) \\ &= D(x)\beta x\alpha y + x\beta d(x)\alpha y + D(y)\alpha x\beta x + x\beta xad(y) \\ &\quad + yad(x)\beta x + y\alpha x\beta d(x) + 2D(x\alpha y\beta x), \quad \forall x, y \in M, \alpha, \beta \in \Gamma. \end{aligned} \quad (3.9)$$

Comparing (3.7) and (3.9) we obtain,

$$2D(x\alpha y\beta x) = 2\{D(x)\alpha y\beta x + x\alpha y\beta d(x) + xad(y)\beta x\}. \quad (3.10)$$

Since M is 2-torsion-free, it gives

$$D(x\alpha y\beta x) = D(x)\alpha y\beta x + xad(y)\beta x. \quad (3.11)$$

(iii) Replace x for $x + z$ in (3.12), we get,

$$\begin{aligned} &D((x + z)\alpha y\beta(x + z)) \\ &= D(x + z)\alpha y\beta(x + z) + (x + z)ad(y\beta(x + z)) \\ \Rightarrow &D(x\alpha y\beta x + x\alpha y\beta z + z\alpha y\beta x + z\alpha y\beta z) \\ &= D(x)\alpha y\beta x + D(z)\alpha y\beta x + D(x)\alpha y\beta z + D(z)\alpha y\beta z + xad(y\beta x) \\ &\quad + xad(y\beta z) + zad(y\beta x + y\beta z) \\ \Rightarrow &D(x\alpha y\beta x + z\alpha y\beta z) + D(x\alpha y\beta z + z\alpha y\beta x) \\ &= (D(x)\alpha y\beta x + D(z)\alpha y\beta z + xad(y\beta x) + zad(y\beta z)) \\ &\quad + (D(z)\alpha y\beta x + D(x)\alpha y\beta z + xad(y\beta z) + zad(y\beta x)) \end{aligned}$$

$$\begin{aligned}
&\Rightarrow D(x\alpha y\beta z + z\alpha y\beta x) \\
&= D(x)\alpha y\beta z + D(z)\alpha y\beta x + xad(y\beta z) + zad(y\beta x) \\
&\Rightarrow D(x\alpha y\beta z + z\beta y\alpha x) \\
&= D(x)\alpha y\beta z + D(z)\alpha y\beta x + xad(y\beta z) + zad(y\beta x), \quad \forall x, y \in M, \alpha, \beta \in \Gamma.
\end{aligned} \tag{3.12}$$

□

Definition 3.5. Let M be a Γ -ring and $D : M \rightarrow M$ be a Jordan generalized derivation with the associated derivation d . Define $G_\alpha(x, y) = D(x\alpha y) - D(x)\alpha y - xad(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Lemma 3.6. The function $G_\alpha(x, y)$ has the following properties:

- (i) $G_\alpha(x, y) + G_\alpha(y, x) = 0$.
- (ii) $G_\alpha(x, y + z) = G_\alpha(x, y) + G_\alpha(x, z)$.
- (iii) $G_\alpha(x + y, z) = G_\alpha(x, z) + G_\alpha(y, z)$.
- (iv) $G_{\alpha+\beta}(x, y) = G_\alpha(x, y) + G_\beta(x, y)$.

Proof. The results easily follow from Lemma 3.4(i). □

Remark 3.7. D is a generalized derivation if and only if $G_\alpha(x, y) = 0$ for all $x, y \in M, \alpha \in \Gamma$.

Theorem 3.8. Let M be a 2-torsion-free semiprime Γ -ring satisfying the condition $(*)$. Let $x, y, z \in M$ and $\beta, \delta \in \Gamma$. Then

- (i) $G_\alpha(x, y)\beta z\delta[x, y]_\alpha = 0$ for all $z \in M$,
- (ii) $G_\alpha(x, y)\beta[x, y]_\alpha = 0$.

