Letter to the Editor

Cusp Forms in $S_4(\Gamma_0(47))$ and the Number of Representations of Positive Integers by Some Direct Sum of Binary Quadratic Forms with Discriminant -47

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A basis of $S_4(\Gamma_0(47))$ is given and the formulas for the number of representations of positive integers by some direct sum of the quadratic forms $x_1^2 + x_1x_2 + 12x_2^2$, $2x_1^2 \pm x_1x_2 + 6x_2^2$, $3x_1^2 \pm x_1x_2 + 4x_2^2$ are determined.

1. Introduction

This paper is the correction of the paper [1].

- (1) It is stated that $\dim S_4(\Gamma_0(47), 1) = 5$ at page 643 in [1]. But this dimension is 11 as stated at page 299 in [2]. Therefore, the coefficients of power series in (2.2), (2.3), (2.5), (2.6), (2.7), (2.8), (2.10), (2.12), (2.14), (2.15), (2.20), and (2.22) have to be calculated up to z^{11} , and Theorem 2.4 and consequently Theorem 2.7 are false since 5 vectors cannot be a basis of 11-dimensional vector space.
- (2) The notations $G_k(\Gamma, \chi)$ and (k, Γ, χ) are both used as if they are different like at line 11 at page 638.
- (3) The definitions of

$$\Gamma_1(N), \Gamma(N) \text{ and } \wp(\tau; Q(X), P_v(X), h) = \sum_{n_i \equiv h_i \pmod{N}} P_v(n_1, n_2, \dots, n_k) z_N^{(1/N)Q(n_1, n_2, \dots, n_k)} \quad (1.1)$$

have never been used in the paper.

- (4) The class number of $\mathbb{Q}(\sqrt{-23})$ is 3; therefore, only F_1, Φ_1 , and their combinations have been examined and a basis of $S_4(\Gamma_0(23), 1)$ could be obtained. The authors in [1] also examined only two quadratic forms

$$F_1 = x_1^2 + x_1x_2 + 12x_2^2, \quad G_1 = 2x_1^2 + x_1x_2 + 6x_2^2 \quad (1.2)$$

and their combinations. But, by simple calculations, it is possible to see that these quadratic forms are not enough to get a basis of $S_4(\Gamma_0(47), 1)$. The class number of $\mathbb{Q}(\sqrt{-47})$ is 5; therefore,

$$F_1 = x_1^2 + x_1x_2 + 12x_2^2, \quad G_1 = 2x_1^2 + x_1x_2 + 6x_2^2, \quad H_1 = 2x_1^2 + x_1x_2 + 6x_2^2 \quad (1.3)$$

and their combinations have to be examined. Only in that case, it is possible to obtain a basis of $S_4(\Gamma_0(47), 1)$ as we have done in the following.

2. Determination of a Basis of $S_4(\Gamma_0(47))$

We can calculate all reduced forms of a positive definite quadratic form

$$Q = ax^2 + bxy + cy^2, \quad a > 0, \quad (2.1)$$

with discriminant $\Delta = -47$ as follows:

$$\begin{aligned} F_1 &= x_1^2 + x_1x_2 + 12x_2^2, & H_1 &= 2x_1^2 + x_1x_2 + 6x_2^2, & G_1 &= 3x_1^2 + x_1x_2 + 4x_2^2, \\ G'_1 &= 3x_1^2 - x_1x_2 + 4x_2^2, & H'_1 &= 2x_1^2 - x_1x_2 + 6x_2^2. \end{aligned} \quad (2.2)$$

Here G'_1 is the inverse of G_1 , and they represent the same integers. Similarly, H'_1 is the inverse of H_1 and they represent the same integers. Therefore, the theta series of H_1 and H'_1 are the same with the theta series of G_1 and G'_1 , respectively. F_1 is the identity element. It can be seen easily that, the group of these quadratic forms is a group of order 5 and can be described as

$$H_1^2 = G'_1, \quad H_1^3 = G_1, \quad H_1^4 = H'_1, \quad H_1^5 = F_1. \quad (2.3)$$

