Research Article

A Combinatorial Note for Harmonic Tensors

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We give another characterization of the annihilator of the space of (dual) harmonic tensors in the group algebra of symmetric group.

1. Introduction and Preliminaries

Let $m, n \in \mathbb{N}$. Let K be an infinite field and V a $2m$ -dimensional symplectic vector space over K equipped with a skew bilinear form $(\ , \)$. The symplectic group $\mathrm{Sp}(V)$ acts naturally on V from the left hand side, and hence on the n -tensor space $V^{\otimes n}$. Let $B_n = B_n(-2m)$ be the Brauer algebra over K with canonical generators $s_1, \dots, s_{n-1}, e_1, \dots, e_{n-1}$ subject to the following relations:

$$\begin{aligned} s_i^2 &= 1, \quad e_i^2 = (-2m)e_i, \quad e_i s_i = s_i e_i = e_i, \quad \forall 1 \leq i \leq n-1, \\ s_i s_j &= s_j s_i, \quad s_i e_j = e_j s_i, \quad e_i e_j = e_j e_i, \quad \forall 1 \leq i < j-1 \leq n-2, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \quad e_i e_{i+1} e_i = e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1}, \quad \forall 1 \leq i \leq n-2, \\ s_i e_{i+1} e_i &= s_{i+1} e_i, \quad e_{i+1} e_i s_{i+1} = e_{i+1} s_i, \quad \forall 1 \leq i \leq n-2. \end{aligned} \tag{1.1}$$

Note that B_n is a K -algebra with dimension $(2n-1)!! = (2n-1) \cdot (2n-3) \cdots 3 \cdot 1$.

The Brauer algebra was first introduced by Brauer (see [1]) when he studied how the n -tensor space decomposes into irreducible modules over the orthogonal group or the symplectic group. There is a right action of B_n on $V^{\otimes n}$ which we now recall. Let δ_{ij} denote the

Kronecker delta. For each integer i with $1 \leq i \leq 2m$, set $i' := 2m + 1 - i$. We fix an ordered basis $\{v_i\}_{i=1}^{2m}$ of V such that

$$(v_i, v_j) = 0 = (v_{i'}, v_{j'}), \quad (v_i, v_{j'}) = \delta_{ij} = -(v_{j'}, v_i), \quad \forall 1 \leq i, j \leq m. \quad (1.2)$$

For any $i, j \in \{1, 2, \dots, 2m\}$, let

$$\varepsilon_{i,j} := \begin{cases} 1 & \text{if } i = j', \ i < j, \\ -1 & \text{if } i = j', \ i > j, \\ 0 & \text{otherwise.} \end{cases} \quad (1.3)$$

For any simple tensor $v_{i_1} \otimes \dots \otimes v_{i_n} \in V^{\otimes n}$, the right action of B_n on $V^{\otimes n}$ is defined on generators by

$$\begin{aligned} (v_{i_1} \otimes \dots \otimes v_{i_n}) s_j &:= -(v_{i_1} \otimes \dots \otimes v_{i_{j-i}} \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes v_{i_{j+2}} \otimes \dots \otimes v_{i_n}), \\ (v_{i_1} \otimes \dots \otimes v_{i_n}) e_j &:= \varepsilon_{i_j, i_{j+1}} v_{i_1} \otimes \dots \otimes v_{i_{j-1}} \otimes \left(\sum_{k=1}^m (v_{k'} \otimes v_k - v_k \otimes v_{k'}) \right) \otimes v_{i_{j+2}} \otimes \dots \otimes v_{i_n}. \end{aligned} \quad (1.4)$$

The s_j acts as a signed transposition, and e_j acts as a signed contraction. It is well known that the centralizer of the image of the group algebra $K\mathrm{Sp}(V)$ in $\mathrm{End}_K(V^{\otimes n})$ is the image of B_n and vice versa. This fact is called Schur-Weyl duality (see [1–3]).