Proof. (i) From Lemma 3.4(iii) we get

$$D(x\alpha y\beta z + z\beta y\alpha x) = D(x)\alpha y\beta z + D(z)\beta y\alpha x + xad(y\beta z) + z\beta d(y\alpha x). \tag{3.13}$$

We set

$$A = x\alpha y\beta z\delta y\alpha x + y\alpha x\beta z\delta x\alpha y. \tag{3.14}$$

Since

$$\begin{aligned}
D(A) &= D(x\alpha y\beta z\delta y\alpha x + y\alpha x\beta z\delta x\alpha y) \\
&= D(x\alpha(y\beta z\delta y\alpha x) + y\alpha(x\beta z\delta x\alpha y)) \\
&= D(x)\alpha y\beta z\delta y\alpha x + xad(y\beta z\delta y\alpha x) + D(y)\alpha x\beta z\delta x\alpha y + yad(x\beta z\delta x\alpha y)
\end{aligned}$$

$$\begin{aligned}
&= D(x)\alpha y\beta z\delta y\alpha x + xad(y)\beta z\delta y\alpha x + x\alpha y\beta d(z\delta y\alpha x) + D(y)\alpha x\beta z\delta x\alpha y \\
&\quad + yad(x)\beta z\delta x\alpha y + y\alpha x\beta d(z\delta x\alpha y), \quad \forall x, y, z \in M, \alpha, \beta, \delta \in \Gamma.
\end{aligned} \tag{3.15}$$

Again

$$\begin{aligned}
D(A) &= D(x\alpha y\beta z\delta y\alpha x + y\alpha x\beta z\delta x\alpha y) \\
&= D((x\alpha y)\beta z\delta(y\alpha x) + (y\alpha x)\beta z\delta(x\alpha y)) \\
&= D(x\alpha y)\beta z\delta y\alpha x + D(y\alpha x)\beta z\delta x\alpha y + x\alpha y\beta d(z\delta y\alpha x) \\
&\quad + y\alpha x\beta d(z\delta x\alpha y), \quad \forall x, y, z \in M, \alpha, \beta, \delta \in \Gamma.
\end{aligned} \tag{3.16}$$

From (3.15) and (3.16) we find,

$$\begin{aligned}
&(D(x\alpha y) - D(x)\alpha y - xad(y))\beta z\delta y\alpha x + (D(y\alpha x) - D(y)\alpha x - yad(x))\beta z\delta x\alpha y = 0. \\
&\implies G_\alpha(x, y)\beta z\delta y\alpha x + G_\alpha(y, x)\beta z\delta x\alpha y = 0, \\
&\implies G_\alpha(x, y)\beta z\delta y\alpha x - G_\alpha(x, y)\beta z\delta x\alpha y = 0, \\
&\implies G_\alpha(x, y)\beta z\delta[y, x]_\alpha = 0, \\
&\implies G_\alpha(x, y)\beta z\delta[x, y]_\alpha = 0, \quad \text{for } x, y \in M, \alpha, \beta, \delta \in \Gamma.
\end{aligned} \tag{3.17}$$

(ii) According to Lemma 3.4(ii) we have,

$$D(x\alpha y\beta z) = D(x)\alpha y\beta z + xad(y\beta z). \tag{3.18}$$

Replace z by $x\alpha y$ in (3.18) we find

$$\begin{aligned}
&D(x\alpha y\beta x\alpha y) = D(x)\alpha y\beta x\alpha y + xad(y\beta x\alpha y) \\
&\implies D((x\alpha y)\beta(x\alpha y)) = D(x)\alpha y\beta x\alpha y + xad(y\beta x\alpha y) \\
&\implies D(x\alpha y)\beta x\alpha y + x\alpha y\beta d(x\alpha y) - D(x)\alpha y\beta x\alpha y - xad(y\beta x\alpha y) = 0 \\
&\implies (D(x\alpha y) - D(x)\alpha y - xad(y))\beta x\alpha y + x\alpha y\beta d(x\alpha y) - x\alpha y\beta d(x\alpha y) = 0 \\
&\implies G_\alpha(x, y)\beta x\alpha y = 0.
\end{aligned} \tag{3.19}$$

Similarly $G_\alpha(x, y)\beta y\alpha x = 0$. Therefore,

$$G_\alpha(x, y)\beta x\alpha y - G_\alpha(x, y)\beta y\alpha x = G_\alpha(x, y)\beta[x, y]_\alpha = 0. \tag{3.20}$$

□

Lemma 3.9. $G_\alpha(x, y) \in Z(M)$, for all $x, y \in M, \alpha \in \Gamma$.

Proof. From Theorem 3.8(ii), we have $G_\alpha(x, y)\beta[x, y]_\alpha = 0$. Therefore due to Lemma 3.4(ii) $G_\alpha(x, y) \in Z(M)$. □

Theorem 3.10. Let M be a 2-torsion-free semiprime Γ -ring satisfying the assumption $(*)$ and $D : M \rightarrow M$ be a Jordan generalized derivation with associated derivation d on M . Then D is a generalized derivation.