We can easily see that for the quadratic forms

$$F_1, G_1, H_1, \quad (2.4)$$

the determinant, the discriminant, and the character are

$$D = 47, \quad \Delta = (-1)^{2/2}47 = -47, \quad \chi(d) = \left(\frac{-47}{d}\right). \quad (2.5)$$

Consequently, their theta series are in

$$M_1\left(\Gamma_0(47), \left(\frac{-47}{d}\right)\right). \quad (2.6)$$

Hence by Theorem 2.1 in [3], F_2 , H_2 , G_2 , $F_1 \oplus H_1$, $F_1 \oplus G_1$, and $H_1 \oplus G_1$ are quadratic forms whose theta series are in

$$M_2(\Gamma_0(47)). \quad (2.7)$$

We immediately obtain the following Corollary by Theorem 2.2 in [3].

Corollary 2.1. *Let Q be a positive definite form of 8 variables whose theta series Θ_Q is in*

$$M_4(\Gamma_0(47)). \quad (2.8)$$

Then the Eisenstein part of Θ_Q is

$$E(q : Q) = 1 + \sum_{n=1}^{\infty} (\alpha \sigma_3(n) q^n + \beta \sigma_3(n) q^{47n}), \quad (2.9)$$

where

$$\rho_4 = \frac{3!}{(2\pi)^4} \zeta(4) = \frac{1}{240}, \quad \alpha = 240 \frac{47^2 - 1}{47^4 - 1} = \frac{24}{221}, \quad \beta = 240 \frac{47^4 - 47^2}{47^4 - 1} = 47^2 \frac{24}{221},$$

$$\begin{aligned} E(q : F_4) &= E(q : F_3 \oplus H_1) = E(q : F_2 \oplus H_2) = E(q : F_1 \oplus H_3) = E(q : H_4) \\ &= E(q : F_3 \oplus G_1) = E(q : F_2 \oplus G_2) = E(q : F_1 \oplus G_3) = E(q : G_4) = E(q : H_3 \oplus G_1) \\ &= E(q : H_2 \oplus G_2) = E(q : H_1 \oplus G_3) = 1 + \frac{24}{221} \sum_{n=1}^{\infty} (q^n + 47^2 q^{47n}) \sigma_3(n) \\ &= 1 + \frac{24}{221} \sum_{n=1}^{\infty} \sigma_3^*(n) q^n \\ &= 1 + \frac{24}{221} q + \frac{24 \cdot 9}{221} q^2 + \frac{24 \cdot 28}{221} q^3 + \frac{24 \cdot 73}{221} q^4 + \frac{24 \cdot 126}{221} q^5 + \frac{24 \cdot 252}{221} q^6 \\ &\quad + \frac{24 \cdot 344}{221} q^7 + \frac{24 \cdot 585}{221} q^8 + \frac{24 \cdot 757}{221} q^9 + \frac{24 \cdot 1134}{221} q^{10} + \frac{24 \cdot 1332}{221} q^{11} + \dots \end{aligned} \quad (2.10)$$

Here

$$\sigma_3^*(n) = \begin{cases} \sigma_3(n) & \text{if } n \geq 1 \text{ and } 47 \nmid n, \\ \sigma_3(n) + 47^2 \sigma_3\left(\frac{n}{47}\right) & \text{if } 47 \mid n. \end{cases} \quad (2.11)$$

Now we will determine the sum of quadratic forms F_1 , H_1 , and G_1 and select 11 spherical functions such that the corresponding cusp forms are linearly independent.

(1) For quadratic form

$$\begin{aligned} 2F_2 &= 2x_1^2 + 2x_1x_2 + 24x_2^2 + 2x_3^2 + 2x_3x_4 + 24x_4^2 \\ &= (x_1, x_2, x_3, x_4) \begin{pmatrix} 2 & 1 & & \\ 1 & 24 & & \\ & & 2 & 1 \\ & & 1 & 24 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \end{aligned} \quad (2.12)$$

the determinant and a cofactor are

$$D = 47^2, \quad A_{11} = 24 \cdot 47. \quad (2.13)$$

By putting $2k = 4$, $Q = F_2$, and appropriate i, j in Theorem 2.1 in [3], we get

$$\varphi_{11} = x_1^2 - \frac{1}{4} \frac{24 \cdot 47}{47^2} 2F_2 = x_1^2 - \frac{12}{47} F_2, \quad (2.14)$$

which will be spherical functions of second order with respect to F_2 .