There is a variant of the above Schur-Weyl duality as we will describe. Let $B_n^{(1)}$ be the two-sided ideal of B_n generated by e_1 . We set

$$W_{1,n} := \left\{ v \in V^{\otimes n} \mid vx = 0, \ \forall x \in B_n^{(1)} \right\}. \quad (1.5)$$

We call $W_{1,n}$ the subspace of *harmonic tensors* or *traceless tensors*. It should be pointed out that this definition coincides with that given in [4] and [11, Section 2.1] by [5, Corollary 2.6]. Note that $B_n/B_n^{(1)} \cong K\mathfrak{S}_n$, the group algebra of the symmetric group $\mathfrak{S}_n$. The right action of B_n on $V^{\otimes n}$ gives rise to a right action of $K\mathfrak{S}_n$ on $W_{1,n}$. We, therefore, have two natural K -algebra homomorphisms

$$\varphi : (K\mathfrak{S}_n)^{\mathrm{op}} \longrightarrow \mathrm{End}_{K\mathrm{Sp}(V)}(W_{1,n}), \quad \psi : K\mathrm{Sp}(V) \longrightarrow \mathrm{End}_{K\mathfrak{S}_n}(W_{1,n}). \quad (1.6)$$

In [4], De Concini and Strickland proved that the dimension of $W_{1,n}$ is independent of the field K and φ is always surjective. Moreover, they showed that φ is an isomorphism if $m \geq n$. When $m < n$, in [4, Theorem 3.5] they also described the kernel of φ , that is, the annihilator of $W_{1,n}$ in the group algebra $K\mathfrak{S}_n$. In this paper, we give another combinatorial characterization of $\mathrm{Ker} \varphi$.

For our aim, we need the notation of *dual harmonic tensors*. Maliakas in [6] proved that $W_{1,n}^*$ has a good filtration when $m \geq n$ by using the theory of rational representations of symplectic group. He claimed that it is also true for arbitrary m . This claim was proved by Hu

in [5] using representations of algebraic groups and canonical bases of quantized enveloping algebras. Furthermore, [5, Corollary 1.6] shows that

$$\frac{V^{\otimes n}}{V^{\otimes n}B_n^{(1)}} \cong W_{1,n}^* \quad (1.7)$$

and, thus, we call $V^{\otimes n}/V^{\otimes n}B_n^{(1)}$ the space of dual harmonic tensors. Therefore, we will only characterize the annihilator of $V^{\otimes n}/V^{\otimes n}B_n^{(1)}$ in the group algebra $K\mathfrak{S}_n$.

2. The Main Results

In this section, we will give an elementary combinatorial characterization of the annihilator of $V^{\otimes n}/V^{\otimes n}B_n^{(1)}$ in the group algebra $K\mathfrak{S}_n$. Besides [4, Theorem 3.5], other characterizations of such annihilator can be found in [7, Theorem 4.2] and [8, Theorem 1.3]. We would like to point out that these approaches depend heavily on invariant theory [4] or representation theory [7, 8]. Therefore, the approach of this paper is more elementary and hence is of independent interest for studying the action of the Brauer algebra $B_n(-2m)$ on n -tensor space $V^{\otimes n}$.

For convenience, we set

$$I(2m, n) := \{(i_1, \dots, i_n) \mid i_j \in \{1, 2, \dots, 2m\}, \forall j\}. \quad (2.1)$$

For any $\underline{i} = (i_1, \dots, i_n) \in I(2m, n)$, we write $v_{\underline{i}} = v_{i_1} \otimes \dots \otimes v_{i_n}$. For $\underline{i} \in I(2m, n)$, an ordered pair (s, t) ($1 \leq s < t \leq n$) is called a *symplectic pair* in $\underline{i}$ if $i_s = i'_t$. Two ordered pairs (s, t) and (u, v) are called disjoint if $\{s, t\} \cap \{u, v\} = \emptyset$. We define the *symplectic length* $\ell_s(v_{\underline{i}})$ to be the maximal number of disjoint symplectic pairs (s, t) in $\underline{i}$ (see [3, Page 198]). Without confusion, we will adopt the same symbol for the image of the canonical generator s_i of the Brauer algebra in the group algebra $K\mathfrak{S}_n$. More or less motivated by the work [9] of Härterich, we have the following proposition.