Proof. In particular, $r\gamma G_\alpha(x, y), G_\alpha(x, y)\gamma r \in Z(M)$ for all $r \in M, \alpha, \gamma \in \Gamma$. This gives

$$\begin{aligned} x\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta y &= G_\alpha(x, y)\delta G_\alpha(x, y)\beta y\alpha x \\ &= y\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x \\ &= y\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta x. \end{aligned} \quad (3.21)$$

Then $4D(x\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta y) = 4D(y\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x)$. Now we will compute each side of this equality by using (3.15) and the above relation,

$$\begin{aligned} 4D(x\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta y) &= 2D(x\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta y + G_\alpha(x, y)\delta G_\alpha(x, y)\beta y\alpha x) \\ &= 2D(x)\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta y + 2xad(G_\alpha(x, y)\delta G_\alpha(x, y)\beta y) \\ &\quad + 2D(G_\alpha(x, y)\delta G_\alpha(x, y)\beta y)\alpha x + 2G_\alpha(x, y)\delta G_\alpha(x, y)\beta yad(x) \\ &= 2D(x)\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta y + D(G_\alpha(x, y)\delta G_\alpha(x, y)\beta y + y\beta G_\alpha(x, y)\delta G_\alpha(x, y))\alpha x \\ &\quad + 2xad(G_\alpha(x, y)\delta G_\alpha(x, y)\beta y) + 2G_\alpha(x, y)\delta G_\alpha(x, y)\beta yad(x) \\ &= 2D(x)\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta y + D(G_\alpha(x, y))\delta G_\alpha(x, y)\beta y\alpha x \\ &\quad + G_\alpha(x, y)\delta d(G_\alpha(x, y))\beta y\alpha x + D(y)\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x \\ &\quad + G_\alpha(x, y)\delta G_\alpha(x, y)\beta d(y)\alpha x + y\beta d(G_\alpha(x, y)\delta G_\alpha(x, y))\alpha x \\ &\quad + 2xad(G_\alpha(x, y)\delta G_\alpha(x, y)\beta y) + 2G_\alpha(x, y)\delta G_\alpha(x, y)\beta yad(x). \end{aligned} \quad (3.22)$$

So we get

$$\begin{aligned} 4D(x\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha y) &= 2D(x)\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha y + D(G_\alpha(x, y))\delta G_\alpha(x, y)\beta y\alpha x \\ &\quad + G_\alpha(x, y)\delta d(G_\alpha(x, y))\beta y\alpha x + D(y)\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x \\ &\quad + G_\alpha(x, y)\delta G_\alpha(x, y)\beta d(y)\alpha x + y\beta d(G_\alpha(x, y)\delta G_\alpha(x, y))\alpha x \\ &\quad + 2xad(G_\alpha(x, y)\delta G_\alpha(x, y)\beta y) + 2G_\alpha(x, y)\delta G_\alpha(x, y)\beta yad(x), \quad x, y \in M, \alpha, \beta, \delta \in \Gamma. \end{aligned} \quad (3.23)$$

Moreover,

$$\begin{aligned} 4D(y\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x) &= 2D(y\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x + G_\alpha(x, y)\delta G_\alpha(x, y)\beta x\alpha y) \\ &= 2D(y)\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x + 2y\beta d(G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x) \\ &\quad + 2D(G_\alpha(x, y)\delta G_\alpha(x, y)\beta x)\alpha y + 2G_\alpha(x, y)\delta G_\alpha(x, y)\beta xad(y) \end{aligned}$$

$$\begin{aligned}
&= 2D(y)\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x + D(G_\alpha(x, y)\delta G_\alpha(x, y)\beta x + x\beta G_\alpha(x, y)\delta G_\alpha(x, y))\alpha y \\
&\quad + 2y\beta d(G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x) + 2G_\alpha(x, y)\delta G_\alpha(x, y)\beta x\alpha d(y) \\
&= 2D(y)\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x + D(G_\alpha(x, y))\delta G_\alpha(x, y)\beta x\alpha y \\
&\quad + G_\alpha(x, y)\delta d(G_\alpha(x, y))\beta x\alpha y + D(x)\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha y \\
&\quad + G_\alpha(x, y)\delta G_\alpha(x, y)\alpha d(x)\beta y + x\alpha d(G_\alpha(x, y)\delta G_\alpha(x, y))\beta y \\
&\quad + 2y\beta d(G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x) + 2G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x\beta d(y).
\end{aligned} \tag{3.24}$$