(2) Similarly, for

$$2H_2 = 4x_1^2 + 2x_1x_2 + 12x_2^2 + 4x_3^2 + 2x_3x_4 + 12x_4^2, \quad (2.15)$$

the determinant and some cofactors are

$$D = 47^2, \quad A_{11} = 12 \cdot 47, \quad A_{12} = -47, \quad A_{13} = 12 \cdot 12. \quad (2.16)$$

By putting $2k = 4$, $Q = H_2$, and appropriate i, j in Theorem 2.1 in [3], we get

$$\begin{aligned} \varphi_{11} &= x_1^2 - \frac{1}{4} \frac{12 \cdot 47}{47^2} 2H_2 = x_1^2 - \frac{6}{47} H_2, & \varphi_{12} &= x_1x_2 + \frac{1}{4} \frac{47}{47^2} 2H_2 = x_1x_2 + \frac{1}{2 \cdot 47} H_2, \\ \varphi_{13} &= x_1x_3 - \frac{1}{4} \frac{12 \cdot 12}{47^2} 2H_2 = x_1x_3 - \frac{72}{47^2} H_2, \end{aligned} \quad (2.17)$$

which will be spherical functions of second order with respect to H_2 .

(3) Similarly, for quadratic form

$$2G_2 = 6x_1^2 + 2x_1x_2 + 8x_2^2 + 6x_3^2 + 2x_3x_4 + 8x_4^2, \quad (2.18)$$

the determinant and some cofactors are

$$D = 47^2, \quad A_{11} = 8 \cdot 47, \quad A_{22} = 6 \cdot 47, \quad A_{33} = 8 \cdot 47, \quad A_{34} = -47. \quad (2.19)$$

By putting $2k = 4$, $Q = G_2$, and appropriate i, j in Theorem 2.1 in [3], we get

$$\begin{aligned} \varphi_{11} &= x_1^2 - \frac{1}{4} \frac{8 \cdot 47}{47^2} 2G_2 = x_1^2 - \frac{4}{47} G_2, & \varphi_{22} &= x_2^2 - \frac{1}{4} \frac{6 \cdot 47}{47^2} 2G_2 = x_2^2 - \frac{3}{47} G_2, \\ \varphi_{33} &= x_3^2 - \frac{1}{4} \frac{8 \cdot 47}{47^2} 2G_2 = x_3^2 - \frac{4}{47} G_2, & \varphi_{34} &= x_3 x_4 + \frac{1}{4} \frac{47}{47^2} 2G_2 = x_3 x_4 + \frac{1}{2 \cdot 47} G_2, \end{aligned} \quad (2.20)$$

which will be spherical functions of second order with respect to G_2 .

(4) Similarly, for quadratic form

$$2(H_1 \oplus G_1) = 4x_1^2 + 2x_1x_2 + 12x_2^2 + 6x_3^2 + 2x_3x_4 + 8x_4^2, \quad (2.21)$$

the determinant and some cofactors are

$$D = 47^2, \quad A_{11} = 12 \cdot 47, \quad A_{22} = 4 \cdot 47, \quad A_{33} = 8 \cdot 47. \quad (2.22)$$

By putting $2k = 4$, $Q = H_1 \oplus G_1$, and appropriate i, j in Theorem 2.1 in [3], we get

$$\begin{aligned} \varphi_{11} &= x_1^2 - \frac{1}{4} \frac{12 \cdot 47}{47^2} 2(H_1 \oplus G_1) = x_1^2 - \frac{6}{47} (H_1 \oplus G_1), \\ \varphi_{22} &= x_2^2 - \frac{1}{4} \frac{4 \cdot 47}{47^2} 2(H_1 \oplus G_1) = x_2^2 - \frac{2}{47} (H_1 \oplus G_1), \\ \varphi_{33} &= x_3^2 - \frac{1}{4} \frac{8 \cdot 47}{47^2} 2(H_1 \oplus G_1) = x_3^2 - \frac{4}{47} (H_1 \oplus G_1), \end{aligned} \quad (2.23)$$

which will be spherical functions of second order with respect to $H_1 \oplus G_1$.