Proposition 2.1. *For any simple tensor $v_{\underline{i}} \in V^{\otimes n}$ there is $v_{\underline{i}}x_{m+1} \in V^{\otimes n}B_n^{(1)}$, where $x_{m+1} = \sum_{w \in \mathfrak{S}_{m+1}} w$.*

Proof. If we have proved the proposition over the base field $\mathbb{Q}$ of rational numbers, it can be restated as a result in $\mathbb{Z}\mathfrak{S}_n$ by restriction since x_{m+1} is a $\mathbb{Z}$ -linear combination of basis elements of $\mathbb{Z}\mathfrak{S}_n$. Applying the specialization functor $K_{\otimes \mathbb{Z}}$, we obtain the present statement. Therefore, we now assume we work on the base field $\mathbb{Q}$.

By the actions of Brauer algebras on n -tensor spaces defined in Section 1, we know that x_{m+1} only acts on the first $m+1$ components of $v_{\underline{i}}$. Hence, we can set $n = m+1$ without loss of the generality. Let $v_{\underline{i}} = v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_{m+1}}$. If the $(m+1)$ -tuple $(i_1, i_2, \dots, i_{m+1})$ has a repeated number, for instance, $i_s = i_r$ with $s < r$, then obviously $v_{\underline{i}}x_{m+1} = v_{\underline{i}}(s, r)x_{m+1} = -v_{\underline{i}}x_{m+1}$ and hence $v_{\underline{i}}x_{m+1} = 0$, where (s, r) is a transposition.

Then, we assume that $i_1, i_2, \dots, i_{m+1}$ are different from each other. Noting that $\dim_{\mathbb{Q}} V = 2m$, there exists at least one symplectic pair in $\underline{i}$. We assume the symplectic length

$\ell_s(v_i) = s$ ($1 \leq s \leq [(m+1)/2]$) and $v_i = v_1 \otimes v_{2m} \otimes v_2 \otimes v_{2m-1} \otimes \cdots \otimes v_s \otimes v_{2m-s+1} \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1}$ without loss of the generality. Then

$$\begin{aligned}
 v_i x_{m+1} &= v_1 \otimes v_{2m} \otimes v_2 \otimes v_{2m-1} \otimes \cdots \otimes v_s \otimes v_{2m-s+1} \\
 &\quad \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1} \\
 &= \frac{1}{2} (v_1 \otimes v_{1'} - v_{1'} \otimes v_1) \otimes v_2 \otimes v_{2'} \otimes \cdots \otimes v_s \otimes v_{s'} \\
 &\quad \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1} \\
 &\equiv \frac{1}{2} \left(\sum_{j=2}^m v_{j'} \otimes v_j - v_j \otimes v_{j'} \right) \otimes v_2 \otimes v_{2'} \otimes \cdots \otimes v_s \otimes v_{s'} \\
 &\quad \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1} \pmod{V^{\otimes(m+1)} B_{m+1}^{(1)}} \\
 &= \left(\sum_{j=m-s+2}^m v_{j'} \otimes v_j \right) \otimes v_2 \otimes v_{2'} \otimes \cdots \otimes v_s \otimes v_{s'} \\
 &\quad \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1}.
 \end{aligned} \tag{2.2}$$

In the following, the notation $\equiv$ always means equivalence mod $V^{\otimes(m+1)} B_{m+1}^{(1)}$. We abbreviate w_j for $v_{j'} \otimes v_j$, noting that $w_j(1, 2) = -v_j \otimes v_{j'}$. By the same procedures, we obtain

$$v_i x_{m+1} \equiv w_{1'} \otimes \cdots \otimes w_{(k-1)'} \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes w_{(k+1)'} \otimes \cdots \otimes w_{s'} \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1}, \tag{2.3}$$

where $1 \leq k \leq s$.