So we get

$$\begin{aligned}
&4D(y\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x) \\
&= 2D(y)\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x + D(G_\alpha(x, y))\delta G_\alpha(x, y)\alpha x\beta y \\
&\quad + G_\alpha(x, y)\delta d(G_\alpha(x, y))\alpha x\beta y + D(x)\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha y \\
&\quad + G_\alpha(x, y)\delta G_\alpha(x, y)\alpha d(x)\beta y + x\alpha d(G_\alpha(x, y)\delta G_\alpha(x, y))\beta y \\
&\quad + 2y\beta d(G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x) + 2G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x\beta d(y), \\
&\quad x, y \in M, \alpha, \beta, \delta \in \Gamma.
\end{aligned} \tag{3.25}$$

Comparing the results of (3.22) and (3.24) and using the above relations

$$\begin{aligned}
G_\alpha(x, y)\beta y\alpha x &= G_\alpha(x, y)\beta y\alpha x \\
&= x\alpha G_\alpha(x, y)\beta y \\
&= x\alpha G_\alpha(x, y)\beta y \\
&= G_\alpha(x, y)\beta x\alpha y, \\
G_\alpha(x, y)\delta d(G_\alpha(x, y))\beta y\alpha x &= d(G_\alpha(x, y))\delta G_\alpha(x, y)\beta y\alpha x \\
&= d(G_\alpha(x, y))\delta G_\alpha(x, y)\alpha x\beta y \\
&= G_\alpha(x, y)\delta d(G_\alpha(x, y))\alpha x\beta y, \\
x\beta G_\alpha(x, y)\delta d(G_\alpha(x, y))\alpha y &= d(G_\alpha(x, y))\alpha x\delta G_\alpha(x, y)\beta y \\
&= d(G_\alpha(x, y))\delta G_\alpha(x, y)\beta x\alpha y \\
&= d(G_\alpha(x, y))\delta G_\alpha(x, y)\alpha y\beta x \\
&= G_\alpha(x, y)\beta y\delta d(G_\alpha(x, y))\alpha x \\
&= y\beta G_\alpha(x, y)\delta d(G_\alpha(x, y))\alpha x,
\end{aligned} \tag{3.26}$$

we obtain

$$\begin{aligned}
&D(x)\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta y + x\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta d(y) \\
&= D(y)\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta x + y\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta d(x),
\end{aligned} \tag{3.27}$$

which gives

$$\varphi_\alpha(x, y)\beta G_\alpha(x, y)\delta G_\alpha(x, y) = \varphi_\alpha(y, x)\alpha G_\alpha(x, y)\delta G_\alpha(x, y), \quad (3.28)$$

where $\varphi_\alpha(x, y)$ stands for $D(x)\alpha y + x\alpha d(y)$.

On the other hand, we have

$$4D(x\alpha y\beta G_\alpha(x, y)\delta G_\alpha(x, y)) = 4D(x\alpha G_\alpha(x, y)\delta y\beta G_\alpha(x, y)). \quad (3.29)$$

Now we use (3.15) and the properties of $G_\alpha(x, y)$, to derive

$$\begin{aligned} 4D(x\alpha y\beta G_\alpha(x, y)\delta G_\alpha(x, y)) &= 2D(x\alpha y\beta G_\alpha(x, y)\delta G_\alpha(x, y) + G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x\beta y) \\ &= 2D(x\alpha y)\beta G_\alpha(x, y)\delta G_\alpha(x, y) + 2x\alpha y\beta d(G_\alpha(x, y)\delta G_\alpha(x, y)) \\ &\quad + 2D(G_\alpha(x, y)\delta G_\alpha(x, y))\alpha x\beta y + 2G_\alpha(x, y)\delta G_\alpha(x, y)\beta d(x\alpha y), \end{aligned} \quad (3.30)$$

which gives

$$\begin{aligned} 4D(x\alpha y\beta G_\alpha(x, y)\delta G_\alpha(x, y)) &= 2D(x\alpha y)\beta G_\alpha(x, y)\delta G_\alpha(x, y) + 2x\alpha y\beta d(G_\alpha(x, y)\delta G_\alpha(x, y)) \\ &\quad + 2D(G_\alpha(x, y)\delta G_\alpha(x, y))\alpha x\beta y \\ &\quad + 2G_\alpha(x, y)\delta G_\alpha(x, y)\beta d(x\alpha y), \quad x, y \in M, \alpha, \beta, \delta \in \Gamma. \end{aligned} \quad (3.31)$$