Now we can determine a basis of $S_4(\Gamma_0(47))$ whose dimension is 11, see [2].

Theorem 2.2. *The following generalized 11 theta series:*

$$\begin{aligned}
\Theta_{F_2, \varphi_{11}}(q) &= \frac{1}{47} \sum_{n=1}^{\infty} \sum_{F_2=n} (47x_1^2 - 12F_2) q^n, \\
\Theta_{H_2, \varphi_{11}}(q) &= \frac{1}{47} \sum_{n=1}^{\infty} \sum_{H_2=n} (47x_1^2 - 6H_2) q^n, \\
\Theta_{H_2, \varphi_{12}}(q) &= \frac{1}{2 \cdot 47} \sum_{n=1}^{\infty} \sum_{H_2=n} (2 \cdot 47x_1x_2 + H_2) q^n, \\
\Theta_{H_2, \varphi_{13}}(q) &= \frac{1}{47^2} \sum_{n=1}^{\infty} \sum_{H_2=n} (47^2x_1x_3 - 72H_2) q^n, \\
\Theta_{G_2, \varphi_{11}}(q) &= \frac{1}{47} \sum_{n=1}^{\infty} \sum_{G_2=n} (47x_1^2 - 4G_2) q^n, \\
\Theta_{G_2, \varphi_{22}}(q) &= \frac{1}{47} \sum_{n=1}^{\infty} \sum_{G_2=n} (47x_2^2 - 3G_2) q^n, \\
\Theta_{G_2, \varphi_{33}}(q) &= \frac{1}{47} \sum_{n=1}^{\infty} \sum_{G_2=n} (47x_3^2 - 4G_2) q^n, \\
\Theta_{G_2, \varphi_{34}}(q) &= \frac{1}{2 \cdot 47} \sum_{n=1}^{\infty} \sum_{G_2=n} (2 \cdot 47x_3x_4 + G_2) q^n, \\
\Theta_{H_1 \oplus G_1, \varphi_{11}}(q) &= \frac{1}{47} \sum_{n=1}^{\infty} \sum_{H_1 \oplus G_1=n} (47x_1^2 - 6(H_1 \oplus G_1)) q^n, \\
\Theta_{H_1 \oplus G_1, \varphi_{22}}(q) &= \frac{1}{47} \sum_{n=1}^{\infty} \sum_{H_1 \oplus G_1=n} (47x_2^2 - 2(H_1 \oplus G_1)), \\
\Theta_{H_1 \oplus G_1, \varphi_{33}}(q) &= \frac{1}{47} \sum_{n=1}^{\infty} \sum_{H_1 \oplus G_1=n} (47x_3^2 - 4(H_1 \oplus G_1))
\end{aligned} \tag{2.24}$$

are a basis of $S_4(\Gamma_0(47))$.

Proof. The series are cusp forms because of Theorem 2.1 in [3]. Moreover, by simple calculations, we have

$$\begin{aligned}
\Theta_{F_2, \varphi_{11}}(q) &= \frac{1}{79} (46q + 192q^2 + 184q^4 + 460q^5 + 368q^8 + 414q^9 + 920q^{10} + 0q^{11} + \dots), \\
\Theta_{H_2, \varphi_{11}}(q) &= \frac{1}{47} (46q^2 + 92q^4 - 144q^6 - 74q^7 - 12q^8 - 178q^9 + 460q^{10} - 152q^{11} + \dots), \\
\Theta_{H_2, \varphi_{12}}(q) &= \frac{1}{2 \cdot 47} (8q^2 + 16q^4 + 24q^6 - 160q^7 + 96q^8 - 80q^9 + 80q^{10} + 464q^{11} + \dots),
\end{aligned}$$