Now we assume for $1 < l \leq s$ that

$$\begin{aligned}
 ((l-1)!) v_i x_{m+1} &\equiv w_{1'} \otimes \cdots \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes w_{k'_1} \otimes \cdots \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes w_{k'_2} \\
 &\quad \otimes \cdots \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes w_{k'_{(l-1)}} \otimes \cdots \otimes w_{s'} \\
 &\quad \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1},
 \end{aligned} \tag{2.4}$$

where the $l - 1$ summands $\sum_{j=m-s+2}^m w_j$ appear at the $(k_1 - 1)$ -th, $(k_2 - 1)$ -th, $\dots$, $(k_{l-1} - 1)$ -th positions ($1 \leq k_1 - 1 < k_2 - 1 < \dots < k_{l-1} - 1 \leq s$), respectively. We want to prove that

$$\begin{aligned}
 (l!)v_{\underline{l}}x_{m+1} &\equiv w_{1'} \otimes \dots \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes w_{k'_1} \otimes \dots \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes w_{k'_2} \\
 &\otimes \dots \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes w_{k'_l} \otimes \dots \otimes w_{s'} \\
 &\otimes v_{s+1} \otimes \dots \otimes v_{m-s+1}x_{m+1},
 \end{aligned} \tag{2.5}$$

where the l summands $\sum_{j=m-s+2}^m w_j$ appear at the $(k_1 - 1)$ -th, $(k_2 - 1)$ -th, $\dots$, $(k_l - 1)$ -th positions ($1 \leq k_1 - 1 < k_2 - 1 < \dots < k_l - 1 \leq s$), respectively. Without loss of the generality, we only need to prove it for the case $1 \leq k_1 - 1 < k_2 - 1 < \dots < k_l - 1 \leq l$. In fact, we have

$$\begin{aligned}
 (2(l-1)!)v_{\underline{l}}x_{m+1} &\equiv v_1 \otimes v_{1'} \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes \dots \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes v_{(l+1)} \otimes v_{(l+1)'} \\
 &\otimes \dots \otimes v_s \otimes v_{s'} \otimes v_{s+1} \otimes \dots \otimes v_{m-s+1}x_{m+1} \\
 &+ \left(\sum_{j=m-s+2}^m w_j \right) \otimes \dots \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes v_l \otimes v_{l'} \\
 &\otimes \dots \otimes v_s \otimes v_{s'} \otimes v_{s+1} \otimes \dots \otimes v_{m-s+1}x_{m+1} \\
 &= (v_1 \otimes v_{1'} + v_l \otimes v_{l'}) \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes \dots \otimes \left(\sum_{j=m-s+2}^m w_j \right) \\
 &\otimes v_{(l+1)} \otimes v_{(l+1)'} \otimes \dots \otimes v_s \otimes v_{s'} \otimes v_{s+1} \otimes \dots \otimes v_{m-s+1}x_{m+1} \\
 &\equiv \left(\sum_{j=2}^{l-1} w_j + \sum_{j=m-s+2}^m w_j \right) \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes \dots \otimes \left(\sum_{j=m-s+2}^m w_j \right) \\
 &\otimes v_{(l+1)} \otimes v_{(l+1)'} \otimes \dots \otimes v_s \otimes v_{s'} \otimes v_{s+1} \otimes \dots \otimes v_{m-s+1}x_{m+1} \\
 &\equiv -(l-2)((l-1)!)v_{\underline{l}}x_{m+1} + \left(\sum_{j=m-s+2}^m w_j \right) \otimes \dots \otimes \left(\sum_{j=m-s+2}^m w_j \right) \\
 &\otimes v_{(l+1)} \otimes v_{(l+1)'} \otimes \dots \otimes v_s \otimes v_{s'} \otimes v_{s+1} \otimes \dots \otimes v_{m-s+1}x_{m+1},
 \end{aligned} \tag{2.6}$$

where the last equivalence follows from the induction hypothesis and the fact $w_j(1,2) = -v_j \otimes v_{j'}$. Hence, we have proved what we desired.