Moreover,

$$\begin{aligned} 4D(x\alpha G_\alpha(x, y)\delta y\beta G_\alpha(x, y)) &= 2D(x\alpha G_\alpha(x, y)\delta y\beta G_\alpha(x, y) + y\beta G_\alpha(x, y)\delta x\alpha G_\alpha(x, y)) \\ &= 2D(G_\alpha(x, y)\alpha x)\delta G_\alpha(x, y)\beta y + 2G_\alpha(x, y)\alpha x\delta d(G_\alpha(x, y)\beta y) \\ &\quad + 2D(G_\alpha(x, y)\beta y)\delta G_\alpha(x, y)\alpha x + 2G_\alpha(x, y)\beta y\delta d(G_\alpha(x, y)\alpha x) \\ &= D(x\alpha G_\alpha(x, y) + G_\alpha(x, y)\alpha x)\delta G_\alpha(x, y)\beta y + 2G_\alpha(x, y)\alpha x\delta d(G_\alpha(x, y)\beta y) \\ &\quad + D(y\beta G_\alpha(x, y) + G_\alpha(x, y)\beta y)\delta G_\alpha(x, y)\alpha x + 2G_\alpha(x, y)\beta y\delta d(G_\alpha(x, y)\alpha x) \\ &= D(x)\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta y + D(G_\alpha(x, y))\delta G_\alpha(x, y)\alpha x\beta y \\ &\quad + x\alpha d(G_\alpha(x, y))\delta G_\alpha(x, y)\beta y + G_\alpha(x, y)\alpha d(x)\delta G_\alpha(x, y)\beta y \\ &\quad + 2G_\alpha(x, y)\alpha x\delta d(G_\alpha(x, y)\beta y) + D(y)\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x \\ &\quad + D(G_\alpha(x, y))\delta G_\alpha(x, y)\beta y\alpha x + y\beta d(G_\alpha(x, y))\delta G_\alpha(x, y)\alpha x \\ &\quad + G_\alpha(x, y)\beta d(y)\delta G_\alpha(x, y)\alpha x + 2G_\alpha(x, y)\beta y\delta d(G_\alpha(x, y)\alpha x). \end{aligned} \quad (3.32)$$

So we obtain

$$\begin{aligned}
 & 4D(x\alpha G_\alpha(x, y)\delta y\beta G_\alpha(x, y)) \\
 &= D(x)\alpha G_\alpha(x, y)\delta G_\alpha(x, y)\beta y + D(G_\alpha(x, y))\delta G_\alpha(x, y)\beta x\alpha y \\
 &\quad + x\alpha d(G_\alpha(x, y))\delta G_\alpha(x, y)\beta y + G_\alpha(x, y)\alpha d(x)\delta G_\alpha(x, y)\beta y \\
 &\quad + 2G_\alpha(x, y)\alpha x\delta d(G_\alpha(x, y)\beta y) + D(y)\beta G_\alpha(x, y)\delta G_\alpha(x, y)\alpha x \\
 &\quad + D(G_\alpha(x, y))\delta G_\alpha(x, y)\beta y\alpha x + y\beta d(G_\alpha(x, y))\delta G_\alpha(x, y)\alpha x \\
 &\quad + G_\alpha(x, y)\beta d(y)\delta G_\alpha(x, y)\alpha x + 2G_\alpha(x, y)\beta y\delta d(G_\alpha(x, y)\alpha x), \\
 &\quad x, y \in M, \alpha, \beta, \delta \in \Gamma.
 \end{aligned} \tag{3.33}$$

Comparing (3.31) and (3.33), we derive

$$\begin{aligned}
 & 2D(x\alpha y)\beta G_\alpha(x, y)\delta G_\alpha(x, y) \\
 &= \varphi_\alpha(x, y)\beta G_\alpha(x, y)\delta G_\alpha(x, y) + \varphi_\alpha(y, x)\beta G_\alpha(x, y)\delta G_\alpha(x, y), \quad x, y \in M, \alpha, \beta, \delta \in \Gamma.
 \end{aligned} \tag{3.34}$$