$$\begin{aligned}
\Theta_{H_2, \varphi_{13}}(q) &= \frac{1}{47^2} \left(8q^2 + 16q^4 + 24q^6 + 28q^7 + 96q^8 + 108q^9 + 80q^{10} + 88q^{11} + \dots \right), \\
\Theta_{G_2, \varphi_{11}}(q) &= \frac{1}{47} \left(46q^3 - 64q^4 - 4q^6 - 36q^7 - 162q^8 + 88q^9 - 132q^{10} + 24q^{11} + \dots \right), \\
\Theta_{G_2, \varphi_{22}}(q) &= \frac{1}{47} \left(-48q^3 + 30q^4 - 98q^6 - 36q^7 + 26q^8 - 100q^9 + 56q^{10} - 164q^{11} + \dots \right), \\
\Theta_{G_2, \varphi_{33}}(q) &= \frac{1}{47} \left(46q^3 - 64q^4 + 90q^6 - 36q^7 - 162q^8 + 840q^9 - 132q^{10} + 24q^{11} + \dots \right), \\
\Theta_{G_2, \varphi_{34}}(q) &= \frac{1}{47} \left(-48q^3 - 64q^4 - 286q^6 - 36q^7 - 162q^8 - 476q^9 - 508q^{10} - 164q^{11} + \dots \right), \\
\Theta_{H_1 \oplus G_1, \varphi_{11}}(q) &= \frac{1}{47} \left(70q^2 - 36q^3 - 48q^4 + 68q^5 - 100q^6 + 10q^7 + 180q^8 - 230q^9 - 344q^{10} \right. \\
&\quad \left. - 152q^{11} + \dots \right), \\
\Theta_{H_1 \oplus G_1, \varphi_{22}}(q) &= \frac{1}{47} \left(-24q^2 - 36q^3 - 48q^4 - 120q^5 - 194q^6 + 10q^7 - 384q^8 - 42q^9 - 344q^{10} \right. \\
&\quad \left. - 340q^{11} + \dots \right), \\
\Theta_{H_1 \oplus G_1, \varphi_{33}}(q) &= \frac{1}{47} \left(-16q^2 + 70q^3 - 32q^4 + 108q^5 - 98q^6 - 56q^7 + 26q^8 - 28q^9 - 104q^{10} \right. \\
&\quad \left. - 164q^{11} + \dots \right).
\end{aligned} \tag{2.25}$$

The determinant of the coefficients matrix is $5321\ 028\ 802\ 318\ 956\ 232\ 704/47^{11}$. So, the set of theta series in the Theorem is a basis of $S_4(\Gamma_0(47))$. $\square$