As a consequence, we immediately get that

$$(s!)v_i x_{m+1} \equiv \left(\sum_{j=m-s+2}^m w_j \right) \otimes \cdots \otimes \left(\sum_{j=m-s+2}^m w_j \right) \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1}. \quad (2.7)$$

However, $m - (m - s + 1) = s - 1$, there must exist a repeated w_j in the right hand side of the above equivalence when written as a linear combination of simple tensors. Therefore, $v_i x_{m+1} \equiv 0$. $\square$

Theorem 2.2. *The annihilator of the space $V^{\otimes n}/V^{\otimes n}B_n^{(1)}$ of dual harmonic tensors in the group algebra $K\mathfrak{S}_n$ is the principal ideal $\langle x_{m+1} \rangle$.*

Proof. We denote $\text{Ann}(V^{\otimes n}/V^{\otimes n}B_n^{(1)})$ as the annihilator of the space $V^{\otimes n}/V^{\otimes n}B_n^{(1)}$ of dual harmonic tensors in the group algebra $K\mathfrak{S}_n$. It follows from Proposition 2.1 that

$$\langle x_{m+1} \rangle \subseteq \text{Ann}\left(\frac{V^{\otimes n}}{V^{\otimes n}B_n^{(1)}}\right). \quad (2.8)$$

On the other hand, by the work of [10], we know that

$$\langle x_{m+1} \rangle = K - \text{Span}\left\{ m_{s,t}^\lambda \mid \lambda \vdash n, \ell(\lambda) > m, s, t \in \text{Std}(\lambda) \right\}, \quad (2.9)$$

where each $m_{s,t}^\lambda$ is the Murphy basis element in [10], and $\text{Std}(\lambda)$ denotes the set of standard λ -tableaux with entries in $\{1, 2, \dots, n\}$. In particular, [5, Theorem 1.8] shows that (see also [4])

$$\begin{aligned} \dim_K \langle x_{m+1} \rangle &= \sum_{\lambda \vdash n, \ell(\lambda) > m} (\dim_K S^\lambda)^2 \\ &= \dim_{\mathbb{Q}} \mathbb{Q}\mathfrak{S}_n - \text{End}_{\mathbb{Q}\text{Sp}(V)}\left(\frac{V^{\otimes n}}{V^{\otimes n}B_n^{(1)}}\right) \\ &= \dim_K K\mathfrak{S}_n - \text{End}_{K\text{Sp}(V)}\left(\frac{V^{\otimes n}}{V^{\otimes n}B_n^{(1)}}\right) \\ &= \dim_K \text{Ann}\left(\frac{V^{\otimes n}}{V^{\otimes n}B_n^{(1)}}\right), \end{aligned} \quad (2.10)$$

where S^λ denotes the Specht module of $K\mathfrak{S}_n$ associated to λ . This completes the proof of the theorem. $\square$

Let $B_n^{(f)}$ be the two-sided ideal of B_n generated by $e_1 e_3 \cdots e_{2f-1}$ with $1 \leq f \leq [n/2]$. Let $X_{m+1} \in B_n$ be the element defined in [7, Page 2912]. We end this note by a conjecture which is connected with the invariant theory of classical groups (see [11, 12]).

Conjecture 2.3. *The annihilator of the space $V^{\otimes n}/V^{\otimes n}B_n^{(f)}$ of dual partially harmonic tensors of valence f in the algebra $B_n/B_n^{(f)}$ is the principal ideal $\langle X_{m+1} + B_n^{(f)} \rangle$.*

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