Finally using (3.31) we get $D(x\alpha y)\beta G_\alpha(x, y)\delta G_\alpha(x, y) = \varphi_\alpha(x, y)\beta G_\alpha(x, y)\delta G_\alpha(x, y)$. But $G_\alpha(x, y) = D(x\alpha y) - \varphi_\alpha(x, y)$. By the semiprimeness of M , we have $\varphi_\alpha(x, y)\beta G_\alpha(x, y) = 0$. Again by the primeness of M , we get $G_\alpha(x, y) = 0$. The proof is complete. $\square$

It is clear that if we let the derivation d to be the zero derivation in the above theorem, we get the following result.

Theorem 3.11. *Let M be a 2-torsion-free semiprime Γ -ring and $D : M \rightarrow M$ be an additive mapping which satisfies $D(x\alpha x) = D(x)\alpha x$ for all $x \in M, \alpha \in \Gamma$. Then D is a left centralizer*

Proof. We have

$$D(x\alpha x) = D(x)\alpha x. \tag{3.35}$$

If we replace x by $x + y$, we get

$$D(x\alpha y + y\alpha x) = D(x)\alpha y + D(y)\alpha x. \tag{3.36}$$

By replacing y with $x\alpha y + y\alpha x$ and using (3.5), we arrive at

$$D(x\alpha(x\alpha y + y\alpha x) + (x\alpha y + y\alpha x)\alpha x) = D(x)\alpha x\alpha y + D(x)\alpha y\alpha x + D(x)\alpha y\alpha x + D(y)\alpha x\alpha x. \tag{3.37}$$

But on the other hand,

$$D(x\alpha x\alpha y + y\alpha x\alpha x) + 2D(x\alpha y\alpha x) = D(x)\alpha x\alpha y + D(y)\alpha x\alpha x + 2D(x\alpha y\alpha x). \tag{3.38}$$

Comparing (3.37) and (3.38) we obtain

$$D(x\alpha y\alpha x) = D(x)\alpha y\alpha x. \quad (3.39)$$

If we linearize (3.39) in x , we get

$$D(x\alpha y\alpha z + z\alpha y\alpha x) = D(x)\alpha y\alpha z + D(z)\alpha y\alpha x. \quad (3.40)$$

Now we shall compute $D(x\alpha y\alpha z\alpha y\alpha x + y\alpha x\alpha z\alpha x\alpha y)$ in two different ways. If we use (3.39) we have

$$D(x\alpha y\alpha z\alpha y\alpha x + y\alpha x\alpha z\alpha x\alpha y) = D(x)\alpha y\alpha z\alpha y\alpha x + D(y)\alpha x\alpha z\alpha x\alpha y. \quad (3.41)$$

But if we use (3.40) we have

$$D(x\alpha y\alpha z\alpha y\alpha x + y\alpha x\alpha z\alpha x\alpha y) = D(x\alpha y)\alpha z\alpha y\alpha x + D(y\alpha x)\alpha z\alpha x\alpha y. \quad (3.42)$$

Comparing (3.41) and (3.42) and introducing a bi-additive mapping $G_\alpha(x, y) = D(x\alpha y) - D(x)\alpha y$ we arrive at

$$G_\alpha(x, y)\alpha z\alpha y\alpha x + G_\alpha(y, x)\alpha z\alpha x\alpha y = 0. \quad (3.43)$$

Equality (3.36) can be rewritten in this notation as $G_\alpha(x, y) = -G_\alpha(y, x)$. Using this fact and (3.43) we obtain

$$G_\alpha(x, y)\alpha z\alpha [x, y]_\alpha = 0. \quad (3.44)$$

Using first Lemma 3.2 and then Lemma 3.1 we have

$$G_\alpha(x, y)\alpha z\alpha [u, v]_\alpha = 0. \quad (3.45)$$

Now fix some $x, y \in M$ and using Lemma 3.3 we get $G_\alpha(x, y) \in Z$.