3. Representation Numbers of n

Proposition 3.1. *The theta series of the quadratic forms are*

$$\begin{aligned}
\Theta_{F_4}(q) &= \Theta_{F_2}(q)\Theta_{F_2}(q) = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 \\
&\quad + 24q^8 + 104q^9 + 144q^{10} + 96q^{11} + \dots, \\
\Theta_{H_4}(q) &= \Theta_{H_2}(q)\Theta_{H_2}(q) = 1 + 8q^2 + 24q^4 + 40q^6 + 8q^7 + 72q^8 + 56q^9 + 144q^{10} \\
&\quad + 144q^{11} + \dots, \\
\Theta_{G_4}(q) &= \Theta_{G_2}(q)\Theta_{G_2}(q) = 1 + 8q^3 + 8q^4 + 32q^6 + 48q^7 + 32q^8 + 80q^9 + 144q^{10} \\
&\quad + 144q^{11} + \dots, \\
\Theta_{F_3 \oplus H_1}(q) &= \Theta_{F_3}(q)\Theta_{H_1}(q) = 1 + 6q + 14q^2 + 20q^3 + 30q^4 + 40q^5 + 38q^6 + 62q^7 \\
&\quad + 98q^8 + 84q^9 + 112q^{10} + 184q^{11} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_2 \oplus H_2}(q) &= \Theta_{F_2}(q) \Theta_{H_2}(q) = 1 + 4q + 8q^2 + 16q^3 + 24q^4 + 24q^5 + 36q^6 + 52q^7 \\
&\quad + 64q^8 + 112q^9 + 144q^{10} + 152q^{11} + \dots, \\
\Theta_{F_1 \oplus H_3}(q) &= \Theta_{F_1}(q) \Theta_{H_3}(q) = 1 + 2q + 6q^2 + 12q^3 + 14q^4 + 24q^5 + 26q^6 + 34q^7 \\
&\quad + 66q^8 + 92q^9 + 136q^{10} + 168q^{11} + \dots, \\
\Theta_{F_3 \oplus G_1}(q) &= \Theta_{F_3}(q) \Theta_{G_1}(q) = 1 + 6q + 12q^2 + 10q^3 + 20q^4 + 60q^5 + 66q^6 + 40q^7 \\
&\quad + 98q^8 + 154q^9 + 108q^{10} + 112q^{11} + \dots, \\
\Theta_{F_2 \oplus G_2}(q) &= \Theta_{F_2}(q) \Theta_{G_2}(q) = 1 + 4q + 4q^2 + 4q^3 + 24q^4 + 40q^5 + 24q^6 + 56q^7 \\
&\quad + 124q^8 + 108q^9 + 112q^{10} + 184q^{11} + \dots, \\
\Theta_{F_1 \oplus G_3}(q) &= \Theta_{F_1}(q) \Theta_{G_3}(q) = 1 + 2q + 6q^3 + 20q^4 + 12q^5 + 18q^6 + 72q^7 + 78q^8 \\
&\quad + 70q^9 + 148q^{10} + 192q^{11} + \dots, \\
\Theta_{H_3 \oplus G_1}(q) &= \Theta_{H_3}(q) \Theta_{G_1}(q) = 1 + 6q^2 + 2q^3 + 14q^4 + 12q^5 + 28q^6 + 30q^7 + 68q^8 \\
&\quad + 58q^9 + 124q^{10} + 120q^{11} + \dots, \\
\Theta_{H_2 \oplus G_2}(q) &= \Theta_{H_2}(q) \Theta_{G_2}(q) = 1 + 4q^2 + 4q^3 + 8q^4 + 16q^5 + 28q^6 + 28q^7 + 68q^8 \\
&\quad + 68q^9 + 112q^{10} + 144q^{11} + \dots, \\
\Theta_{H_1 \oplus G_3}(q) &= \Theta_{H_1}(q) \Theta_{G_3}(q) = 1 + 2q^2 + 6q^3 + 6q^4 + 12q^5 + 32q^6 + 26q^7 + 56q^8 + 94q^9 \\
&\quad + 108q^{10} + 136q^{11} + \dots, \\
\Theta_{F_2 \oplus H_1 \oplus G_1}(q) &= \Theta_{F_2}(q) \cdot \Theta_{H_1 \oplus G_1}(q) = 1 + 4q + 6q^2 + 10q^3 + 22q^4 + 28q^5 + 40q^6 + 74q^7 \\
&\quad + 76q^8 + 82q^9 + 148q^{10} + 168q^{11} + \dots, \\
\Theta_{F_1 \oplus H_2 \oplus G_1}(q) &= \Theta_{H_2}(q) \cdot \Theta_{F_1 \oplus G_1}(q) = 1 + 2q + 4q^2 + 10q^3 + 12q^4 + 20q^5 + 38q^6 + 44q^7 \\
&\quad + 66q^8 + 98q^9 + 108q^{10} + 152q^{11} + \dots, \\
\Theta_{F_1 \oplus H_1 \oplus G_2}(q) &= \Theta_{G_2}(q) \cdot \Theta_{F_1 \oplus H_1}(q) = 1 + 2q + 2q^2 + 8q^3 + 14q^4 + 16q^5 + 38q^6 + 54q^7 \\
&\quad + 54q^8 + 104q^9 + 144q^{10} + 144q^{11} + \dots,
\end{aligned} \tag{3.1}$$