In particular, $r\beta G_\alpha(x, y), G_\alpha(x, y)\beta r \in Z$ for all $r \in M$. This gives

$$\begin{aligned} x\beta G_\alpha(x, y)\delta G_\alpha(x, y)\beta y &= G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y\Gamma x \\ &= y\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x \\ &= y\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x. \end{aligned} \quad (3.46)$$

Therefore $4D(x\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y) = 4D(y\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x)$. Both sides of this equality will be computed in few steps using (3.36),

$$\begin{aligned}
& 2D(x\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y + G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y\Gamma x) \\
&= 2D(y\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x + G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x\Gamma y), \\
& 2D(x)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y + 2D(G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y)\Gamma x \\
&= 2D(y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x + 2D(G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x)\Gamma y, \\
& 2D(x)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y + D(G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y + y\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y))\Gamma x \\
&= 2D(y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x + D(G_\alpha(x, y))\Gamma G_\alpha(x, y)\Gamma x + x\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y, \\
& 2D(x)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y + D(G_\alpha(x, y))\Gamma G_\alpha(x, y)\Gamma y\Gamma x + D(y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x \\
&= 2D(y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x + D(G_\alpha(x, y))\Gamma G_\alpha(x, y)\Gamma x\Gamma y \\
&\quad + D(x)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y, \\
& D(x)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y + D(G_\alpha(x, y))\Gamma G_\alpha(x, y)\Gamma y\Gamma x \\
&= D(y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x + D(G_\alpha(x, y))\Gamma G_\alpha(x, y)\Gamma x\Gamma y.
\end{aligned} \tag{3.47}$$

Since $G_\alpha(x, y)\Gamma y\Gamma x = G_\alpha(x, y)\Gamma y\Gamma x = x\Gamma G_\alpha(x, y)\Gamma y = x\Gamma G_\alpha(x, y)\Gamma y = G_\alpha(x, y)\Gamma x\Gamma y$, we obtain

$$D(x)\Gamma G_\alpha(x, y)\Gamma y = D(y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x. \tag{3.48}$$

On the other hand, we also have

$$\begin{aligned}
& 4D(x\Gamma y\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)) = 4D(x\Gamma G_\alpha(x, y)\Gamma y\Gamma G_\alpha(x, y)), \\
& 2D(x\Gamma y\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y) + G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x\Gamma y) \\
&= 2D(x\Gamma G_\alpha(x, y)\Gamma y\Gamma G_\alpha(x, y) + y\Gamma G_\alpha(x, y)\Gamma x\Gamma G_\alpha(x, y)), \\
& 2D(x\Gamma y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y) + 2D(G_\alpha(x, y))\Gamma G_\alpha(x, y)\Gamma x\Gamma y \\
&= 2D(G_\alpha(x, y)\Gamma x)\Gamma G_\alpha(x, y)\Gamma y + 2D(G_\alpha(x, y)\Gamma y)\Gamma G_\alpha(x, y)\Gamma x, \\
& 2D(x\Gamma y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y) + 2D(G_\alpha(x, y))\Gamma G_\alpha(x, y)\Gamma x\Gamma y \\
&= D(x\Gamma G_\alpha(x, y) + G_\alpha(x, y)\Gamma x)\Gamma G_\alpha(x, y)\Gamma y + D(y\Gamma G_\alpha(x, y) + G_\alpha(x, y)\Gamma y)\Gamma G_\alpha(x, y)\Gamma x, \\
& 2D(x\Gamma y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y) + 2D(G_\alpha(x, y))\Gamma G_\alpha(x, y)\Gamma x\Gamma y \\
&= D(x)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma y + D(G_\alpha(x, y))\Gamma G_\alpha(x, y)\Gamma x\Gamma y \\
&\quad + D(y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma x + D(G_\alpha(x, y))\Gamma G_\alpha(x, y)\Gamma x\Gamma y, \\
& 2D(x\Gamma y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y) \\
&= D(x)\Gamma y\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y) + D(y)\Gamma x\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y).
\end{aligned} \tag{3.49}$$

Finally using (3.48) we arrive at $D(x\Gamma y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y) = D(x)\Gamma y\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)$, but this means that $G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y) = 0$. Hence,

$$\begin{aligned} G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma M\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y) &= G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma M = 0, \\ G_\alpha(x, y)\Gamma M\Gamma G_\alpha(x, y) &= G_\alpha(x, y)\Gamma G_\alpha(x, y)\Gamma M = 0, \end{aligned} \quad (3.50)$$

which implies $G_\alpha(x, y) = 0$. The proof is complete. $\square$

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