and the subtraction of the any one of these theta series by the Eisenstein series

$$\begin{aligned}
E(q : F_4) &= \dots = E(q : F_1 \oplus H_1 \oplus G_2) = 1 + \frac{24}{221} \sum_{n=1}^{\infty} \sigma_3^*(n) q^n \\
&= 1 + \frac{24}{221} q + \frac{24 \cdot 9}{221} q^2 + \frac{24 \cdot 28}{221} q^3 + \frac{24 \cdot 73}{221} q^4 + \frac{24 \cdot 126}{221} q^5 + \frac{24 \cdot 252}{221} q^6 \\
&\quad + \frac{24 \cdot 344}{221} q^7 + \frac{24 \cdot 585}{221} q^8 + \frac{24 \cdot 757}{221} q^9 + \frac{24 \cdot 1134}{221} q^{10} + \frac{24 \cdot 1332}{221} q^{11} + \dots
\end{aligned} \tag{3.2}$$

is a linear combinations of the theta series in the preceding theorem. The coefficients are given in table [4].

Proof. By determination of solutions of

$$F_1 = n, \quad H_1 = n, \quad G_1 = n \quad \text{for } n = 1, 2, \dots, 11, \quad (3.3)$$

we easily calculate the theta series

$$\begin{aligned} & \Theta_{F_1}, \Theta_{H_1}, \Theta_{G_1}, \Theta_{F_2}, \Theta_{F_3}, \Theta_{F_4}, \Theta_{H_2}, \Theta_{H_3}, \Theta_{H_4}, \Theta_{G_2}, \Theta_{G_3}, \Theta_{G_4}, \Theta_{F_1 \oplus H_1}, \Theta_{F_3 \oplus H_1}, \Theta_{F_2 \oplus H_2}, \\ & \Theta_{F_1 \oplus H_3}, \Theta_{F_1 \oplus G_1}, \Theta_{F_3 \oplus G_1}, \Theta_{F_4}, \Theta_{F_2 \oplus G_2}, \Theta_{F_1 \oplus G_3}, \Theta_{H_1 \oplus G_1}, \Theta_{H_3 \oplus G_1}, \Theta_{H_2 \oplus G_2}, \Theta_{H_1 \oplus G_3}, \\ & \Theta_{F_2 \oplus H_1 \oplus G_1}, \Theta_{F_1 \oplus H_2 \oplus G_1}, \Theta_{F_1 \oplus H_1 \oplus G_2}. \end{aligned} \quad (3.4)$$

For the second part, now let us look at the case:

$$\begin{aligned} \Theta_{F_4}(q) - E(q : F) &= \frac{1744}{221}q + \frac{5088}{221}q^2 + \frac{6400}{221}q^3 + \frac{3552}{221}q^4 + \frac{7584}{221}q^5 + \frac{15168}{221}q^6 \\ &+ \frac{5888}{221}q^7 - \frac{672}{221}q^8 + \frac{4816}{221}q^9 + \frac{4608}{221}q^{10} - \frac{10752}{221}q^{11} + \dots \\ &= c_1 \Theta_{F_2, \varphi_{11}}(q) + c_2 \Theta_{H_2, \varphi_{11}}(q) + c_3 \Theta_{H_2, \varphi_{12}}(q) + c_4 \Theta_{H_2, \varphi_{13}}(q) \\ &+ c_5 \Theta_{G_2, \varphi_{11}}(q) + c_6 \Theta_{G_2, \varphi_{22}}(q) + c_7 \Theta_{G_2, \varphi_{33}}(q) + c_8 \Theta_{G_2, \varphi_{34}}(q) \\ &+ c_9 \Theta_{H_1 \oplus G_1, \varphi_{11}}(q) + c_{10} \Theta_{H_1 \oplus G_1, \varphi_{22}}(q) + c_{11} \Theta_{H_1 \oplus G_1, \varphi_{33}}(q). \end{aligned} \quad (3.5)$$

By equating the coefficients of q^n in both sides for $n = 1, 2, 3, \dots, 11$, we get an equation in coefficients

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 & c_{10} & c_{11} \end{pmatrix}. \quad (3.6)$$

We repeat the same procedure for the other cases. At the end, by solving 11 linear equations in 11 variables, we get the coefficients in table [4]. $\square$

Corollary 3.2. *The representation numbers for the quadratic forms*

$$\begin{aligned} \Omega &= F_4, H_4, H'_4, G_4, G'_4, F_3 \oplus H_1, F_3 \oplus H'_1, F_2 \oplus H_2, F_2 \oplus H'_2, F_1 \oplus H_3, F_1 \oplus H'_3, \\ &F_3 \oplus G_1, F_3 \oplus G'_1, F_2 \oplus G_2, F_2 \oplus G'_2, F_1 \oplus G_3, F_1 \oplus G'_3, H_3 \oplus G_1, H'_3 \oplus G_1, H_3 \oplus G'_1, H'_3 \oplus G'_1, \\ &H_2 \oplus G_2, H'_2 \oplus G_2, H_2 \oplus G'_2, H'_2 \oplus G'_2, H_1 \oplus G_3, H'_1 \oplus G_3, H_1 \oplus G'_3, H'_1 \oplus G'_3, \\ &F_2 \oplus H_1 \oplus G_1, F_1 \oplus H_2 \oplus G_1, F_1 \oplus H_1 \oplus G_2 \end{aligned} \quad (3.7)$$

are

$$\begin{aligned}
 r(n; H) = & \frac{24}{221} \sigma_3^*(n) + \frac{c_1}{47} \sum_{F_2=n} (47x_1^2 - 12n) + \frac{c_2}{47} \sum_{H_2=n} (47x_1^2 - 6n) \\
 & + \frac{c_3}{2 \cdot 47} \sum_{H_2=n} (2 \cdot 47x_1x_2 + n) + \frac{c_4}{47^2} \sum_{H_2=n} (47^2x_1x_3 - 72n) + \frac{c_5}{47} \sum_{G_2=n} (47x_1^2 - 4n) \\
 & + \frac{c_6}{47} \sum_{G_2=n} (47x_2^2 - 3n) + \frac{c_7}{47} \sum_{G_2=n} (47x_3^2 - 4n) + \frac{c_8}{2 \cdot 47} \sum_{G_2=n} (2 \cdot 47x_3x_4 + n) \\
 & + \frac{c_9}{47} \sum_{H_1 \oplus G_1=n} (47x_1^2 - 6n) + \frac{c_{10}}{47} \sum_{H_1 \oplus G_1=n} (47x_2^2 - 2n) + \frac{c_{11}}{47} \sum_{H_1 \oplus G_1=n} (47x_3^2 - 4n).
 \end{aligned} \tag{3.8}$$

The coefficients

$$\begin{aligned}
 & (c_1 \ c_2 \ c_3 \ c_4 \ c_5 \ c_6), \\
 & (c_7 \ c_8 \ c_9 \ c_{10} \ c_{11})
 \end{aligned} \tag{3.9}$$

corresponding to the quadratic form Ω are given in [4].

Proof. It follows from the preceding theorem. $\square$

References

- [1] A. Tekcan and O. Bizim, "On the number of representations of positive integers by quadratic forms as the basis of the space $S_4(\Gamma_0(47), 1)$," *International Journal of Mathematics and Mathematical Sciences*, no. 9–12, pp. 637–646, 2004.
- [2] T. Miyake, *Modular Forms*, Springer, Berlin, Germany, 1989.
- [3] B. Kendirli, "Cusp Forms in $S_4(\Gamma_0(79))$ and the number of representations of positive integers by some direct sum of binary quadratic forms with discriminant -79 ," *Bulletin of the Korean Mathematical Society*. In press.
- [4] <http://math.fatih.edu.tr/~bkendirli/Announcements.htm>.